Thai Journal of Mathematics
Vol. 18, No. 1 (2020),
Pages 296 - 314

ISSN 1686-0209

APPLICATIONS OF GENERALIZED PICTURE FUZZY SOFT SET IN CONCEPT SELECTION

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Abstract The incorrect identification of a good concept for a particular product leads to an increase in design modification, which increases the functional cost and generating time, therefore the concept selection is a crucial process in the new product development (NPD). In this paper, we used the generalized picture fuzzy soft set to obtain the optimal design concept. We introduced the b-picture fuzzy soft set ($bPFSS$) and generalized b-picture fuzzy soft set ($GbPFSS$) on the basis of the bijective soft set. We introduced the lower-and-substitution and upper-or-substitution operations for $GbPFSS$ and discuss their basic properties. The $GbPFSSs$ are used to illustrate the mapping from customers requirements to design concepts. We proposed an algorithm for choosing best design concept using the upper-or-substitution operations for $GbPFSS$. After we discuss a case study for the design concept of laptops for one and two customers separately.

MSC: 94D05; 03E72

Keywords: Generalized picture fuzzy soft sets; concept selection, bijective soft sets

Submission date: 28.11.2019 / Acceptance date: 30.01.2020

1. Introduction

Due to continual change in customers requirements, apace development in new technologies, and increased global competition, the companies develop new products with minimum costs and superior quality [1]. The incorrect identification of a good concept for a particular product leads to an increase in design modification, which increases the functional cost and generating time, therefore the concept selection is a crucial process in NPD [2]. The design concept involves the complex decision making at early stages because of imprecise, subjective and vague data. Based on the discernment of design experts, the design concept process is effective by incomplete and imprecise data and information provided in a qualitative form [3]. Due to many factors, it can be considered

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Published by The Mathematical Association of Thailand.

as the group decision making problem because it relies on the subjective and vague decision maker judgements, linguistic customers requirements, customer satisfaction level, the trade-off between design criteria, and the performance of design alternatives. Therefore, the concept selection process is subject to various degrees of uncertainty from these factors [4]. There is a need of an effective model for the concept selection to deals with the subjective and vague decision maker judgements, linguistic customers requirements, customer satisfaction level, the trade-off between design criteria, and the performance of design alternatives.

Many classical theories such as fuzzy set theory [5], probability theory, vague set theory [6], rough set theory [7], intuitionistic fuzzy set [8], and interval mathematics [9] are well known and effectively model uncertainty. These approaches show their inherent difficulties as pointed out by Molodtsov [10], because of intensive quantity and type of uncertainties. In [10], Molodtsov defines the soft set which is an absolutely new logical instrument for dealing uncertainties. Nowadays, many authors work to hybridize the different models with the soft set and achieved results in many applicable theories. Agarwal defines the generalized intuitionistic fuzzy soft set ($GIFSS$) [11] which has some problems pointed out by Feng [12] and redefined $GIFSS$. In [13], Coung defines the picture fuzzy set which is an extension of the fuzzy soft set and intuitionistic fuzzy set. In [14], Yang defines the picture fuzzy soft set and applied them to decision-making problems. In [15], Jabir khan defines the generalized picture fuzzy soft set and applied them to decision making problems. For study more about decision making, we refer to [16–22].

The bijective soft sets define by Gong [23] are important in concept selection. Actually bijective soft set mapped every element to only one parameter. In the design concept perspective, one target meets to one target value from each requirement. For example, the size of the mobile phone is large or small at one time, and it can not be both at once.

1.1. Related Work

In NPD, concept selection has a fundamental and important role. Different decision making approaches have developed by researchers on concept selection. Concept evaluation method is categorized into five categories by King and Sivaloganathan which are, namely, quality function deployment (QFD), utility theory, graphical tools, analytic hierarchical process (AHP), and fuzzy logic method [24]. The analytic hierarchical process for the design problem was introduced by Marsh [25]. The analytic network process (ANP) based concept selection was made by Ayag and Ozdemir [26]. The AHP and TOPSIS based design concept evaluation are made by Lin [27]. In [27], according to Lin, the conventional decision making techniques like AHP is unable to capture the vague and subjective judgments of decision makers in concept selection. Based on integrating Delphi method and fuzzy theory into the AHP method, a comparison for design alternatives under a subjective, vague and uncertain environment is made by Sii and Wang [28]. To select the best design concept from the developed designs, a fuzzy multi-layer graph-based model is proposed to resolve the conflict of experts judgments and opinions by Jenab [29]. By combining quality function deployment and group decision making, an integrated method was proposed in order to improve the effectiveness of concept evaluation process by Zhang and Chu [30].

A VIKOR method for interval numbers for selecting the best alternative in the presence of conflicting criteria is proposed by Sayadi [31]. Akay [32] used the interval type-2 fuzzy sets for selecting the best concept. An integrated method is proposed by Chen and

Tsao, which is based on interval value fuzzy sets and TOPSIS to calculate the measures for the relative importance of the design parameters and effects of the generated design alternatives on these parameters [33].

For capturing uncertain, vague and subjective information many researchers have also integrated rough set theory and decision making methods. For computing relative importance rating of design criteria and rule mining, rough set theory was adopted in the decision-making methods by Zhu [2]. Two new concepts namely, rough number and rough boundary interval were proposed by Zhai to capture subjective and linguistic assessments in QFD [34]. An integrating rough number based AHP and rough number based VIKOR method are proposed by Zhu for concept evaluation [2]. By combining rough numbers with AHP and rough numbers with TOPSIS, a novel concept evaluation methodology was proposed by Song [35]. An integrated method for concept selection based on rough numbers and VIKOR method was proposed by Tiwari [36]. For performing the concept evaluation process effectively, a TOPSIS method for vague sets was proposed by Geng [37]. In [38], Khizar proposed an algorithm for best concept selection using bijective soft sets, generalized intuitionistic fuzzy soft set and int-AND-product operation on generalized intuitionistic fuzzy soft set. In [39], a promising framework is developed based on soft sets, TOPSIS and the Shannon entropy which aggregates concept selection on design parameters values by merging acceptable and satisfactory level requirements of the customers by Khizar.

1.2. Motivation and Organization of the Paper

There are various degrees of uncertainties in the concept selection process to deals with the subjective and vague decision maker judgments, linguistic customers requirements, customer satisfaction level, the trade-off between design criteria, and the performance of design alternatives. Therefore, only the soft set and fuzzy preferences of customers are not enough because only membership function is not enough to represent the uncertainty. Also, the membership and non-membership functions of intuitionistic fuzzy set are not enough to model uncertainty, therefore, picture fuzzy environment is more suitable for this type of uncertainties. In picture fuzzy environment, we have an extra input function, namely, neutral membership function. So the uncertainty can be model with more accuracy and the range of decision maker has expanded to three membership functions. We also use the operation, namely, upper-or-substitution operation which aggregates the information from from different design concepts and get maximum value of membership function.

The paper organized as follows. Section 1 and 2 consists of introduction and preliminaries. Section 3 presents the definitions of $bPFSS$, $GbPFSS$, lower-and-substitution, and upper-or-substitution operations between $GbPFSS$ and some properties of these operations. Section 4 consists of the methodology, where we discuss in details the design concepts generation with an example, representing the mapping of customers requirements to the design concepts. The algorithm to get the best design concept using $GbPFSS$ and some related concepts is proposed and the case study of laptops for one and two customers are discussed separately in section 5. The discussion and conclusion are presented in sections 6 and 7.

2. Preliminaries

A fuzzy set is defined by Zadeh [5], which handles uncertainty based on the view of gradualness effectively.

Definition 2.1. [5] A membership function $\xi_{\hat{\mathcal{A}}} : \hat{\mathcal{Y}} \to [0,1]$ defines the fuzzy set $\hat{\mathcal{A}}$ over the $\hat{\mathcal{Y}}$, where $\xi_{\hat{\mathcal{A}}}(y)$ particularized the membership of an element $y \in \hat{\mathcal{Y}}$ in fuzzy set $\hat{\mathcal{A}}$.

A soft set is defined by Molodtsov [10], which provides an effective framework to dealings with imprecision with the parametric point of view, i.e. each element is judged by some criteria of attributes.

Definition 2.2. [10] Let $\hat{\mathcal{Y}}$ be a universal set, $\hat{\mathcal{P}}$ a parameter space, $\hat{\mathcal{A}} \subseteq \hat{\mathcal{P}}$ and $P(\hat{\mathcal{Y}})$ the power set of $\hat{\mathcal{Y}}$. A pair $(\hat{\mathcal{T}}, \hat{\mathcal{A}})$ is called a soft set over $\hat{\mathcal{Y}}$, where $\hat{\mathcal{T}}$ is a set valued mapping given by $\hat{\mathcal{T}} : \hat{\mathcal{A}} \to P(\hat{\mathcal{Y}})$.

In the design concept perspective the bijective soft sets define by Gong [23], have importance because a bijective soft set mapped every element to only one parameter i.e., in design concept one target meet to one only target value from each requirement. For example, the size of a laptop is large or small at one time, and it can not be both at once.

Definition 2.3. [23] Let $(\hat{\mathcal{T}}, \hat{\mathcal{A}})$ be a soft set over $\hat{\mathcal{Y}}$, $\hat{\mathcal{A}} \subseteq \hat{\mathcal{P}}$ a parameter set. Then $(\hat{\mathcal{T}}, \hat{\mathcal{A}})$ is called the bijective soft set if the following two properties are satisfies:

1. $\cup_{h \in \hat{\mathcal{A}}} \hat{\mathcal{T}}(h) = \hat{\mathcal{Y}}$.
2. If $p \neq q$ be any two attributes in $\hat{\mathcal{A}}$, then it must be the case that $\hat{\mathcal{T}}(p) \cap \hat{\mathcal{T}}(q) = \emptyset$.

In [13], Cuong defines the PFS, which is an extension of fuzzy set and applicable in many real life problems. By adding an extra membership function, namely, the degree of the neutral membership function, the picture fuzzy set is obtained. Basically, the model of the picture fuzzy set may be adequate in situations when we face human opinions involving more answers of the type: yes, abstain, no, refusal. Voting can be a good example of picture fuzzy set because it involves the situation of more answers of the type: yes, abstain, no, refusal.

Definition 2.4. [13] A PFS $\hat{\mathcal{A}}$ over the universe $\hat{\mathcal{Y}}$ is defined as

$$\hat{\mathcal{A}} = \{(y, \xi_{\hat{\mathcal{A}}}, \eta_{\hat{\mathcal{A}}}, \upsilon_{\hat{\mathcal{A}}}) | y \in \hat{\mathcal{Y}}\},$$

where $\xi_{\hat{\mathcal{A}}} : \hat{\mathcal{Y}} \to [0,1]$, $\eta_{\hat{\mathcal{A}}} : \hat{\mathcal{Y}} \to [0,1]$ and $\vartheta_{\hat{\mathcal{A}}} : \hat{\mathcal{Y}} \to [0,1]$ are the degree of positive membership, neutral membership and degree of negative membership, respectively. Furthermore, it is required that $0 \leq \xi_{\hat{\mathcal{A}}} + \eta_{\hat{\mathcal{A}}} + \upsilon_{\hat{\mathcal{A}}} \leq 1$. Then for $y \in \hat{\mathcal{Y}}$, $\pi_{\hat{\mathcal{A}}}(y) = 1 - (\xi_{\hat{\mathcal{A}}}(y) + \eta_{\hat{\mathcal{A}}}(y) + \upsilon_{\hat{\mathcal{A}}}(y))$ is called the degree of refusal membership of y in $\hat{\mathcal{A}}$. For PFS $(\xi_{\hat{\mathcal{A}}}(y), \eta_{\hat{\mathcal{A}}}(y), \upsilon_{\hat{\mathcal{A}}}(y))$ are said to picture fuzzy value (PFV) or picture fuzzy number (PFN) and each PFV can be denoted by $q = (\xi_q, \eta_q, \upsilon_q)$, where ξ_q, η_q and $\upsilon_q \in [0,1]$, with condition that $0 \leq \xi_q + \eta_q + \upsilon_q \leq 1$.

Definition 2.5. [23] Let $\hat{\mathcal{A}}$ and $\hat{\mathcal{B}}$ be two $PFSs$ over $\hat{\mathcal{Y}}$. Then their containment, union, intersection and complement are defined as follows:

1. $\hat{\mathcal{A}} \subset \hat{\mathcal{B}}$, if $\xi_{\hat{\mathcal{A}}} \leq \xi_{\hat{\mathcal{B}}}$, $\eta_{\hat{\mathcal{A}}} \leq \eta_{\hat{\mathcal{B}}}$ and $\vartheta_{\hat{\mathcal{A}}} \geq \vartheta_{\hat{\mathcal{B}}}$, $\forall y \in \hat{\mathcal{Y}}$,
2. $\hat{\mathcal{A}} \cup \hat{\mathcal{B}} = \{(y, \max(\xi_{\hat{\mathcal{A}}}, \xi_{\hat{\mathcal{B}}}), \min(\eta_{\hat{\mathcal{A}}}, \eta_{\hat{\mathcal{B}}}), \min(\vartheta_{\hat{\mathcal{A}}}, \vartheta_{\hat{\mathcal{B}}})) | \forall y \in \hat{\mathcal{Y}}\}$,

3. $\hat{\mathcal{A}} \cap \hat{\mathcal{B}} = \{(y, \min(\xi_{\hat{\mathcal{A}}}, \xi_{\hat{\mathcal{B}}}), \min(\eta_{\hat{\mathcal{A}}}, \eta_{\hat{\mathcal{B}}}), \max(\vartheta_{\hat{\mathcal{A}}}, \vartheta_{\hat{\mathcal{B}}}))|\forall y \in \hat{\mathcal{Y}}\}$,
4. $\hat{\mathcal{A}}^c = \{(y, \vartheta_{\hat{\mathcal{A}}}, \eta_{\hat{\mathcal{A}}}, \xi_{\hat{\mathcal{A}}})|y \in \hat{\mathcal{Y}}\}$.

In [14], Yang defines the $PFSS$ as follows.

Definition 2.6. Let $\hat{\mathcal{Y}}$ be a universal set, $\hat{\mathcal{P}}$ a parameter space, $\hat{\mathcal{A}} \subset \hat{\mathcal{P}}$ and $PF(\hat{\mathcal{Y}})$ the set of all $PFSs$ over $\hat{\mathcal{Y}}$. A pair $(\hat{\mathcal{T}}, \hat{\mathcal{A}})$ is called a $PFSS$ over $\hat{\mathcal{Y}}$, where $\hat{\mathcal{T}}$ is a set valued mapping given by $\hat{\mathcal{T}} : \hat{\mathcal{A}} \to PF(\hat{\mathcal{Y}})$.

Yang also defines the and-operation and or-operation for $PFSS$ in [14].

Definition 2.7. Let $\hat{\mho}_1 = (\hat{\mathcal{T}}, \hat{\mathcal{A}})$ and $\hat{\mho}_2 = (\hat{\mathcal{S}}, \hat{\mathcal{B}})$ be two $PFSSs$ over $\hat{\mathcal{Y}}$. Then the "and-operation" is denoted as the $PFSS$ $(\hat{\mathcal{F}}, \hat{\mathcal{C}}) = (\hat{\mathcal{T}}, \hat{\mathcal{A}}) \vartriangle (\hat{\mathcal{S}}, \hat{\mathcal{B}})$, where $\hat{\mathcal{C}} = \hat{\mathcal{A}} \times \hat{\mathcal{B}}$ and defined as

$$(\hat{\mathcal{F}}, \hat{\mathcal{C}}) = \{(y, \min\{\xi_{\hat{\mathcal{T}}(p)}(y), \xi_{\hat{\mathcal{S}}(q)}(y)\}, \min\{\eta_{\hat{\mathcal{T}}(p)}(y), \eta_{\hat{\mathcal{S}}(q)}(y)\},$$
$$\max\{\upsilon_{\hat{\mathcal{T}}(p)}(y), \upsilon_{\hat{\mathcal{S}}(q)}(y)\})|(p,q) \in \hat{\mathcal{C}}\}.$$

Definition 2.8. Let $\hat{\mho}_1 = (\hat{\mathcal{T}}, \hat{\mathcal{A}})$ and $\hat{\mho}_2 = (\hat{\mathcal{S}}, \hat{\mathcal{B}})$ be two $PFSSs$ over $\hat{\mathcal{Y}}$. Then the "or-operation" is denoted as the $PFSS$ $(\hat{\mathcal{F}}, \hat{\mathcal{C}}) = (\hat{\mathcal{T}}, \hat{\mathcal{A}}) \lozenge (\hat{\mathcal{S}}, \hat{\mathcal{B}})$, where $\hat{\mathcal{C}} = \hat{\mathcal{A}} \times \hat{\mathcal{B}}$ and defined as

$$(\hat{\mathcal{F}}, \hat{\mathcal{C}}) = \{(y, \max\{\xi_{\hat{\mathcal{T}}(p)}(y), \xi_{\hat{\mathcal{S}}(q)}(y)\}, \min\{\eta_{\hat{\mathcal{T}}(p)}(y), \eta_{\hat{\mathcal{S}}(q)}(y)\},$$
$$\min\{\upsilon_{\hat{\mathcal{T}}(p)}(y), \upsilon_{\hat{\mathcal{S}}(q)}(y)\})|(p,q) \in \hat{\mathcal{C}}\}.$$

In [15], Jabir defines the $GPFSS$. The idea of $GPFSS$ is very encouraging in decision-making since it considers how to capitalize an additional picture fuzzy input from the director to minimize any possible perversion in the data provided by evaluating specialists.

Definition 2.9. Let $\hat{\mathcal{Y}}$ be a universal set, $\hat{\mathcal{A}} \subset \hat{\mathcal{P}}$ a parametric set. By a $GPFSS$ we mean a triple $(\hat{\mathcal{T}}, \hat{\mathcal{A}}, \hat{\rho})$, where $(\hat{\mathcal{T}}, \hat{\mathcal{A}})$ is a $PFSS$ over $\hat{\mathcal{Y}}$ and $\hat{\rho} : \hat{\mathcal{A}} \to PF(\hat{\mathcal{A}})$ is a PFS in $\hat{\mathcal{A}}$.

Where $(\hat{\mathcal{F}}, \hat{\mathcal{A}})$ is called basic picture fuzzy soft set ($BPFSS$) and $\hat{\rho}$ is called the parametric picture fuzzy set ($PPFS$).

We define the expected score value from PFS as follows.

Definition 2.10. Let $p = (\xi_p, \eta_p, \upsilon_p)$ be a picture fuzzy value PFV, then the expected score value is define as

$$\delta(p) = \frac{\xi_p - \eta_p - \upsilon_p + 1}{2}. \tag{2.1}$$

3. Operations on Generalized b-Picture Fuzzy Soft Set

In this section, we define the b-picture fuzzy soft set ($bPFSS$), generalized b-picture fuzzy soft set $GbPFSS$, lower-and-substitution and upper-or-substitution operations between $GbPFSS$ and some properties of these operations.

Definition 3.1. Let $(T, \hat{\mathcal{A}})$ be a bijective soft set over $\hat{\mathcal{Y}}$. Then the b-picture fuzzy soft set $bPFSS$, a picture fuzzy representation of $(T, \hat{\mathcal{A}})$, is denoted as $(\hat{\mathcal{T}}, \hat{\mathcal{A}})$ and defined as

1. $\xi_{\hat{\mathcal{A}}}(y) = 0$, $\eta_{\hat{\mathcal{A}}}(y) = 0$ and $\upsilon_{\hat{\mathcal{A}}}(y) = 1$, for each $y \notin T(h)$ for all $h \in \hat{\mathcal{A}}$,

2. $\xi_{\hat{\mathcal{A}}}(y) \in [0,1]$, $\eta_{\hat{\mathcal{A}}}(y) \in [0,1]$ and $\upsilon_{\hat{\mathcal{A}}}(y) \in [0,1]$, for each $y \in T(h)$ $\forall h \in \hat{\mathcal{A}}$, such that $0 \leq \xi_{\hat{\mathcal{A}}}(y) + \eta_{\hat{\mathcal{A}}}(y) + \upsilon_{\hat{\mathcal{A}}}(y) \leq 1$.

Definition 3.2. A $GbPFSS$ over $\hat{\mathcal{Y}}$ is denoted and defined as $(\hat{\mathcal{T}}, \hat{\mathcal{A}}, \hat{\rho})$, where $(\hat{\mathcal{T}}, \hat{\mathcal{A}})$ is the $bPFSS$ over $\hat{\mathcal{Y}}$ and $\hat{\rho}$ is a PFS in $\hat{\mathcal{A}}$.

Now we define lower-and-substitution and upper-or-substitution operations between $GbPFSSs$.

Definition 3.3. Let $\hat{\mho}_1 = (\hat{\mathcal{T}}, \hat{\mathcal{A}}, \hat{\rho})$ and $\hat{\mho}_2 = (\hat{\mathcal{S}}, \hat{\mathcal{B}}, \hat{\sigma})$ be two $GbPFSSs$ over $\hat{\mathcal{Y}}$. Then the lower-and-substitution operation is defined as the $GbPFSS$ $(\hat{\mathcal{F}}, \hat{\mathcal{C}}, \hat{\tau}) = (\hat{\mathcal{T}}, \hat{\mathcal{A}}, \hat{\rho}) \vartriangle (\hat{\mathcal{S}}, \hat{\mathcal{B}}, \hat{\sigma})$, where $\hat{\mathcal{C}} = \hat{\mathcal{A}} \times \hat{\mathcal{B}}$, such that $(\hat{\mathcal{F}}, \hat{\mathcal{C}}) = (\hat{\mathcal{T}}, \hat{\mathcal{A}}) \vartriangle (\hat{\mathcal{S}}, \hat{\mathcal{B}})$ and

$$\hat{\tau} = \{(h, min\{\xi_{\hat{\mathcal{A}}}(p), \xi_{\hat{\sigma}}(q)\}, min\{\eta_{\hat{\rho}}(p), \eta_{\hat{\sigma}}(q)\}, max\{\upsilon_{\hat{\rho}}(p), \upsilon_{\hat{\sigma}}(q)\}) | h = (p,q) \in \hat{\mathcal{C}}\}.$$

Definition 3.4. Let $\hat{\mho}_1 = (\hat{\mathcal{T}}, \hat{\mathcal{A}}, \hat{\rho})$ and $\hat{\mho}_2 = (\hat{\mathcal{S}}, \hat{\mathcal{B}}, \hat{\sigma})$ be two $GbPFSSs$ over $\hat{\mathcal{Y}}$. Then the upper-or-substitution operation is defined as the $GbPFSS$ $(\hat{\mathcal{F}}, \hat{\mathcal{C}}, \hat{\tau}) = (\hat{\mathcal{T}}, \hat{\mathcal{A}}, \hat{\rho}) \Diamond (\hat{\mathcal{S}}, \hat{\mathcal{B}}, \hat{\sigma})$, where $\hat{\mathcal{C}} = \hat{\mathcal{A}} \times \hat{\mathcal{B}}$, such that $(\hat{\mathcal{F}}, \hat{\mathcal{C}}) = (\hat{\mathcal{T}}, \hat{\mathcal{A}}) \Diamond (\hat{\mathcal{S}}, \hat{\mathcal{B}})$ and

$$\hat{\tau} = \{(h, max\{\xi_{\hat{\mathcal{A}}}(p), \xi_{\hat{\sigma}}(q)\}, min\{\eta_{\hat{\rho}}(p), \eta_{\hat{\sigma}}(q)\}, min\{\upsilon_{\hat{\rho}}(p), \upsilon_{\hat{\sigma}}(q)\}) | h = (p,q) \in \hat{\mathcal{C}}\}.$$

Now we discuss some properties of the lower-and-substitution and the upper-or-substitution operations.

Theorem 3.5. *Let $(\hat{\mathcal{T}}, \hat{\mathcal{A}}, \hat{\rho})$ and $(\hat{\mathcal{S}}, \hat{\mathcal{B}}, \hat{\sigma})$ be two $GbPFSSs$ over $\hat{\mathcal{Y}}$. Then we have*

1. $[(\hat{\mathcal{T}}, \hat{\mathcal{A}}, \hat{\rho}) \vartriangle (\hat{\mathcal{S}}, \hat{\mathcal{B}}, \hat{\sigma})]^c = (\hat{\mathcal{T}}, \hat{\mathcal{A}}, \hat{\rho})^c \Diamond (\hat{\mathcal{S}}, \hat{\mathcal{B}}, \hat{\sigma})^c$;
2. $[(\hat{\mathcal{T}}, \hat{\mathcal{A}}, \hat{\rho}) \Diamond (\hat{\mathcal{S}}, \hat{\mathcal{B}}, \hat{\sigma})]^c = (\hat{\mathcal{T}}, \hat{\mathcal{A}}, \hat{\rho})^c \vartriangle (\hat{\mathcal{S}}, \hat{\mathcal{B}}, \hat{\sigma})^c$.

Proof. (1) Suppose that $(\hat{\mathcal{F}}, \hat{\mathcal{C}}, \hat{\tau}) = (\hat{\mathcal{T}}, \hat{\mathcal{A}}, \hat{\rho}) \vartriangle (\hat{\mathcal{S}}, \hat{\mathcal{B}}, \hat{\sigma})$, where $\hat{\mathcal{C}} = \hat{\mathcal{A}} \times \hat{\mathcal{B}}$. Then $(\hat{\mathcal{T}}, \hat{\mathcal{A}}) \vartriangle (\hat{\mathcal{S}}, \hat{\mathcal{B}}) = (\hat{\mathcal{F}}, \hat{\mathcal{C}})$ and $\hat{\rho} \vartriangle \hat{\sigma} = \hat{\tau}$. Therefore $((\hat{\mathcal{T}}, \hat{\mathcal{A}}) \vartriangle (\hat{\mathcal{S}}, \hat{\mathcal{B}}))^c = (\hat{\mathcal{F}}, \hat{\mathcal{C}})^c = (\hat{\mathcal{F}}^c, \hat{\mathcal{C}})$ and $(\hat{\rho} \vartriangle \hat{\sigma})^c = \hat{\tau}^c$. Take $(p,q) \in \hat{\mathcal{C}} = \hat{\mathcal{A}} \times \hat{\mathcal{B}}$, therefore, $\hat{\mathcal{F}}^c(p,q) = (\hat{\mathcal{F}}(p,q))^c = (\hat{\mathcal{T}}(p) \cap \hat{\mathcal{S}}(q))^c = \hat{\mathcal{T}}^c(p) \cup \hat{\mathcal{S}}^c(q)$ and also for

$$\hat{\tau}(p,q) = \hat{\rho}(p) \vartriangle \hat{\sigma}(q) = \{(h, min\{\xi_{\hat{\rho}}(p), \xi_{\hat{\sigma}}(q)\}, min\{\eta_{\hat{\rho}}(p), \eta_{\hat{\sigma}}(q)\}, \\ max\{\upsilon_{\hat{\rho}}(p), \upsilon_{\hat{\sigma}}(q)\}) | (p,q) \in \hat{\mathcal{C}}\},$$

we have

$$\hat{\tau}^c(p,q) = (\hat{\rho}(p) \vartriangle \hat{\sigma}(q))^c = \{(h, max\{\upsilon_{\hat{\rho}}(p), \upsilon_{\hat{\sigma}}(q)\}, min\{\eta_{\hat{\rho}}(p), \eta_{\hat{\sigma}}(q)\}, \\ min\{\xi_{\hat{\rho}}(p), \xi_{\hat{\sigma}}(q)\}) | (p,q) \in \hat{\mathcal{C}}\}.$$

Again let $(\hat{\mathcal{T}}, \hat{\mathcal{A}}, \hat{\rho})^c \Diamond (\hat{\mathcal{S}}, \hat{\mathcal{B}}, \hat{\sigma})^c = (\hat{\mathcal{O}}, \hat{\mathcal{C}}, \hat{\gamma})$, therefore, $(\hat{\mathcal{O}}, \hat{\mathcal{C}}) = (\hat{\mathcal{T}}, \hat{\mathcal{A}})^c \Diamond (\hat{\mathcal{S}}, \hat{\mathcal{B}})^c$ and $\hat{\gamma}^c = \hat{\rho}^c \Diamond \hat{\sigma}^c$. For $(p,q) \in \hat{\mathcal{C}} = \hat{\mathcal{A}} \times \hat{\mathcal{B}}$, we have $\hat{\mathcal{O}}(p,q) = \hat{\mathcal{T}}^c(p) \cup \hat{\mathcal{S}}^c(q)$ and also for

$$\hat{\gamma}^c(p,q) = \hat{\rho}^c(p) \Diamond \hat{\sigma}^c(q) = \{(h, max\{\upsilon_{\hat{\rho}}(p), \upsilon_{\hat{\sigma}}(q)\}, min\{\eta_{\hat{\rho}}(p), \eta_{\hat{\sigma}}(q)\}, \\ min\{\xi_{\hat{\rho}}(p), \xi_{\hat{\sigma}}(q)\}) | (p,q) \in \hat{\mathcal{C}}\}.$$

Hence $(\hat{\mathcal{F}}, \hat{\mathcal{C}}, \hat{\tau})^c = (\hat{\mathcal{O}}, \hat{\mathcal{C}}, \hat{\gamma})$. Proved.

(2) The second part can be proved in a similar way. ∎

Theorem 3.6. *Let $(\hat{\mathcal{T}}, \hat{\mathcal{A}}, \hat{\rho})$, $(\hat{\mathcal{S}}, \hat{\mathcal{B}}, \hat{\sigma})$ and $(\hat{\mathcal{F}}, \hat{\mathcal{C}}, \hat{\tau})$ be three $GbPFSSs$ over $\hat{\mathcal{Y}}$. Then we have*

1. $(\hat{\mathcal{T}}, \hat{\mathcal{A}}, \hat{\rho}) \vartriangle [(\hat{\mathcal{S}}, \hat{\mathcal{B}}, \hat{\sigma}) \vartriangle (\hat{\mathcal{F}}, \hat{\mathcal{C}}, \hat{\tau})] = [(\hat{\mathcal{T}}, \hat{\mathcal{A}}, \hat{\rho}) \vartriangle (\hat{\mathcal{S}}, \hat{\mathcal{B}}, \hat{\sigma})] \vartriangle (\hat{\mathcal{F}}, \hat{\mathcal{C}}, \hat{\tau})$;

2. $(\hat{\mathcal{T}}, \hat{\mathcal{A}}, \hat{\rho})\Diamond[(\hat{\mathcal{S}}, \hat{\mathcal{B}}, \hat{\sigma})\Diamond(\hat{\mathcal{F}}, \hat{\mathcal{C}}, \hat{\tau})] = [(\hat{\mathcal{T}}, \hat{\mathcal{A}}, \hat{\rho})\Diamond(\hat{\mathcal{S}}, \hat{\mathcal{B}}, \hat{\sigma})]\Diamond(\hat{\mathcal{F}}, \hat{\mathcal{C}}, \hat{\tau})$.

Proof. Both properties cab easily be obtained from Definitions 3.3 and 3.4. ∎

Definition 3.7. Let $\hat{\mho} = (\hat{\mathcal{T}}, \hat{\mathcal{A}}, \hat{\rho})$ be a $GbPFSS$ over $\hat{\mathcal{Y}}$ and $\hat{\mathcal{A}}$ is finite. Then for all $y \in \hat{\mathcal{Y}}$, the expected score value of y in $GbPFSS$ is defined as the finite number

$$\Lambda_{\hat{\mho}}(y) = \frac{1}{|\hat{\mathcal{A}}|} \sum_{h \in \hat{\mathcal{A}}} \delta_{\hat{\mathcal{T}}(h)}(y).\delta_{\hat{\rho}}(h) \in [0, 1], \tag{3.1}$$

where $\delta_{\hat{\mathcal{T}}(h)}(y)$ and $\delta_{\hat{\rho}}(h)$ are defined as

$$\delta_{\hat{\mathcal{T}}(h)}(y) = \frac{\xi_{\hat{\mathcal{T}}(h)}(y) - \eta_{\hat{\mathcal{T}}(h)}(y) - \upsilon_{\hat{\mathcal{T}}(h)}(y) + 1}{2},$$

$$\delta_{\hat{\rho}}(h) = \frac{\xi_{\hat{\rho}}(h) - \eta_{\hat{\rho}}(h) - \upsilon_{\hat{\rho}}(h) + 1}{2}.$$

4. Methodology

The effectiveness of the NPD process increases during the concept selection if we include the customer's preferences and expertise. In the proposed method, we use the b-picture fuzzy soft set $bPFSS$ to represent the mapping among customers requirements and design concept. The dependencies hidden knowledge of design concept on customers requirements is represented as a membership, neutral and non-membership functions. The methodology consists of the following steps:

4.1. Design Concepts Generation

In [40], Tiwari proposed a complete method of generation of design concept by evolving customers requirements on design attributes and make an effort to meet customers demands and requirements. The following steps are included:

i. For any product, the customer requirements (CRs) are generated by a comprehensive field survey of the market and denoted as $[\mathcal{CR}_1, \mathcal{CR}_2, ..., \mathcal{CR}_k]$.

ii. Based on the customers requirements, design parameters (DPs) are established. Observing the CRs, it is possible that the DPs have more than one values. The set of DPs is represented as $[\mathcal{DP}_1, \mathcal{DP}_2, ..., \mathcal{DP}_m]$.

iii. Each DP have one or more than one values, which are called the design parameters values (DPVs). Based on the requirements of different customers, these DPVs represent target specifications for the product and denoted as the vector set $\mathcal{DP}_p = \{\Upsilon_{pq}\}$, where $p \in \{1, 2, ..., m\}$ and $q \in \{1, 2, ..., n\}$.

iv. Design concepts are generated by the suitable combinations of design parameters values DPVs of each DP by the designer and the dependencies of each DPV can be easily analysed.

The following example helps us to understand the complete process of generating the design concepts.

Example 4.1. Using laptops specifications, a company presents design concepts of laptops. On the basis of customers survey, customers demands are investigated. Based on the customers demands, DPs are specified by the designers {Hard Derive/RAM, Battery Life, Operational Cost}. The DPVs are identified as, Hard Derive/RAM={Minimum, Medium, Maximum}=$\{\Upsilon_{11}, \Upsilon_{12}, \Upsilon_{13}\}$, Battery Life={Low, Medium, Heigh}=$\{\Upsilon_{21}, \Upsilon_{22}, \Upsilon_{23}\}$, and Operational Cost={Cheap, Expensive}= $\{\Upsilon_{31}, \Upsilon_{32}\}$. From suitable combinations of DPVs from each DP, six concept designs are generated which are represented as follows:

$$y_1 = \{\Upsilon_{11}, \Upsilon_{21}, \Upsilon_{31}\}, y_2 = \{\Upsilon_{11}, \Upsilon_{22}, \Upsilon_{32}\}, y_3 = \{\Upsilon_{12}, \Upsilon_{23}, \Upsilon_{31}\},$$
$$y_4 = \{\Upsilon_{12}, \Upsilon_{22}, \Upsilon_{32}\}, y_5 = \{\Upsilon_{12}, \Upsilon_{23}, \Upsilon_{32}\}, y_6 = \{\Upsilon_{13}, \Upsilon_{23}, \Upsilon_{32}\}.$$

Then the set of all design concepts is represented by $\hat{\mathcal{Y}} = \{y_1, y_2, y_3, y_4, y_5, y_6\}$.

4.2. Representing the Mapping of Customer Requirements to Design Concepts

All the generated DCs can be stored in a design repository, or it can be a database, knowledge template. From the definition of soft set, the set of all DCs (design repository) forms a common universe $\hat{\mathcal{Y}}$, where the DPs set represents the parameters set. Design concepts can be characterised by soft set over $\hat{\mathcal{Y}}$ and we represents the corresponding soft sets as $(\hat{\mathcal{T}}, \mathcal{DP}_i)$ $i \in \{1, 2, ..., m\}$. It means we obtain m soft sets and actually the corresponding soft sets are bijective soft sets over a common universe $\hat{\mathcal{Y}}$.

Next, the picture fuzzy representation of the aforementioned bijective soft sets is discussed, where each DPV of a DP may consist of membership, neutral and non-membership values in their dependent design concepts. For explanation, if we consider the DPV $\Upsilon_{13} = 1000GB$ in Example 4.1. The designer has an opinion that the hard disk should be $700GB$ in design concept and between $550GB$ to $700GB$ is satisfactory (remains neutral) but not less than $500GB$. So the best way to represent this as membership, neutral and non-membership values. Here we add an extra value (neutral value/satisfactory) because in each decision making process there are many factors and situations where the designer not sure (due to less information, knowledge).

Definition 4.2. Let $(T, \mathcal{DP}_a)$ be a bijective soft set over $\hat{\mathcal{Y}}$. Then b-picture fuzzy soft set ($bPFSS$), a picture fuzzy representation of $(T, \mathcal{DP}_a)$, is denoted as $(\hat{\mathcal{T}}, \mathcal{DP}_a)$ and defined as for all $y \in \hat{\mathcal{Y}}$

1. $\xi_{\mathcal{DP}_a}(y) = 0$, $\eta_{\mathcal{DP}_a}(y) = 0$ and $\upsilon_{\mathcal{DP}_a}(y) = 1$, for each $y \notin T(\Upsilon_{at})$, $\forall \Upsilon_{at} \in \mathcal{DP}_a$,
2. $\xi_{\mathcal{DP}_a}(y) \in [0, 1]$, $\eta_{\mathcal{DP}_a}(y) \in [0, 1]$ and $\upsilon_{\mathcal{DP}_a}(y) \in [0, 1]$, for each $y \in T(\Upsilon_{at})$, $\forall \Upsilon_{at} \in \mathcal{DP}_a$, such that $0 \leq \xi_{\mathcal{DP}_a}(y) + \eta_{\mathcal{DP}_a}(y) + \upsilon_{\mathcal{DP}_a}(y) \leq 1$.

Example 4.3. Each DP in Example 4.1 can be represented as a bijective soft set in the following way:

1. $(T, \mathcal{DP}_1) = \{T(\Upsilon_{11}) = \{y_1, y_2\}, T(\Upsilon_{12}) = \{y_3, y_4, y_5\}, T(\Upsilon_{13}) = \{y_6\}\}$,
2. $(T, \mathcal{DP}_2) = \{T(\Upsilon_{21}) = \{y_1\}, T(\Upsilon_{22}) = \{y_2, y_4\}, T(\Upsilon_{23}) = \{y_3, y_5, y_6\}\}$,
3. $(T, \mathcal{DP}_3) = \{T(\Upsilon_{31}) = \{y_1, y_3\}, T(\Upsilon_{32}) = \{y_2, y_4, y_5, y_6\}$.

To give precise information about DPVs, designer specifies the belongingness, neutral and non-belongingness values to the each design concept with respect to a DPV. So bPFSS can be defined to show the observed dependencies of design concepts on DPVs. For instance,

$$\hat{\mathcal{T}}(\Upsilon_{11}) = \{(0.5, 0.2, 0.3)/y_1, (0.6, 0.1, 0.2)/y_2, (0.0, 0.0, 1.0)/y_3, (0.0, 0.0, 1.0)/y_4, (0.0, 0.0, 1.0)/y_5, (0.0, 0.0, 1.0)/y_6\},$$

$$\hat{\mathcal{T}}(\Upsilon_{12}) = \{(0.0, 0.0, 1.0)/y_1, (0.0, 0.0, 1.0)/y_2, (0.3, 0.2, 0.5)/y_3, (0.4, 0.3, 0.3)/y_4, (0.7, 0.1, 0.2)/y_5, (0.0, 0.0, 1.0)/y_6\},$$

$$\hat{\mathcal{T}}(\Upsilon_{13}) = \{(0.0, 0.0, 1.0)/y_1, (0.0, 0.0, 1.0)/y_2, (0.0, 0.0, 1.0)/y_3, (0.0, 0.0, 1.0)/y_4, (0.0, 0.0, 1.0)/y_5, (0.6, 0.2, 0.2)/y_6\},$$

means that the evaluation to Υ_{11}, Υ_{12}, and Υ_{13} in design concepts is concerned, the designer believes that, y_3, y_4, y_5, and y_6 are unreliable for Υ_{11}, while y_1 and y_2 are 50% and 60% reliable, 20% and 10% neutral (satisfactory) and 30% and 20% unreliable. In the same way, y_1, y_2, and y_6 are unreliable for Υ_{12}, while y_3, y_4, and y_5 are 30%, 40% and 70% reliable, 20%, 30% and 10% neutral (not sure) and 50%, 30% and 20% unreliable. Similarly, y_1, y_2, y_3, y_4, and y_5 are unreliable for Υ_{13}, while y_6 is 60% reliable, 20% neutral (satisfactory) and 20% unreliable.

Now on the customers demands, a PFS is captured and a generalized b-picture fuzzy soft set is defined over $\hat{\mathcal{Y}}$. Based on the requirements of the one or more than one customers, set of attributes values are identified. Define PFS on the set of attributes values of each customer that describes the membership, neutral and non-membership values of the attributes. From Example 4.1, based on the customers demands, DPs set is specified by the designers as {hard derive/RAM, battery life, operational cost} and DPVs are identified as, hard derive/RAM = {minimum, medium, maximum}, battery life={low, medium, heigh}, and operational cost={cheap, expensive}. Let the set $[\hat{\mathcal{R}}] = \{maximum, heigh, expansive\}$ represents the requirements of the customers analysed by the designer, then the PFS on the $[\hat{\mathcal{R}}]$ is defined like, if $\hat{\rho}(maximum) = \{0.6, 0.2, 0.2\}$, then it represents that the designer believes that the evaluation to DPV "maximum" is concerned, 60% of the demands are reliable, 20% are neutral (satisfactory) and 20% are unreliable.

Definition 4.4. A generalized b-picture fuzzy soft set $\hat{\mho} = (\hat{\mathcal{T}}, \mathcal{DP}_a, \hat{\rho}_a)$ over $\hat{\mathcal{Y}}$ is consists of the $bPFSS$ $(\hat{\mathcal{T}}, \mathcal{DP}_a)$ over $\hat{\mathcal{Y}}$ and a PFS $\hat{\rho}$ in $\mathcal{DP}_a$. If $a = 1, ..., n$ then there will be n generalized b-picture fuzzy soft set over a common universe $\hat{\mathcal{Y}}$.

5. An Applications of GbPFSS in Concept Selection

In this section, we proposed an algorithm for best concept selection from different design concepts. Also, we give numerical examples to strengthen our proposed method for one and two customers separately. We can also use this method to finite number of customers.

Firstly we proposed an algorithm, using the aforementioned $bPFSSs$ on design repository and PFS on the customer's requirements, we define $GbPFSSs$ on the set on customers requirements. Perform an upper-or-substitution operation on all $GbPFSSs$ and compute the expected score value by using Definition 3.7. By ranking design concepts, we get the best design concept with the max expected score value which effectively and productively meet the set of requirements for one or finite number of customers. We input the customer's demands or requirements and get the best concept design in the proposed algorithm.

5.1. **Algorithm**

1- Attributes values are captured on the requirements of k different customers and the values are presented in the form of sets $\hat{\mathcal{R}}_1, \hat{\mathcal{R}}_2, ..., \hat{\mathcal{R}}_k$.
2- Define a PFS $\hat{\rho}_a = \{(\Upsilon_{ap}, \xi_{\hat{\rho}_a}(\Upsilon_{ap}), \eta_{\hat{\rho}_a}(\Upsilon_{ap}), \upsilon_{\hat{\rho}_a}(\Upsilon_{ap}))|\Upsilon_{ap} \in \hat{\mathcal{R}}_a\}$ for each $\hat{\mathcal{R}}_a$ $(a = 1, 2, ..., k)$ and $p = 1, 2, ...n$.
3- For all $\hat{\mathcal{R}}_a$ $(a = 1, 2, ..., k)$, $bPFSSs$ $(\hat{\mathcal{T}}, \mathcal{DP}_a)$ are represented over previously identified dependencies of design concept on design attributes values.
4- Represents $GbPFSSs$ $\hat{\mho}_a = (\hat{\mathcal{T}}, \mathcal{DP}_a, \hat{\rho}_a)$, for each $\hat{\mathcal{R}}_a$ $(a = 1, 2, ..., k)$.
5- Compute upper-or-substitution operation on $\hat{\mho}_1, \hat{\mho}_2, ..., \hat{\mho}_k$ and obtain $GbPFSS$ $\hat{\mho}$.
6- Using Definition 3.7, find the expected score values $\Lambda_{\hat{\mho}}$ from the $GbPFSS$ $\hat{\mho}$.
7- Rank the alternatives on the basis of $\Lambda_{\hat{\mho}}$ and chose optimal with the maximum value i.e., choose y_k if $\Lambda_{\hat{\mho}}(y_k) = max\{\Lambda_{\hat{\mho}}(y_i)|y_i \in \hat{\mathcal{Y}}\}$.
8- Any one alternative y_k can be chosen if k has more than one value.

5.2. **Generation of Design Concept for Laptops**

We strengthen our proposed algorithm by a case study for best concept selection of laptops from several design concepts. Before apply the algorithm, we need to do some early calculations, which we have done in this section. The following steps are used to generate the design concepts for laptops.

1. For the design concepts of laptops, requirements of customers are identified.
2. The designer choose the four DPs and represent in a set which carrying out the functional demands of the customers such that $DP = [\mathcal{DP}_1, \mathcal{DP}_2, \mathcal{DP}_3, \mathcal{DP}_4]$, where $\mathcal{DP}_1$ = Hard Disk/RAM, $\mathcal{DP}_2$ = Battery Life, $\mathcal{DP}_3$ = Display, and $\mathcal{DP}_4$ = Operational Expenses.
3. Now, based on the customers requirements, the target specifications of each DP can be represented as range values or DPVs. The DPVs in case of the laptops of the respective DPs are as follows:

$$\mathcal{DP}_1 = \{\Upsilon_{11}, \Upsilon_{12}, \Upsilon_{13}\} = \{low, medium, high\},$$
$$\mathcal{DP}_2 = \{\Upsilon_{21}, \Upsilon_{22}, \Upsilon_{23}\} = \{low, medium, high\},$$
$$\mathcal{DP}_3 = \{\Upsilon_{31}, \Upsilon_{32}\} = \{low, high\},$$
$$\mathcal{DP}_4 = \{\Upsilon_{41}, \Upsilon_{42}\} = \{cheap, costly\}.$$

4. From suitable combinations of DPVs of each DP, seven concept designs are generated which are represented as follows:

$$y_1 = \{\Upsilon_{11}, \Upsilon_{21}, \Upsilon_{31}, \Upsilon_{41}\} = \{low, low, low, cheap\},$$
$$y_2 = \{\Upsilon_{12}, \Upsilon_{22}, \Upsilon_{32}, \Upsilon_{42}\} = \{medium, medium, high, costly\},$$
$$y_3 = \{\Upsilon_{12}, \Upsilon_{23}, \Upsilon_{31}, \Upsilon_{42}\} = \{medium, high, low, costly\},$$
$$y_4 = \{\Upsilon_{12}, \Upsilon_{23}, \Upsilon_{32}, \Upsilon_{42}\} = \{medium, high, high, costly\},$$
$$y_5 = \{\Upsilon_{13}, \Upsilon_{22}, \Upsilon_{31}, \Upsilon_{41}\} = \{high, medium, low, cheap\},$$
$$y_6 = \{\Upsilon_{13}, \Upsilon_{22}, \Upsilon_{32}, \Upsilon_{41}\} = \{high, medium, high, cheap\},$$
$$y_7 = \{\Upsilon_{13}, \Upsilon_{23}, \Upsilon_{32}, \Upsilon_{42}\} = \{high, high, high, costly\}.$$

Then the set of all design concepts is represented by $\hat{\mathcal{Y}} = \{y_1, y_2, y_3, y_4, y_5, y_6, y_7\}$ and known as a universe set.
5. In the next step, we define the bijective soft set $(T, \mathcal{DP}_a)$ $(a = 1, 2, 3, 4)$ from $\mathcal{DP}_a$ to $P(\hat{\mathcal{Y}})$, which represents the association of DPVs with design concepts. For each a we get:

1. $(T, \mathcal{DP}_1) = \{T(\Upsilon_{11}) = \{y_1\}, T(\Upsilon_{12}) = \{y_2, y_3, y_4\}, T(\Upsilon_{13}) = \{y_5, y_6, y_7\}\}$,
2. $(T, \mathcal{DP}_2) = \{T(\Upsilon_{21}) = \{y_1\}, T(\Upsilon_{22}) = \{y_2, y_5, y_6\}, T(\Upsilon_{23}) = \{y_3, y_4, y_7\}\}$,
3. $(T, \mathcal{DP}_3) = \{T(\Upsilon_{31}) = \{y_1, y_3, y_5\}, T(\Upsilon_{32}) = \{y_2, y_4, y_6, y_7\}$,
4. $(T, \mathcal{DP}_4) = \{T(\Upsilon_{41}) = \{y_1, y_5, y_6\}, T(\Upsilon_{42}) = \{y_2, y_3, y_4, y_7\}$.

6. To give precise information about DPVs, designer specifies the belongingness, neutral and non-belongingness values to the each design concept with respect to a DPV. So $bPFSSs$ can be defined to show the observed dependencies of design concepts on DPVs. In the case of laptops, the $bPFSSs$ are represented as follows:

1. $(\hat{\mathcal{T}}, \mathcal{DP}_1) = \{\hat{\mathcal{T}}(\Upsilon_{11}), \hat{\mathcal{T}}(\Upsilon_{12}), \hat{\mathcal{T}}(\Upsilon_{13})\}$,
2. $(\hat{\mathcal{T}}, \mathcal{DP}_2) = \{\hat{\mathcal{T}}(\Upsilon_{21}), \hat{\mathcal{T}}(\Upsilon_{22}), \hat{\mathcal{T}}(\Upsilon_{23})\}$,
3. $(\hat{\mathcal{T}}, \mathcal{DP}_3) = \{\hat{\mathcal{T}}(\Upsilon_{31}), \hat{\mathcal{T}}(\Upsilon_{32})\}$,
4. $(\hat{\mathcal{T}}, \mathcal{DP}_4) = \{\hat{\mathcal{T}}(\Upsilon_{41}), \hat{\mathcal{T}}(\Upsilon_{42})\}$.

Here

$$\begin{aligned}\hat{\mathcal{T}}(\Upsilon_{11}) = \{&(0.5, 0.2, 0.3)/y_1, (0.0, 0.0, 1.0)/y_2, (0.0, 0.0, 1.0)/y_3, (0.0, 0.0, 1.0)/y_4, \\ &(0.0, 0.0, 1.0)/y_5, (0.0, 0.0, 1.0)/y_6, (0.0, 0.0, 1.0)/y_7\},\end{aligned}$$

$$\hat{\mathcal{T}}(\Upsilon_{12}) = \{(0.0, 0.0, 1.0)/y_1, (0.7, 0.1, 0.2)/y_2, (0.3, 0.2, 0.5)/y_3, (0.4, 0.3, 0.3)/y_4, (0.0, 0.0, 1.0)/y_5, (0.0, 0.0, 1.0)/y_6, (0.0, 0.0, 1.0)/y_7\},$$
$$\hat{\mathcal{T}}(\Upsilon_{13}) = \{(0.0, 0.0, 1.0)/y_1, (0.0, 0.0, 1.0)/y_2, (0.0, 0.0, 1.0)/y_3, (0.0, 0.0, 1.0)/y_4, (0.4, 0.1, 0.3)/y_5, (0.6, 0.2, 0.2)/y_6, (0.8, 0.1, 0.1)/y_7\},$$
$$\hat{\mathcal{T}}(\Upsilon_{21}) = \{(0.5, 0.1, 0.4)/y_1, (0.0, 0.0, 1.0)/y_2, (0.0, 0.0, 1.0)/y_3, (0.0, 0.0, 1.0)/y_4, (0.0, 0.0, 1.0)/y_5, (0.0, 0.0, 1.0)/y_6, (0.0, 0.0, 1.0)/y_7\},$$
$$\hat{\mathcal{T}}(\Upsilon_{22}) = \{(0.0, 0.0, 1.0)/y_1, (0.5, 0.3, 0.2)/y_2, (0.0, 0.0, 1.0)/y_3, (0.0, 0.0, 1.0)/y_4, (0.7, 0.2, 0.1)/y_5, (0.5, 0.3, 0.2)/y_6, (0.0, 0.0, 1.0)/y_7\},$$
$$\hat{\mathcal{T}}(\Upsilon_{23}) = \{(0.0, 0.0, 1.0)/y_1, (0.0, 0.0, 1.0)/y_2, (0.8, 0.1, 0.1)/y_3, (0.5, 0.1, 0.4)/y_4, (0.0, 0.0, 1.0)/y_5, (0.0, 0.0, 1.0)/y_6, (0.6, 0.1, 0.3)/y_7\},$$
$$\hat{\mathcal{T}}(\Upsilon_{31}) = \{(0.4, 0.2, 0.4)/y_1, (0.0, 0.0, 1.0)/y_2, (0.6, 0.2, 0.1)/y_3, (0.0, 0.0, 1.0)/y_4, (0.7, 0.1, 0.2)/y_5, (0.0, 0.0, 1.0)/y_6, (0.0, 0.0, 1.0)/y_7\},$$
$$\hat{\mathcal{T}}(\Upsilon_{32}) = \{(0.0, 0.0, 1.0)/y_1, (0.4, 0.1, 0.5)/y_2, (0.0, 0.0, 1.0)/y_3, (0.6, 0.1, 0.2)/y_4, (0.0, 0.0, 1.0)/y_5, (0.5, 0.2, 0.3)/y_6, (0.7, 0.2, 0.1)/y_7\},$$
$$\hat{\mathcal{T}}(\Upsilon_{41}) = \{(0.6, 0.1, 0.3)/y_1, (0.0, 0.0, 1.0)/y_2, (0.0, 0.0, 1.0)/y_3, (0.0, 0.0, 1.0)/y_4, (0.5, 0.4, 0.1)/y_5, (0.9, 0.0, 0.1)/y_6, (0.0, 0.0, 1.0)/y_7\},$$
$$\hat{\mathcal{T}}(\Upsilon_{42}) = \{(0.0, 0.0, 1.0)/y_1, (0.4, 0.1, 0.4)/y_2, (0.5, 0.3, 0.2)/y_3, (0.7, 0.1, 0.2)/y_4, (0.0, 0.0, 1.0)/y_5, (0.0, 0.0, 1.0)/y_6, (0.8, 0.0, 0.2)/y_7\}.$$

5.3. **Selection of Design Concept of Laptops for One Customer**

To support our algorithm, we discussed the case study of design concept of laptops for one and two customers separately. We already generated the $bPFSSs$ in 5.2. Now the designer analyses the evaluation already done and give PFS on the set of attributes. It completes the formation of $GbPFSS$. After we perform upper-or-substitution operation on $GbPFSS$ and expected score values are calculated and on the basis of expected score values rank the alternatives, and select the optimal which has the highest expected score value. To get an optimal design concept for a single customer following steps have proceeded:

1. The designer identifies the DP set $[\hat{\mathcal{R}}] = \{\Upsilon_{12}, \Upsilon_{23}, \Upsilon_{31}, \Upsilon_{41}\}$ on the basis of customers demands.
2. A PFS on $[\hat{\mathcal{R}}]$ is defined by the designer as

$$\hat{\rho} = \{(0.5, 0.2, 0.3)/\Upsilon_{12}, (0.7, 0.1, 0.2)/\Upsilon_{23}, (0.4, 0.3, 0.3)/\Upsilon_{31}, (0.6, 0.2, 0.2)/\Upsilon_{41}\}.$$

It means that while evaluating the Υ_{12}, the designer is 50% trustable (excellent), 20% neutral (satisfactory), and the 30% not trustable (suspicious). Similarly, $\hat{\rho}(\Upsilon_{23})$, $\hat{\rho}(\Upsilon_{31})$ and $\hat{\rho}(\Upsilon_{41})$ are obtained.
3. From step (6) of Section 5.2, a $bPFSS$ $\Gamma = (\hat{\mathcal{T}}, [\hat{\mathcal{R}}])$ is defined on $[\hat{\mathcal{R}}]$, which completes the formulation of $GbPFSS$ $\hat{\mho} = (\hat{\mathcal{T}}, [\hat{\mathcal{R}}], \hat{\rho})$ on the set of attributes values of the customer. Table 1 representing the tabular form of $GbPFSS$ of $[\hat{\mathcal{R}}]$.

Table 1. The $GbPFSS$ $(\hat{\mathcal{T}}, [\hat{\mathcal{R}}], \hat{\rho})$

$\hat{\mathcal{Y}}$	Υ_{12}	Υ_{23}	Υ_{31}	Υ_{41}
y_1	(0.0,0.0,1.0)	(0.0,0.0,1.0)	(0.4,0.2,0.4)	(0.6,0.1,0.3)
y_2	(0.7,0.1,0.2)	(0.0,0.0,1.0)	(0.0,0.0,1.0)	(0.0,0.0,1.0)
y_3	(0.3,0.2,0.5)	(0.8,0.1,0.1)	(0.6,0.2,0.1)	(0.0,0.0,1.0)
y_4	(0.4,0.3,0.3)	(0.5,0.1,0.4)	(0.0,0.0,1.0)	(0.0,0.0,1.0)
y_5	(0.0,0.0,1.0)	(0.0,0.0,1.0)	(0.7,0.1,0.2)	(0.5,0.4,0.1)
y_6	(0.0,0.0,1.0)	(0.0,0.0,1.0)	(0.0,0.0,1.0)	(0.9,0.0,0.1)
y_7	(0.0,0.0,1.0)	(0.6,0.1,0.3)	(0.0,0.0,1.0)	(0.0,0.0,1.0)
$\hat{\rho}$	(0.5,0.2,0.3)	(0.7,0.1,0.2)	(0.4,0.3,0.3)	(0.6,0.2,0.2)

Table 2. $(\hat{\mathcal{T}}, [\hat{\mathcal{R}}], \hat{\rho})\Diamond(\hat{\mathcal{T}}, [\hat{\mathcal{R}}], \hat{\rho}) = (\hat{\mathcal{F}}, [\hat{\mathcal{P}}], \hat{\tau})$

$\hat{\mathcal{Y}}$	$(\Upsilon_{12}, \Upsilon_{12})$	$(\Upsilon_{12}, \Upsilon_{23})$	$(\Upsilon_{12}, \Upsilon_{31})$	$(\Upsilon_{12}, \Upsilon_{41})$	$(\Upsilon_{23}, \Upsilon_{12})$	$(\Upsilon_{23}, \Upsilon_{23})$	$(\Upsilon_{23}, \Upsilon_{31})$	$(\Upsilon_{23}, \Upsilon_{41})$
y_1	(0.0,0.0,1.0)	(0.0,0.0,1.0)	(0.4,0.0,0.4)	(0.6,0.0,0.3)	(0.0,0.0,1.0)	(0.0,0.0,1.0)	(0.4,0.0,0.4)	(0.6,0.0,0.3)
y_2	(0.7,0.1,0.2)	(0.7,0.0,0.2)	(0.7,0.0,0.2)	(0.7,0.0,0.2)	(0.7,0.0,0.2)	(0.0,0.0,1.0)	(0.0,0.0,1.0)	(0.0,0.0,1.0)
y_3	(0.3,0.2,0.5)	(0.8,0.1,0.1)	(0.6,0.2,0.1)	(0.3,0.0,0.5)	(0.8,0.1,0.1)	(0.8,0.1,0.1)	(0.8,0.1,0.1)	(0.8,0.0,0.1)
y_4	(0.4,0.3,0.3)	(0.5,0.1,0.3)	(0.4,0.0,0.3)	(0.4,0.0,0.3)	(0.5,0.1,0.3)	(0.5,0.1,0.4)	(0.5,0.0,0.4)	(0.5,0.0,0.4)
y_5	(0.0,0.0,1.0)	(0.0,0.0,1.0)	(0.7,0.0,0.2)	(0.5,0.0,0.1)	(0.0,0.0,1.0)	(0.0,0.0,1.0)	(0.7,0.0,0.2)	(0.5,0.0,0.1)
y_6	(0.0,0.0,1.0)	(0.0,0.0,1.0)	(0.0,0.0,1.0)	(0.9,0.0,0.1)	(0.0,0.0,1.0)	(0.0,0.0,1.0)	(0.0,0.0,1.0)	(0.9,0.0,0.1)
y_7	(0.0,0.0,1.0)	(0.6,0.0,0.3)	(0.0,0.0,1.0)	(0.0,0.0,1.0)	(0.6,0.0,0.3)	(0.6,0.1,0.3)	(0.6,0.0,0.3)	(0.6,0.0,0.3)
$\hat{\tau}$	(0.5,0.2,0.3)	(0.7,0.1,0.2)	(0.4,0.2,0.3)	(0.6,0.2,0.2)	(0.7,0.1,0.2)	(0.7,0.1,0.2)	(0.7,0.1,0.2)	(0.7,0.1,0.2)
$\hat{\mathcal{Y}}$	$(\Upsilon_{31}, \Upsilon_{12})$	$(\Upsilon_{31}, \Upsilon_{23})$	$(\Upsilon_{31}, \Upsilon_{31})$	$(\Upsilon_{31}, \Upsilon_{41})$	$(\Upsilon_{41}, \Upsilon_{12})$	$(\Upsilon_{41}, \Upsilon_{23})$	$(\Upsilon_{41}, \Upsilon_{31})$	$(\Upsilon_{41}, \Upsilon_{41})$
y_1	(0.4,0.0,0.4)	(0.4,0.0,0.4)	(0.4,0.2,0.4)	(0.6,0.1,0.3)	(0.6,0.0,0.3)	(0.6,0.0,0.3)	(0.6,0.1,0.3)	(0.6,0.1,0.3)
y_2	(0.7,0.0,0.2)	(0.0,0.0,1.0)	(0.0,0.0,1.0)	(0.0,0.0,1.0)	(0.7,0.0,0.2)	(0.0,0.0,1.0)	(0.0,0.0,1.0)	(0.0,0.0,1.0)
y_3	(0.6,0.2,0.1)	(0.8,0.1,0.1)	(0.6,0.2,0.1)	(0.6,0.0,0.1)	(0.3,0.0,0.5)	(0.8,0.0,0.1)	(0.6,0.0,0.1)	(0.0,0.0,1.0)
y_4	(0.4,0.0,0.3)	(0.5,0.0,0.4)	(0.0,0.0,1.0)	(0.0,0.0,1.0)	(0.4,0.0,0.3)	(0.5,0.0,0.4)	(0.0,0.0,1.0)	(0.0,0.0,1.0)
y_5	(0.7,0.0,0.2)	(0.7,0.0,0.2)	(0.7,0.1,0.2)	(0.7,0.1,0.1)	(0.5,0.0,0.1)	(0.5,0.0,0.1)	(0.7,0.1,0.1)	(0.5,0.4,0.1)
y_6	(0.0,0.0,1.0)	(0.0,0.0,1.0)	(0.0,0.0,1.0)	(0.9,0.0,0.1)	(0.9,0.0,0.1)	(0.9,0.0,0.1)	(0.9,0.0,0.1)	(0.9,0.0,0.1)
y_7	(0.0,0.0,1.0)	(0.6,0.0,0.3)	(0.0,0.0,1.0)	(0.0,0.0,1.0)	(0.0,0.0,1.0)	(0.6,0.0,0.3)	(0.0,0.0,1.0)	(0.0,0.0,1.0)
$\hat{\tau}$	(0.5,0.2,0.3)	(0.7,0.1,0.2)	(0.4,0.3,0.3)	(0.6,0.2,0.2)	(0.6,0.2,0.2)	(0.7,0.1,0.2)	(0.6,0.2,0.2)	(0.6,0.2,0.2)

4. Calculate upper-or-substitution operation on $GbPFSS$ $(\hat{\mathcal{T}}, [\hat{\mathcal{R}}], \hat{\rho})$, i.e., $(\hat{\mathcal{T}}, [\hat{\mathcal{R}}], \hat{\rho})\Diamond(\hat{\mathcal{T}}, [\hat{\mathcal{R}}], \hat{\rho}) = (\hat{\mathcal{F}}, [\hat{\mathcal{P}}], \hat{\tau})$, where $[\hat{\mathcal{P}}] = [\hat{\mathcal{R}}] \times [\hat{\mathcal{R}}]$ as shown in the Table 2.

5. In the next step, we calculate the expected score value of each $y_i \in \hat{\mathcal{Y}}$ by using definition 3.7 as follows:

$$\Theta_{\hat{\mho}}(y_k) = \frac{1}{16} \sum_{(\Upsilon_{ij}, \Upsilon_{i'j'}) \in [\hat{\mathcal{R}}] \times [\hat{\mathcal{R}}]} \delta_{\hat{\mathcal{T}}(\Upsilon_{ij}, \Upsilon_{i'j'})}(y_k) . \delta_{\hat{\tau}(\Upsilon_{ij}, \Upsilon_{i'j'})},$$

where $k = 1, 2, ..., 7$ and the details are in Table 3.

6. From Table 3, the y_3 is the best concept selection of laptops for one customer and we get the rank of all alternatives

$$y_3 \succ y_5 \succ y_1 \succ y_4 \succ y_6 \succ y_7 \succ y_2.$$

TABLE 3. Decision Values for One Customer

$\hat{\mathcal{Y}}$	$\Theta_{\hat{\mathcal{T}}(h)}(y_1)$	$\Theta_{\hat{\mathcal{T}}(h)}(y_2)$	$\Theta_{\hat{\mathcal{T}}(h)}(y_3)$	$\Theta_{\hat{\mathcal{T}}(h)}(y_4)$	$\Theta_{\hat{\mathcal{T}}(h)}(y_5)$	$\Theta_{\hat{\mathcal{T}}(h)}(y_6)$	$\Theta_{\hat{\mathcal{T}}(h)}(y_7)$	$\Theta_{\hat{\rho}(h)}$
$(\Upsilon_{12}, \Upsilon_{12})$	0.00	0.70	0.30	0.40	0.00	0.00	0.00	0.50
$(\Upsilon_{12}, \Upsilon_{23})$	0.00	0.75	0.80	0.55	0.00	0.00	0.65	0.70
$(\Upsilon_{12}, \Upsilon_{31})$	0.50	0.75	0.65	0.55	0.75	0.00	0.00	0.45
$(\Upsilon_{12}, \Upsilon_{41})$	0.65	0.75	0.40	0.55	0.70	0.90	0.00	0.60
$(\Upsilon_{23}, \Upsilon_{12})$	0.00	0.75	0.80	0.55	0.00	0.00	0.65	0.70
$(\Upsilon_{23}, \Upsilon_{23})$	0.00	0.00	0.80	0.50	0.00	0.00	0.60	0.70
$(\Upsilon_{23}, \Upsilon_{31})$	0.50	0.00	0.80	0.55	0.75	0.00	0.65	0.70
$(\Upsilon_{23}, \Upsilon_{41})$	0.65	0.00	0.85	0.55	0.70	0.90	0.65	0.70
$(\Upsilon_{31}, \Upsilon_{12})$	0.50	0.75	0.65	0.55	0.75	0.00	0.00	0.50
$(\Upsilon_{31}, \Upsilon_{23})$	0.50	0.00	0.80	0.55	0.75	0.00	0.65	0.70
$(\Upsilon_{31}, \Upsilon_{31})$	0.40	0.00	0.65	0.00	0.70	0.00	0.00	0.40
$(\Upsilon_{31}, \Upsilon_{41})$	0.60	0.00	0.75	0.00	0.75	0.90	0.00	0.60
$(\Upsilon_{41}, \Upsilon_{12})$	0.65	0.75	0.40	0.55	0.70	0.90	0.00	0.60
$(\Upsilon_{41}, \Upsilon_{23})$	0.65	0.00	0.85	0.55	0.70	0.90	0.65	0.70
$(\Upsilon_{41}, \Upsilon_{31})$	0.60	0.00	0.75	0.00	0.75	0.90	0.00	0.60
$(\Upsilon_{41}, \Upsilon_{41})$	0.60	0.00	0.00	0.00	0.50	0.90	0.00	0.60
$\Lambda_{\hat{\mho}}(y)$	0.25656	0.18828	0.39984	0.25266	0.31641	0.24750	0.19688	–

5.4. Concept Selection for Two Customers

A case study of design concept of laptops for two customers is discussed to support our proposed algorithm. We already generated the $bPFSSs$ in 5.2. Now the designer analyses the evaluation already done and give $PFSs$ on the set of attributes. It completes the formation of $GbPFSSs$ $(\hat{\mathcal{T}}, [\hat{\mathcal{R}}], \hat{\rho})$ and $(\hat{\mathcal{T}}, [\hat{\mathcal{Q}}], \hat{\sigma})$. After we perform upper-or-substitution operation on $GbPFSSs$ and expected score values are calculated and on the basis of expected score values rank the alternatives, and select the optimal which has the highest expected score value. To get an optimal design concept for two customers following steps have proceeded:

1. The designer identifies the DPs sets $[\hat{\mathcal{R}}]$ and $[\hat{\mathcal{Q}}]$ on the basis of customers demands as follows

$$[\hat{\mathcal{R}}] = \{\Upsilon_{12}, \Upsilon_{23}, \Upsilon_{32}, \Upsilon_{41}\},\ [\hat{\mathcal{Q}}] = \{\Upsilon_{13}, \Upsilon_{22}, \Upsilon_{32}, \Upsilon_{42}\}.$$

2. The $PFSs$ on $[\hat{\mathcal{R}}]$ and $[\hat{\mathcal{Q}}]$ are defined by the designer as follows

$$\hat{\rho} = \{(0.5, 0.2, 0.3)/\Upsilon_{12}, (0.7, 0.1, 0.1)/\Upsilon_{23}, (0.6, 0.2, 0.2)/\Upsilon_{32}, (0.5, 0.1, 0.4)/\Upsilon_{41}\},$$
$$\hat{\sigma} = \{(0.7, 0.1, 0.2)/\Upsilon_{13}, (0.8, 0.1, 0.1)/\Upsilon_{22}, (0.4, 0.1, 0.5)/\Upsilon_{32}, (0.5, 0.1, 0.4)/\Upsilon_{42}\}.$$

It means that while evaluating the Υ_{12}, the designer is 50% trustable (excellent), 20% neutral (satisfactory), and the 30% not trustable (suspicious).

3. From step (6) of Section 5.2, the $bPFSSs$ $\Gamma_1 = (\hat{\mathcal{T}}, [\hat{\mathcal{R}}])$ and $\Gamma_2 = (\hat{\mathcal{S}}, [\hat{\mathcal{Q}}])$ are defined on $[\hat{\mathcal{R}}]$ and $[\hat{\mathcal{Q}}]$, which completes the formulation of $GbPFSSs$ $\hat{\mho}_1 = (\hat{\mathcal{T}}, [\hat{\mathcal{R}}], \hat{\rho})$ and $\hat{\mho}_2 = (\hat{\mathcal{T}}, [\hat{\mathcal{Q}}], \hat{\sigma})$ on the set of attributes values of the customers. Tables 4 and 5 representing the tabular form of $GbPFSSs$ of $[\hat{\mathcal{R}}]$ and $[\hat{\mathcal{Q}}]$.

4. Calculate upper-or-substitution operation on $GbPFSSs$ $(\hat{\mathcal{T}}, [\hat{\mathcal{R}}], \hat{\rho})$ and $(\hat{\mathcal{S}}, [\hat{\mathcal{Q}}], \hat{\sigma})$, i.e., $(\hat{\mathcal{T}}, [\hat{\mathcal{R}}], \hat{\rho})\lozenge(\hat{\mathcal{S}}, [\hat{\mathcal{Q}}], \hat{\sigma}) = (\hat{\mathcal{F}}, [\hat{\mathcal{P}}], \hat{\tau})$, where $[\hat{\mathcal{P}}] = [\hat{\mathcal{R}}] \times [\hat{\mathcal{Q}}]$ as shown in the Table 6.

Table 4. The $GbPFSS$ $(\hat{\mathcal{T}}, [\hat{\mathcal{R}}], \hat{\rho})$

$\hat{\mathcal{Y}}$	Υ_{12}	Υ_{23}	Υ_{32}	Υ_{41}
y_1	(0.0,0.0,1.0)	(0.0,0.0,1.0)	(0.0,0.0,1.0)	(0.6,0.1,0.3)
y_2	(0.7,0.1,0.2)	(0.0,0.0,1.0)	(0.4,0.1,0.5)	(0.0,0.0,1.0)
y_3	(0.3,0.2,0.5)	(0.8,0.1,0.1)	(0.0,0.0,1.0)	(0.0,0.0,1.0)
y_4	(0.4,0.3,0.3)	(0.5,0.1,0.4)	(0.6,0.1,0.2)	(0.0,0.0,1.0)
y_5	(0.0,0.0,1.0)	(0.0,0.0,1.0)	(0.0,0.0,1.0)	(0.5,0.4,0.1)
y_6	(0.0,0.0,1.0)	(0.0,0.0,1.0)	(0.5,0.2,0.3)	(0.9,0.0,0.1)
y_7	(0.0,0.0,1.0)	(0.6,0.1,0.3)	(0.7,0.2,0.1)	(0.0,0.0,1.0)
$\hat{\sigma}$	(0.5,0.2,0.3)	(0.7,0.1,0.1)	(0.6,0.2,0.2)	(0.5,0.1,0.4)

Table 5. The $GbPFSS$ $(\hat{\mathcal{S}}, [\hat{\mathcal{Q}}], \hat{\sigma})$

$\hat{\mathcal{Y}}$	Υ_{13}	Υ_{22}	Υ_{32}	Υ_{42}
y_1	(0.0,0.0,1.0)	(0.0,0.0,1.0)	(0.0,0.0,1.0)	(0.0,0.0,1.0)
y_2	(0.0,0.0,1.0)	(0.5,0.3,0.2)	(0.4,0.1,0.5)	(0.4,0.1,0.4)
y_3	(0.0,0.0,1.0)	(0.0,0.0,1.0)	(0.0,0.0,1.0)	(0.5,0.3,0.2)
y_4	(0.0,0.0,1.0)	(0.0,0.0,1.0)	(0.6,0.1,0.2)	(0.7,0.1,0.2)
y_5	(0.4,0.1,0.3)	(0.7,0.2,0.1)	(0.0,0.0,1.0)	(0.0,0.0,1.0)
y_6	(0.6,0.2,0.2)	(0.5,0.3,0.2)	(0.5,0.2,0.3)	(0.0,0.0,1.0)
y_7	(0.8,0.1,0.1)	(0.0,0.0,1.0)	(0.7,0.2,0.1)	(0.8,0.0,0.2)
$\hat{\rho}$	(0.7,0.1,0.2)	(0.8,0.1,0.1)	(0.4,0.1,0.5)	(0.5,0.1,0.4)

5. In the next step, we calculate the expected score values of each $y_i \in \hat{\mathcal{Y}}$ by using definition 3.7 as follows:

$$\Theta_{\hat{\mho}}(y_k) = \frac{1}{16} \sum_{(\Upsilon_{ij}, \Upsilon_{i'j'}) \in [\hat{\mathcal{R}}] \times [\hat{\mathcal{Q}}]} \delta_{\hat{\mathcal{T}}(\Upsilon_{ij}, \Upsilon_{i'j'})}(y_k).\delta_{\hat{\tau}(\Upsilon_{ij}, \Upsilon_{i'j'})},$$

where $k = 1, 2, ..., 7$ and the details are in Table 7.
6. From Table 7, the y_7 is the best concept selection of laptops for two customers and we get the rank of all alternatives

$$y_7 \succ y_1 \succ y_2 \succ y_4 \succ y_6 \succ y_3 \succ y_5.$$

TABLE 6. $(\hat{\mathcal{T}}, [\hat{\mathcal{R}}], \hat{\rho}) \Diamond (\hat{\mathcal{S}}, [\hat{\mathcal{Q}}], \hat{\sigma}) = (\hat{\mathcal{F}}, [\hat{\mathcal{P}}], \hat{\tau})$

$\hat{\mathcal{Y}}$	$(\Upsilon_{12}, \Upsilon_{13})$	$(\Upsilon_{12}, \Upsilon_{22})$	$(\Upsilon_{12}, \Upsilon_{32})$	$(\Upsilon_{12}, \Upsilon_{42})$	$(\Upsilon_{23}, \Upsilon_{13})$	$(\Upsilon_{23}, \Upsilon_{22})$	$(\Upsilon_{23}, \Upsilon_{32})$	$(\Upsilon_{23}, \Upsilon_{42})$
y_1	(0.0,0.0,1.0)	(0.0,0.0,1.0)	(0.0,0.0,1.0)	(0.0,0.0,1.0)	(0.0,0.0,1.0)	(0.0,0.0,1.0)	(0.0,0.0,1.0)	(0.0,0.0,1.0)
y_2	(0.7,0.0,0.2)	(0.7,0.1,0.2)	(0.7,0.1,0.2)	(0.7,0.1,0.2)	(0.0,0.0,1.0)	(0.5,0.0,0.2)	(0.4,0.0,0.5)	(0.4,0.0,0.4)
y_3	(0.3,0.0,0.5)	(0.3,0.0,0.5)	(0.3,0.0,0.5)	(0.5,0.2,0.2)	(0.8,0.0,0.1)	(0.8,0.0,0.1)	(0.8,0.0,0.1)	(0.8,0.1,0.1)
y_4	(0.4,0.0,0.3)	(0.4,0.0,0.3)	(0.6,0.1,0.2)	(0.7,0.1,0.2)	(0.5,0.0,0.4)	(0.5,0.0,0.4)	(0.6,0.1,0.2)	(0.7,0.1,0.2)
y_5	(0.4,0.0,0.3)	(0.7,0.0,0.1)	(0.0,0.0,1.0)	(0.0,0.0,1.0)	(0.4,0.0,0.3)	(0.7,0.0,0.1)	(0.0,0.0,1.0)	(0.0,0.0,1.0)
y_6	(0.6,0.0,0.2)	(0.5,0.0,0.2)	(0.5,0.0,0.3)	(0.0,0.0,1.0)	(0.6,0.0,0.2)	(0.5,0.0,0.2)	(0.5,0.0,0.3)	(0.0,0.0,1.0)
y_7	(0.8,0.0,0.1)	(0.0,0.0,1.0)	(0.7,0.0,0.1)	(0.8,0.0,0.2)	(0.8,0.1,0.1)	(0.6,0.0,0.3)	(0.7,0.1,0.1)	(0.8,0.0,0.2)
$\hat{\tau}$	(0.7,0.1,0.2)	(0.8,0.1,0.1)	(0.5,0.1,0.3)	(0.5,0.1,0.3)	(0.7,0.1,0.1)	(0.8,0.1,0.1)	(0.7,0.1,0.1)	(0.7,0.1,0.1)
$\hat{\mathcal{Y}}$	$(\Upsilon_{32}, \Upsilon_{13})$	$(\Upsilon_{32}, \Upsilon_{22})$	$(\Upsilon_{32}, \Upsilon_{32})$	$(\Upsilon_{32}, \Upsilon_{42})$	$(\Upsilon_{41}, \Upsilon_{13})$	$(\Upsilon_{41}, \Upsilon_{22})$	$(\Upsilon_{41}, \Upsilon_{32})$	$(\Upsilon_{41}, \Upsilon_{42})$
y_1	(0.0,0.0,1.0)	(0.0,0.0,1.0)	(0.0,0.0,1.0)	(0.0,0.0,1.0)	(0.6,0.0,0.3)	(0.6,0.0,0.3)	(0.6,0.0,0.3)	(0.6,0.0,0.3)
y_2	(0.4,0.0,0.5)	(0.5,0.1,0.2)	(0.4,0.1,0.5)	(0.4,0.1,0.4)	(0.0,0.0,1.0)	(0.5,0.0,0.2)	(0.4,0.0,0.5)	(0.4,0.0,0.4)
y_3	(0.0,0.0,1.0)	(0.0,0.0,1.0)	(0.0,0.0,1.0)	(0.5,0.0,0.2)	(0.0,0.0,1.0)	(0.0,0.0,1.0)	(0.0,0.0,1.0)	(0.5,0.0,0.2)
y_4	(0.6,0.0,0.2)	(0.6,0.0,0.2)	(0.6,0.1,0.2)	(0.7,0.1,0.2)	(0.0,0.0,1.0)	(0.0,0.0,1.0)	(0.6,0.0,0.2)	(0.7,0.0,0.2)
y_5	(0.4,0.0,0.3)	(0.7,0.0,0.1)	(0.0,0.0,1.0)	(0.0,0.0,1.0)	(0.5,0.1,0.1)	(0.7,0.2,0.1)	(0.5,0.0,0.1)	(0.5,0.0,0.1)
y_6	(0.6,0.2,0.2)	(0.5,0.2,0.2)	(0.5,0.2,0.3)	(0.5,0.0,0.3)	(0.9,0.0,0.1)	(0.9,0.0,0.1)	(0.9,0.0,0.1)	(0.9,0.0,0.1)
y_7	(0.8,0.1,0.1)	(0.7,0.0,0.1)	(0.7,0.2,0.1)	(0.8,0.0,0.1)	(0.8,0.0,1.0)	(0.0,0.0,1.0)	(0.7,0.0,0.1)	(0.8,0.0,0.2)
$\hat{\tau}$	(0.7,0.1,0.2)	(0.8,0.1,0.1)	(0.6,0.1,0.2)	(0.6,0.1,0.2)	(0.7,0.1,0.2)	(0.8,0.1,0.1)	(0.5,0.1,0.4)	(0.5,0.1,0.4)

TABLE 7. Decision Values for Two Customer

$\hat{\mathcal{Y}}$	$\Theta_{\hat{\mathcal{T}}(h)}(y_1)$	$\Theta_{\hat{\mathcal{T}}(h)}(y_2)$	$\Theta_{\hat{\mathcal{T}}(h)}(y_3)$	$\Theta_{\hat{\mathcal{T}}(h)}(y_4)$	$\Theta_{\hat{\mathcal{T}}(h)}(y_5)$	$\Theta_{\hat{\mathcal{T}}(h)}(y_6)$	$\Theta_{\hat{\mathcal{T}}(h)}(y_7)$	$\Theta_{\hat{\rho}(h)}$
$(\Upsilon_{12}, \Upsilon_{13})$	0.00	0.75	0.40	0.55	0.55	0.65	0.85	0.70
$(\Upsilon_{12}, \Upsilon_{22})$	0.00	0.70	0.40	0.55	0.80	0.65	0.00	0.80
$(\Upsilon_{12}, \Upsilon_{32})$	0.00	0.70	0.40	0.65	0.00	0.60	0.80	0.55
$(\Upsilon_{12}, \Upsilon_{42})$	0.00	0.70	0.55	0.70	0.00	0.00	0.80	0.55
$(\Upsilon_{23}, \Upsilon_{13})$	0.00	0.00	0.85	0.55	0.55	0.70	0.80	0.75
$(\Upsilon_{23}, \Upsilon_{22})$	0.00	0.65	0.85	0.55	0.80	0.65	0.65	0.80
$(\Upsilon_{23}, \Upsilon_{32})$	0.00	0.45	0.85	0.65	0.00	0.60	0.75	0.75
$(\Upsilon_{23}, \Upsilon_{42})$	0.00	0.50	0.80	0.70	0.00	0.00	0.80	0.75
$(\Upsilon_{32}, \Upsilon_{13})$	0.00	0.45	0.00	0.70	0.55	0.60	0.80	0.70
$(\Upsilon_{32}, \Upsilon_{22})$	0.00	0.60	0.00	0.70	0.80	0.55	0.80	0.80
$(\Upsilon_{32}, \Upsilon_{32})$	0.00	0.40	0.00	0.65	0.00	0.50	0.70	0.65
$(\Upsilon_{32}, \Upsilon_{42})$	0.00	0.45	0.65	0.70	0.00	0.60	0.85	0.65
$(\Upsilon_{41}, \Upsilon_{13})$	0.60	0.00	0.00	0.00	0.65	0.90	0.40	0.70
$(\Upsilon_{41}, \Upsilon_{22})$	0.60	0.65	0.00	0.00	0.70	0.90	0.00	0.80
$(\Upsilon_{41}, \Upsilon_{32})$	0.65	0.45	0.00	0.70	0.70	0.90	0.80	0.50
$(\Upsilon_{41}, \Upsilon_{42})$	0.65	0.50	0.65	0.75	0.70	0.90	0.80	0.50
$\Lambda_{\hat{\mho}}(y)$	0.44031	0.41625	0.30109	0.38031	0.27656	0.33938	0.96875	–

6. Discussion

The incorrect identification of a good concept for a particular product leads to increase in design modification, which increases the functional cost and generating time, therefore the concept selection is a crucial process in the new product development (NPD). The picture fuzzy environment is important because there are various degrees of uncertainties in the concept selection process to deals with the subjective and vague decision maker judgments, linguistic customers requirements, customer satisfaction level, the trade-off between design criteria, and the performance of design alternatives. If we talk about the method proposed by Tiwari [40], while computing the correlation table for two or more customers some design concept repeats and it may be a result of an inappropriate design concepts for one or more customers [38]. In [36], mainly the soft sets are used, although, the technique is quite useful but soft sets can't model the uncertainty separately. Therefore, its better to use picture fuzzy environment to capture the uncertainties in the concept selection. If we compare our method with the method proposed in [38], since picture fuzzy set is a further generalization of the intuitionistic fuzzy set. So the PFS contains more information (degree of positive membership, degree of neutral membership, degrees of negative membership and degrees of refusal membership) than intuitionistic fuzzy set (both membership degree and non membership degree). So our method is actually the generalization of the method proposed in [38].

7. Conclusion

In this paper, we used the generalized picture fuzzy soft set to obtain the optimal design concept. We introduced the $bPFSS$ and $GbPFSS$ on the basis of bijective soft set. We introduced the lower-and-substitution and upper-or-substitution operations for $GbPFSS$ and discuss the De Morgan's laws, and their basic properties. The $bPFSSs$ are used to illustrate the mapping from customers requirements to design concepts. We proposed an algorithm for choosing optimal design concept using the upper-or-substitution operations for $GbPFSS$ and some related concepts. The reason to choose upper-or-substitution operations for $GbPFSS$ is that because it aggregates the information from design concepts by taking maximum value from membership function. After we discuss a case study for the design concept of laptops for one and two customers separately.

In the future directions, we find the best concept selection using TOPSIS and VIKOR for $GbPFSS$. Also it is interesting to find the applications of $GbPFSS$ for multi attribute classification or sorting problems. Also, we consider it for pattern recognition and medical diagnosis by defining similarity measures and entropy on $GPFSSs$.

Acknowledgements

The authors acknowledge the financial support provided by the Petchra Pra Jom Klao Scholarship and King Mongkuts University of Technology Thonburi, Bangkok, Thailand.

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