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A NEW GUARANTEED COST CONTROL FOR ASYMPTOTIC STABILIZATION OF NEURAL NETWORK WITH MIXED TIME-VARYING DELAYS VIA FEEDBACK CONTROL

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Abstract A new guaranteed cost control for asymptotic stability of the neural network with mixed time-varying delays and feedback control is studied. The considered mixed time-delays are both discrete and distributed time-varying delays. The proposed conditions allow us to design the state feedback controllers which stabilize the closed-loop system. By constructing an appropriate Lyapunov-Krasovskii functional includes double integral term and triple integral term, utilizing Writinger-based integral inequality, extended reciprocally convex inequality and Jensen integral inequality, new delay-dependent sufficient conditions for the existence of guaranteed cost control are given in terms of linear matrix inequalities (LMIs). Furthermore, we design new quadratic cost functions and minimize their upper bound. Finally, numerical examples are given to illustrate the effectiveness of the theoretical results.

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1. Introduction

In the past decades, Neural networks (NNs) have been extensively studied due to their wide applications in various fields, for instance, associative memory, signal processing and image processing. The stability of the delayed neural networks (DNNs) has attracted a large number of researchers [1] and some stability criteria have been reported in [2–4]. The stability criteria improved for DNNs can be separated into delay-dependent ones and delay-independent ones. Compared to the latter, the delay-dependent stability criteria, which include the information of time delay, usually have less conservative, especially

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when applied to DNNs with small delay. Thus, more attentions have been paid to delay-dependent stability analysis and its main objective is to reduce the conservatism of the obtained stability condition. In the language of control variables, we call the disturbance functions as control variables. During the last decade, the population models with feedback controls have been extensively studied in many articles, which see [5–7]. Guaranteed cost control problem has the advantage of providing an upper bound on a given system performance index and thus the system performance degradation incurred by the uncertainties or time delays is guaranteed to be less than this bound. The guaranteed cost control was first put forward by Chang and Peng [8] and introduced by a lot of authors [9]. Optimal cost controller for linear system with mixed time-varying delays state and control has been considered in [10]. Novel criteria for finite-time stabilization and guaranteed cost control of delayed neural networks is studied in [11].

For Lyapunov functional approach to delay-dependent stability, the conservatism is related to the selecting of the Lyapunov-Krasovskii functional (LKF) and attending with its derivative. By constructing a LKF is an effective way to decrease conservatism of stability result, and various types of LKF have been reported, for example multiple integrals based LKF [12, 13], activation function based LKF [14], and so on. The conservatism of the Jensen inequality has been analyzed in [15]. In addition, an alternative inequality reducing the gap of the Jensen inequality has been proposed in [16] based on the Wirtinger inequality. The Wirtinger-based integral inequality, combined with the reciprocally convex optimization in [17]. The free-weighting-based inequality in [18].

In this paper, we investigate the problems of asymptotic stability of neural networks via the feedback control. Moreover, we study the optimal cost control problem for a class of neural network with mixed time-varying delays. By applying the Lyapunov-Krasovskii functional includes double integral term, triple integral term are employed. Jensen inequality, Wirtinger inequality, convex combination idea, Newton-Leibniz formula and zero equation are used. A performance measure for the system is considered by a new quadratic cost function. The main contributions of this paper are given as follows:

- A new quadratic cost function

$$
\begin{aligned}
J \leq & \int_0^\infty \Big[x^T(t) Z_1 x(t) + x^T(t-\tau(t)) Z_2 x(t-\tau(t)) + u^T(t) Z_4 u(t) \\
& + \Big(\int_{t-\tau_1(t)}^t x^T(s) ds \Big) Z_3 \Big(\int_{t-\tau_1(t)}^t x(s) ds \Big) \Big] dt,
\end{aligned}
$$

  is first proposed to analyze the problem of guaranteed cost control for a class of neural network with mixed time-varying delays.
- The upper bound of given quadratic cost functions is minimized by guaranteed cost control technique.

The feedback controllers are designed to satisfy with asymptotically stable. We provide the sufficient conditions for existence of the feedback guaranteed cost control in terms of LMIs, which can be determined by utilizing MATLABs LMI control toolbox. Numerical examples are presented to illustrate the effectiveness of our method.

2. Preliminaries

We introduce the following neural network with time-varying delays via feedback control in the following form

$$\begin{aligned}\dot{x}(t) &= -Ax(t) + W_0 f(x(t)) + W_1 g(x(t-\tau(t))) \\ &\quad + W_2 \int_{t-\tau_1(t)}^{t} h(x(s))ds + u(t), \\ x(t) &= \phi(t), \quad t \in [-\tau, 0],\end{aligned} \tag{2.1}$$

where $x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T \in \mathbb{R}^n$ is the neuron state vector, $f(x(t))$, $g(x(t))$, $h(x(t)) \in \mathbb{R}^n$ are the neuron activation functions, $A = \text{diag}\{a_1, a_2, \cdots, a_n\}$ is a diagonal matrix with $a_i > 0$, $i = 1, 2, \cdots, n$, W_0, W_1 and W_2 denote the connection weight matrix, the discretely delayed connection weight matrix, and the distributively delayed connection weight matrix, respectively, $\phi(t) \in \mathcal{C}[[-\tau, 0], \mathbb{R}^n]$ is the initial function. The state feedback controller is in the from

$$u(t) = Kx(t). \tag{2.2}$$

By substituting equation (2.2) into equation (2.1), we get

$$\begin{aligned}\dot{x}(t) &= (K - A)x(t) + W_0 f(x(t)) + W_1 g(x(t-\tau(t))) \\ &\quad + W_2 \int_{t-\tau_1(t)}^{t} h(x(s))ds,\end{aligned} \tag{2.3}$$

where the time-varying delay functions $\tau(t)$ and $\tau_1(t)$, satisfy the conditions

$$0 \leq \tau(t) \leq \tau, \tag{2.4}$$

$$\dot{\tau}(t) \leq \mu, \tag{2.5}$$

$$0 \leq \tau_1(t) \leq \tau_1. \tag{2.6}$$

Moreover, throughout this research, we define the following new nonlinear quadratic cost function of the associated system (2.1) as follows:

$$\begin{aligned}J &\leq \int_0^{\infty} \Big[x^T(t)Z_1x(t) + x^T(t-\tau(t))Z_2x(t-\tau(t)) + u^T(t)Z_4u(t) \\ &\quad + \Big(\int_{t-\tau_1(t)}^{t} x^T(s)ds \Big) Z_3 \Big(\int_{t-\tau_1(t)}^{t} x(s)ds \Big) \Big] dt,\end{aligned} \tag{2.7}$$

where $Z_1, Z_2, Z_3 \in \mathbb{R}^{n \times n}$ and $Z_4 \in \mathbb{R}^{m \times m}$ are positive definite matrices.

The guaranteed cost control problem to be addressed in this section is formulated as follows.

Definition 2.1. Consider the control system (2.1). If there exist a continuous stabilizing state feedback control law $u^*(t) = Kx(t)$ and a positive number J^* such that the zero solution of the closed-loop system (2.3) is asymptotically stable and the value (2.7) satisfies $J(u^*) \leq J^*$ then the cost value J^* is a guaranteed cost value, $u^*(t)$ is a guaranteed cost controller of the system.

The following lemmas are introduced for deriving the main result.

Lemma 2.2. *(Cauchy inequality [19]). For any symmetric positive definite matrix $N \in M^{n\times n}$ and $x, y \in \mathbb{R}^n$ we have*

$$\pm 2x^T y \leq x^T N x + y^T N^{-1} y.$$

Lemma 2.3. *(Schur complement lemma [19]). Given constant symmetric matrices X, Y and Z with appropriate dimensions satisfying $X = X^T, Y = Y^T > 0$, then $X + Z^T Y^{-1} Z < 0$ if and only if*

$$\begin{bmatrix} X & Z^T \\ * & -Y \end{bmatrix} < 0 \quad or \quad \begin{bmatrix} -Y & Z \\ * & X \end{bmatrix} < 0.$$

Lemma 2.4. [20]. *For a positive definite matrix $R \in \mathbb{R}^{n\times n}$, for any continuously differentiable function $x : [\alpha, \beta] \to \mathbb{R}^n$, the following inequality holds:*

$$\int_\alpha^\beta \dot{x}^T(s) R \dot{x}(s) ds \geq \frac{1}{\beta-\alpha}\chi_1^T R \chi_1 + \frac{3}{\beta-\alpha}\chi_2^T R \chi_2 + \frac{5}{\beta-\alpha}\chi_3^T R \chi_3,$$

where

$$\begin{aligned} \chi_1 &= x(\beta) - x(\alpha), \quad \chi_2 = x(\beta) + x(\alpha) - \frac{2}{\beta-\alpha}\int_\alpha^\beta x(s)ds, \\ \chi_3 &= x(\beta) - x(\alpha) + \frac{6}{\beta-\alpha}\int_\alpha^\beta x(s)ds - \frac{12}{(\beta-\alpha)^2}\int_\alpha^b \int_u^\beta x(s)dsdu. \end{aligned}$$

Lemma 2.5. [21]. *For a positive definite matrix $R \in \mathbb{R}^{n\times n}$, scalars $\beta > \alpha \geq 0$ and vector $x : [\alpha, \beta] \to \mathbb{R}^n$ such that the integration concerned is well-defined, then*

$$\int_\alpha^\beta (s-\alpha)x^T(s) R x(s) ds \geq \frac{2}{(\beta-\alpha)^2}\int_\alpha^\beta (s-\alpha)x^T(s)dsR\int_\alpha^\beta (s-\alpha)x(s)ds.$$

Lemma 2.6. *(Wirtinger-based integral inequality [21]). For a positive definite matrix $R \in \mathbb{R}^{n\times n}$ and any differentiable function $x : [\alpha, \beta] \to \mathbb{R}^n$, the following inequality holds:*

$$\int_\alpha^\beta \dot{x}^T(s) R \dot{x}(s) ds \geq \frac{1}{\beta-\alpha}\begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix}^T \begin{bmatrix} R & 0 \\ 0 & 3R \end{bmatrix} \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix},$$

where

$$\varphi_1 = x(\beta) - x(\alpha), \quad \varphi_2 = x(\beta) + x(\alpha) - \frac{2}{\beta-\alpha}\int_\alpha^\beta x(s)ds.$$

Lemma 2.7. *(Extended reciprocally convex inequality [21]). For positive matrices $R_1, R_2 \in \mathbb{R}^{n\times n}$, if there exist symmetric matrices $X_1, X_2 \in \mathbb{R}^{n\times n}$ and any matrices $Y_1, Y_2 \in \mathbb{R}^{n\times n}$ such that*

$$\begin{bmatrix} R_1 - X_1 & -Y_1 \\ * & R_2 \end{bmatrix} > 0, \begin{bmatrix} R_1 & -Y_2 \\ * & R_2 - X_2 \end{bmatrix} > 0,$$

then the following inequality holds for all $\alpha \in [0, 1]$

$$\begin{bmatrix} \frac{1}{\alpha}R_1 & 0 \\ * & \frac{1}{1-\alpha}R_2 \end{bmatrix} \geq \begin{bmatrix} R_1 & 0 \\ * & R_2 \end{bmatrix} + (1-\alpha)\begin{bmatrix} X_1 & Y_2 \\ * & 0 \end{bmatrix} + \alpha\begin{bmatrix} 0 & Y_1 \\ * & X_2 \end{bmatrix}.$$

Lemma 2.8. [22]. *For any constant matrix $M > 0$, a positive definite matrix $R > 0$, the following inequalities hold for all continuously differentiable functions $x : [a, b] \to \mathbb{R}^n$, the following inequality holds:*

$$-(b-a)\int_{-b}^{-a} x^T(s)Mx(s)ds \le -\Big(\int_{-b}^{-a} x(s)ds\Big)^T M\Big(\int_{-b}^{-a} x(s)ds\Big) - 3\Theta^T M\Theta,$$

where

$$\Theta = \int_{-b}^{-a} x(s)ds - \frac{2}{b-a}\int_{-b}^{-a}\int_{-b}^{s} x(u)duds.$$

Lemma 2.9. [22]. *For any constant matrix $M > 0$, a positive definite matrix $R > 0$, the following inequality holds for all continuously differentiable functions $x : [a, b] \to \mathbb{R}^n$, the following inequality holds:*

$$\begin{aligned}
&-\frac{(b-a)^2}{2}\int_a^b\int_a^s x^T(u)Mx(u)duds \\
&\le -\Big(\int_a^b\int_a^s x(u)duds\Big)^T M\Big(\int_a^b\int_a^s x(u)duds\Big) - 2\Theta^T M\Theta,
\end{aligned}$$

where

$$\Theta = \int_a^b\int_a^s x(s)duds - \frac{3}{b-a}\int_a^b\int_a^s\int_a^u x(u)dudvds.$$

Lemma 2.10. *(Jensen's Inequality [21]). For a positive definite matrix $R \in \mathbb{R}^{n\times n}$, scalars $\alpha < \beta$, and vector $x : [\alpha, \beta] \to \mathbb{R}^n$ such that the integration concerned is well defined, then*

$$(\beta-\alpha)\int_\alpha^\beta x^T(s)Rx(s)ds \ge \int_\alpha^\beta x^T(s)dsR\int_\alpha^\beta x(s)ds.$$

3. Main Results

In this section, based on LyapunovKrasovskii stability theory, the guaranteed cost control of asymptotic stability for a neural network with mixed time-varying delays is studied. The following theorem presents a sufficient condition for the existence of the guaranteed cost control laws for the asymptotically stable of the neural network (2.3). To simplify the representation, we introduce some notations as follows:

$$\begin{aligned}
\nu_1^T(t) &= \frac{1}{\tau(t)}\int_{t-\tau(t)}^{t} x^T(s)ds, \quad \nu_2^T(t) = \frac{1}{\tau-\tau(t)}\int_{t-\tau}^{t-\tau(t)} x^T(s)ds,\\
\omega_1^T(t) &= \frac{2}{\tau^2(t)}\int_{t-\tau(t)}^{t}\int_{t-\tau(t)}^{s} x^T(u)duds,\\
\omega_2^T(t) &= \frac{2}{(\tau-\tau(t))^2}\int_{t-\tau}^{t-\tau(t)}\int_{t-\tau}^{s} x^T(u)duds, \quad \eta_1^T(t) = \int_{t-\tau}^{t} x^T(s)ds,\\
\eta_2^T(t) &= \int_{t-\tau}^{t}\int_{u}^{t} x^T(s)dsdu, \quad \chi(t) = [x^T(t) \quad \eta_1^T(t) \quad \eta_2^T(t)]^T,\\
\xi(t) &= \Big[x^T(t) \quad x^T(t-\tau(t)) \quad \dot{x}^T(t) \quad x^T(t-\tau) \quad \nu_1^T(t) \quad \nu_2^T(t) \quad \omega_1^T(t) \quad \omega_2^T(t)\\
&\quad \eta_1^T(t) \quad \eta_2^T(t) \quad f^T(x(t)) \quad f^T(x(t-\tau)) \quad g^T(x(t)) \quad g^T(x(t-\tau(t))) \quad h^T(x(t))\\
&\quad \int_{t-\tau_1(t)}^{t} h^T(x(s))ds\Big]^T,\\
\Pi_1 &= [e_1^T, e_9^T, e_{10}^T]^T, \Pi_2 = [e_3^T, e_1^T - e_4^T, \tau e_1^T - e_9^T]^T, \Pi_3 = e_1 - e_4,\\
\Pi_4 &= e_1 + e_4 - \frac{2}{\tau}e_9, \Pi_5 = e_1 - e_4 + \frac{6}{\tau}e_9 - \frac{12}{\tau^2}e_{10}, \Pi_6 = e_1 - e_5, \Pi_7 = e_2 - e_6,\\
\Pi_8 &= [e_1^T - e_2^T, e_1^T + e_2^T - 2e_5^T]^T,\\
\Pi_9 &= [e_1^T - e_2^T, e_1^T + e_2^T - 2e_5^T, e_2^T - e_4^T, e_2^T + e_4^T - 2e_6^T]^T, \Pi_{10} = e_5 - e_7,
\end{aligned}$$

$$\begin{aligned}
\Pi_{11} &= e_6 - e_8, \Pi_{12} = e_5 - e_2, \Pi_{13} = e_6 - e_4, \Pi_{14} = \frac{1}{2}e_1 + e_5 - \frac{3}{2}e_7,\\
\Pi_{15} &= \frac{1}{2}e_2 + e_6 - \frac{3}{2}e_8, \Pi_{16} = e_1 - e_5, \Pi_{17} = e_2 - e_6, \Pi_{18} = e_2 - e_4,\\
\Pi_{19} &= e_1 - e_2, \Pi_{20} = [e_1^T, e_{11}^T]^T, \Pi_{21} = [e_1^T, e_{14}^T]^T, \Pi_{22} = [e_1^T, e_{15}^T]^T,\\
e_i &= [0_{n\times(i-1)n}, I_n, 0_{n\times(16-i)n}] \; for \; i = 1, 2, \ldots, 16, \quad \lambda_1 = \lambda_{\min}(P),\\
\lambda_2 &= (1 + \tau^2 + \frac{\tau^4}{4})\lambda_{\max}(P) + \tau\lambda_{\max}(M) + \tau\lambda_{\max}(Q) + \frac{\tau^3}{2}\lambda_{\max}(R)\\
&\quad + \frac{\tau^3}{3}\lambda_{\max}(S) + \frac{\tau^3}{2}\lambda_{\max}(U) + \frac{\tau^2}{2}\lambda_{\max}(V) - \frac{\tau^3}{6}\lambda_{\max}(Z) - \frac{\tau^3}{6}\lambda_{\max}(W)\\
&\quad - \tau\lambda_{\max}(F^T L_1 F) + \tau\lambda_{\max}(G^T L_2 G) + \frac{\tau^2}{2}\lambda_{\max}(H^T L_3 H).
\end{aligned}$$

Theorem 3.1. *Consider scalars $\tau > 0, \tau_1 > 0$ and $\mu \geq 0$. If there exist symmetric positive definite matrices $P \in \mathbb{R}^{3n\times 3n}, M, Q, R, S, U, V, Z, W, L_1, L_2, L_3, N_1, N_2 \in \mathbb{R}^{n\times n}$ and positive diagonal matrices $D_1, D_2, D_3 \in \mathbb{R}^{n\times n}$ such that the following LMIs hold:*

$$\begin{aligned}
\tilde{G}_j &> 0, \quad (j = 1, 2),\\
\begin{bmatrix} \tilde{G}_1 - J_1 & -J_3 \\ * & \tilde{G}_2 \end{bmatrix} &\geq 0, \begin{bmatrix} \tilde{G}_1 & -J_4 \\ * & \tilde{G}_2 - J_2 \end{bmatrix} \geq 0,\\
\tau\Xi_2 + \Xi_3 &< 0, \qquad (3.1)\\
\tau\Xi_1 + \Xi_3 &< 0, \qquad (3.2)
\end{aligned}$$

where

$$
\begin{aligned}
\tilde{G}_1 &= \begin{bmatrix} 2S+U & 0 \\ * & 6S+3U \end{bmatrix}, \\
\tilde{G}_2 &= \begin{bmatrix} U & 0 \\ * & 3U \end{bmatrix}, \\
\Xi_1 &= -\frac{1}{\tau}\Pi_9^T \begin{bmatrix} 0 & J_3 \\ * & J_2 \end{bmatrix} \Pi_9 - e_5^T V e_5 - 3\Pi_{10}^T V \Pi_{10} - \frac{1}{2\tau}\Pi_{18}^T Z \Pi_{18}, \\
\Xi_2 &= -\frac{1}{\tau}\Pi_9^T \begin{bmatrix} J_1 & J_4 \\ * & 0 \end{bmatrix} \Pi_9 - e_6^T V e_6 - 3\Pi_{11}^T V \Pi_{11} - \frac{1}{2\tau}\Pi_{19}^T Z \Pi_{19}, \\
\Xi_3 &= 2\Pi_1^T P \Pi_2 + e_1^T M e_1 - (1-\mu) e_2^T M e_2 + e_1^T Q e_1 - e_4^T Q e_4 + \tau^2 e_3^T R e_3 - \\
&\quad \Pi_3^T R \Pi_3 - 3\Pi_4^T R \Pi_4 - 5\Pi_5^T R \Pi_5 + \tau^2 e_3^T S e_3 - 4\Pi_6^T S \Pi_6 - 4\Pi_7^T S \Pi_7 \\
&\quad +\Pi_8^T \begin{bmatrix} 2S & 0 \\ * & 6S \end{bmatrix} \Pi_8 - \Pi_9^T \begin{bmatrix} \tilde{G}_1 & 0 \\ * & \tilde{G}_2 \end{bmatrix} \Pi_9 + \tau^2 e_3^T U e_3 + \tau e_1^T V e_1 + \frac{\tau^2}{4} e_3^T Z e_3 \\
&\quad -\Pi_{12}^T Z \Pi_{12} - \Pi_{13}^T Z \Pi_{13} - 2\Pi_{14}^T Z \Pi_{14} - 2\Pi_{15}^T Z \Pi_{15} + \frac{\tau^2}{4} e_3^T W e_3 - \Pi_{16}^T W \Pi_{16} \\
&\quad -\Pi_{17}^T W \Pi_{17} - 2\Pi_{14}^T W \Pi_{14} - 2\Pi_{15}^T W \Pi_{15} + e_{11}^T L_1 e_{11} - e_{12}^T L_1 e_{12} + e_{13}^T L_2 e_{13} \\
&\quad -(1-\mu) e_{14}^T L_2 e_{14} + \tau_1 e_{15}^T L_3 e_{15} - e_{16}^T L_3 e_{16} - 2e_1^T N_1 e_3 - 2e_1^T N_1 A e_1 \\
&\quad +2e_1^T N_1 W_0 e_{11} + 2e_1^T N_1 W_1 e_{14} + 2e_1^T N_1 W_2 e_{16} + 2e_1^T N_1 K e_1 - 2e_3^T N_2 e_3 \\
&\quad -2e_3^T N_2 A e_1 + 2e_3^T N_2 W_0 e_{11} + 2e_3^T N_2 W_1 e_{14} + 2e_3^T N_2 W_2 e_{16} + 2e_3^T N_2 K e_1
\end{aligned}
$$

$$
\begin{aligned}
&+\Pi_{20}^T \begin{bmatrix} -F_1 D_1 & F_2 D_1 \\ * & -D_1 \end{bmatrix} \Pi_{20} + \Pi_{21}^T \begin{bmatrix} -G_1 D_2 & G_2 D_2 \\ * & -D_2 \end{bmatrix} \Pi_{21} \\
&+\Pi_{22}^T \begin{bmatrix} -H_1 D_3 & H_2 D_3 \\ * & -D_3 \end{bmatrix} \Pi_{22}.
\end{aligned}
$$

Then, the system (2.3) is asymptotically stable.
The upper bound of the quadratic cost function (2.7) is as follows:

$$J^* = \lambda_2 \|\phi\|_c^2. \tag{3.3}$$

Proof. Consider a LyapunovKrasovskii functional candidate

$$V(t, x_t) = \sum_{i=1}^{7} V_i(t, x_t),$$

where

$$
\begin{aligned}
V_1(t,x_t) &= \chi^T(t)P\chi(t),\\
V_2(t,x_t) &= \int_{t-\tau(t)}^{t} x^T(s)Mx(s)ds,\\
V_3(t,x_t) &= \int_{t-\tau}^{t} x^T(s)Qx(s)ds + \tau\int_{t-\tau}^{t}\int_{u}^{t} \dot{x}^T(s)R\dot{x}(s)dsdu,\\
V_4(t,x_t) &= \int_{t-\tau}^{t} (\tau-t+s)^2\dot{x}^T(s)S\dot{x}(s)ds,\\
V_5(t,x_t) &= \tau\int_{t-\tau}^{t} (\tau-t+s)\dot{x}^T(s)U\dot{x}(s)ds,\\
V_6(t,x_t) &= \int_{-\tau}^{0}\int_{t+s}^{0} x^T(\theta)Vx(\theta)d\theta ds + \frac{1}{2}\int_{t-\tau}^{t}\int_{t-\tau}^{u}\int_{\theta}^{t} \dot{x}^T(s)Z\dot{x}(s)dsd\theta du\\
&\quad +\frac{1}{2}\int_{t-\tau}^{t}\int_{t-\tau}^{u}\int_{\theta}^{t} \dot{x}^T(s)W\dot{x}(s)dsd\theta du,\\
V_7(t,x_t) &= \int_{t-\tau}^{t} f^T(x(s))L_1f(x(s))ds + \int_{t-\tau(t)}^{t} g^T(x(s))L_2g(x(s))ds\\
&\quad +\int_{-\tau_1}^{0}\int_{t+s}^{t} h^T(x(\theta))L_3h(x(\theta))d\theta ds.
\end{aligned}
$$

Taking the derivative of $V_i(t,x_t)$ along the solution of system (2.3) yields

$$
\begin{aligned}
\dot{V}_1(t,x_t) &= 2\chi^T(t)P\dot{\chi}(t), && (3.4)\\
\dot{V}_2(t,x_t) &= x^T(t)Mx(t) - (1-\dot{\tau}(t))x^T(t-\tau(t))Mx(t-\tau(t))\\
&\leq x^T(t)Mx(t) - (1-\mu)x^T(t-\tau(t))Mx(t-\tau(t)), && (3.5)\\
\dot{V}_3(t,x_t) &= x^T(t)Qx(t) - x^T(t-\tau)Qx(t-\tau) + \tau^2\dot{x}^T(t)R\dot{x}(t)\\
&\quad -\tau\int_{t-\tau}^{t} \dot{x}^T(s)R\dot{x}(s)ds, && (3.6)\\
\dot{V}_4(t,x_t) &= -2\int_{t-\tau}^{t} (\tau-t+s)\dot{x}^T(s)S\dot{x}(s)ds + \tau^2\dot{x}^T(t)S\dot{x}(t)
\end{aligned}
$$

$$
\begin{aligned}
&= \tau^2\dot{x}^T(t)S\dot{x}(t) - 2\int_{t-\tau}^{t-\tau(t)}(\tau - t + s)\dot{x}^T(s)S\dot{x}(s)ds \\
&\quad -2\int_{t-\tau(t)}^{t}(\tau - t + s)\dot{x}^T(s)S\dot{x}(s)ds + 2\int_{t-\tau(t)}^{t}\tau(t)\dot{x}^T(s)S\dot{x}(s)ds \\
&\quad -2\int_{t-\tau(t)}^{t}\tau(t)\dot{x}^T(s)S\dot{x}(s)ds \\
&= \tau^2\dot{x}^T(t)S\dot{x}(t) - 2\int_{t-\tau}^{t-\tau(t)}(\tau - t + s)\dot{x}^T(s)S\dot{x}(s)ds \\
&\quad -2\int_{t-\tau(t)}^{t}(\tau - \tau(t))\dot{x}^T(s)S\dot{x}(s)ds \\
&\quad -2\int_{t-\tau(t)}^{t}(\tau(t) - t + s)\dot{x}^T(s)S\dot{x}(s)ds, \qquad (3.7)
\end{aligned}
$$

$$
\begin{aligned}
\dot{V}_5(t,x_t) &= -\tau\int_{t-\tau(t)}^{t}\dot{x}^T(s)U\dot{x}(s)ds - \tau\int_{t-\tau}^{t-\tau(t)}\dot{x}^T(s)U\dot{x}(s)ds \\
&\quad +\tau^2\dot{x}^T(t)U\dot{x}(t), \qquad (3.8)\\
\dot{V}_6(t,x_t) &= \tau x^T(t)Vx(t) - \int_{t-\tau}^{t}x^T(\theta)Vx(\theta)d\theta + \frac{\tau^2}{4}\dot{x}^T(t)Z\dot{x}(t) \\
&\quad -\frac{1}{2}\int_{t-\tau}^{t}\int_{t-\tau}^{\theta}\dot{x}^T(s)Z\dot{x}(s)dsd\theta + \frac{\tau^2}{4}\dot{x}^T(t)W\dot{x}(t) \\
&\quad -\frac{1}{2}\int_{t-\tau}^{t}\int_{\theta}^{t}\dot{x}^T(s)W\dot{x}(s)dsd\theta, \qquad (3.9)\\
\dot{V}_7(t,x_t) &= f^T(x(t))L_1f(x(t)) - f^T(x(t-\tau))L_1f(x(t-\tau)) \\
&\quad +g^T(x(t))L_2g(x(t)) - (1-\dot{\tau}(t))g^T(x(t-\tau(t)))L_2g(x(t-\tau(t))) \\
&\quad +\tau_1h^T(x(t))L_3h(x(t)) - \int_{t-\tau_1}^{t}h^T(x(s))L_3h(x(s))ds. \qquad (3.10)
\end{aligned}
$$

Utilizing Lemma 2.4, we obtain that

$$
\begin{aligned}
&-\tau\int_{t-\tau}^{t}\dot{x}^T(s)R\dot{x}(s)ds \\
&\quad \le -[x(t)-x(t-\tau)]^TR[x(t)-x(t-\tau)] - 3\left[x(t)+x(t-\tau)-\frac{2}{\tau}\eta_1(t)\right]^T R\times \\
&\quad \left[x(t)+x(t-\tau)-\frac{2}{\tau}\eta_1(t)\right] - 5\left[x(t)-x(t-\tau)+\frac{6}{\tau}\eta_1(t)-\frac{12}{\tau^2}\eta_2(t)\right]^T R\times \\
&\quad \left[x(t)-x(t-\tau)+\frac{6}{\tau}\eta_1(t)-\frac{12}{\tau^2}\eta_2(t)\right]. \qquad (3.11)
\end{aligned}
$$

Applying Lemma 2.5, we have

$$
-2\int_{t-\tau}^{t-\tau(t)}(\tau - t + s)\dot{x}^T(s)S\dot{x}(s)ds - 2\int_{t-\tau(t)}^{t}(\tau(t) - t + s)\dot{x}^T(s)S\dot{x}(s)ds
$$

$$\leq -\frac{4}{\tau^2(t)}\Big[\int_{t-\tau(t)}^{t}(\tau-t+s)\dot{x}(s)ds\Big]^T S\Big[\int_{t-\tau(t)}^{t}(\tau-t+s)\dot{x}(s)ds\Big]$$
$$-\frac{4}{(\tau-\tau(t))^2}\Big[\int_{t-\tau}^{t-\tau(t)}(\tau-t+s)\dot{x}(s)ds\Big]^T S\Big[\int_{t-\tau}^{t-\tau(t)}(\tau-t+s)\dot{x}(s)ds\Big] \quad (3.12)$$

It follows from Lemma 2.6 that

$$\begin{aligned}
-2(\tau-\tau(t))\int_{t-\tau(t)}^{t}\dot{x}^T(s)S\dot{x}(s)ds \leq\ & -\frac{2(\tau-\tau(t))}{\tau}\begin{bmatrix}x(t)-x(t-\tau(t))\\ x(t)+x(t-\tau(t))-2\nu_1(t)\end{bmatrix}^T \times \\
& \begin{bmatrix}S & 0\\ 0 & 3S\end{bmatrix}\begin{bmatrix}x(t)-x(t-\tau(t))\\ x(t)+x(t-\tau(t))-2\nu_1(t)\end{bmatrix} \\
=\ & -\frac{\tau}{\tau(t)}\begin{bmatrix}x(t)-x(t-\tau(t))\\ x(t)+x(t-\tau(t))-2\nu_1(t)\end{bmatrix}^T \times \\
& \begin{bmatrix}2S & 0\\ 0 & 6S\end{bmatrix}\begin{bmatrix}x(t)-x(t-\tau(t))\\ x(t)+x(t-\tau(t))-2\nu_1(t)\end{bmatrix} \\
& +\begin{bmatrix}x(t)-x(t-\tau(t))\\ x(t)+x(t-\tau(t))-2\nu_1(t)\end{bmatrix}^T\begin{bmatrix}2S & 0\\ 0 & 6S\end{bmatrix} \\
& \begin{bmatrix}x(t)-x(t-\tau(t))\\ x(t)+x(t-\tau(t))-2\nu_1(t)\end{bmatrix}, \\
-\tau\int_{t-\tau(t)}^{t}\dot{x}^T(s)U\dot{x}(s)ds \leq\ & -\frac{\tau}{\tau(t)}\begin{bmatrix}x(t)-x(t-\tau(t))\\ x(t)+x(t-\tau(t))-2\nu_1(t)\end{bmatrix}^T \times \\
& \begin{bmatrix}U & 0\\ 0 & 3U\end{bmatrix}\begin{bmatrix}x(t)-x(t-\tau(t))\\ x(t)+x(t-\tau(t))-2\nu_1(t)\end{bmatrix} \qquad (3.13)
\end{aligned}$$

and

$$\begin{aligned}
-\tau\int_{t-\tau}^{t-\tau(t)}\dot{x}^T(s)U\dot{x}(s)ds \leq\ & -\frac{\tau}{\tau-\tau(t)}\begin{bmatrix}x(t-\tau(t))-x(t-\tau)\\ x(t-\tau(t))-x(t-\tau)-2\nu_2(t)\end{bmatrix}^T \times \\
& \begin{bmatrix}U & 0\\ 0 & 3U\end{bmatrix}\begin{bmatrix}x(t-\tau(t))-x(t-\tau)\\ x(t-\tau(t))-x(t-\tau)-2\nu_2(t)\end{bmatrix}.
\end{aligned}$$

By using Lemma 2.7, we get

$$\begin{aligned}
& -\frac{\tau}{\tau(t)}\begin{bmatrix}x(t)-x(t-\tau(t))\\ x(t)+x(t-\tau(t))-2\nu_1(t)\end{bmatrix}^T\begin{bmatrix}2S+U & 0\\ 0 & 6S+3U\end{bmatrix}\times \\
& \begin{bmatrix}x(t)-x(t-\tau(t))\\ x(t)+x(t-\tau(t))-2\nu_1(t)\end{bmatrix} \\
& -\frac{\tau}{\tau-\tau(t)}\begin{bmatrix}x(t-\tau(t))-x(t-\tau)\\ x(t-\tau(t))-x(t-\tau)-2\nu_2(t)\end{bmatrix}^T\begin{bmatrix}U & 0\\ 0 & 3U\end{bmatrix}\times \\
& \begin{bmatrix}x(t-\tau(t))-x(t-\tau)\\ x(t-\tau(t))-x(t-\tau)-2\nu_2(t)\end{bmatrix} \\
& \leq \begin{bmatrix}x(t)-x(t-\tau(t))\\ x(t)+x(t-\tau(t))-2\nu_1(t)\\ x(t-\tau(t))-x(t-\tau)\\ xt-\tau(t))+x(t-\tau)-2\nu_2(t)\end{bmatrix}^T\begin{bmatrix}\hat{G}_1 & 0\\ * & \hat{G}_2\end{bmatrix}\times
\end{aligned}$$

$$\begin{bmatrix} x(t)-x(t-\tau(t)) \\ x(t)+x(t-\tau(t))-2\nu_1(t) \\ x(t-\tau(t))-x(t-\tau) \\ x(t-\tau(t))+x(t-\tau)-2\nu_2(t) \end{bmatrix} - \frac{\tau-\tau(t)}{\tau} \begin{bmatrix} x(t)-x(t-\tau(t)) \\ x(t)+x(t-\tau(t))-2\nu_1(t) \\ x(t-\tau(t))-x(t-\tau) \\ x(t-\tau(t))+x(t-\tau)-2\nu_2(t) \end{bmatrix}^T \times$$
$$\begin{bmatrix} J_1 & J_4 \\ * & 0 \end{bmatrix} \begin{bmatrix} x(t)-x(t-\tau(t)) \\ x(t)+x(t-\tau(t))-2\nu_1(t) \\ x(t-\tau(t))-x(t-\tau) \\ x(t-\tau(t))+x(t-\tau)-2\nu_2(t) \end{bmatrix} - \frac{\tau(t)}{\tau} \times$$
$$\begin{bmatrix} x(t)-x(t-\tau(t)) \\ x(t)+x(t-\tau(t))-2\nu_1(t) \\ x(t-\tau(t))-x(t-\tau) \\ x(t-\tau(t))+x(t-\tau)-2\nu_2(t) \end{bmatrix}^T \begin{bmatrix} 0 & J_3 \\ * & J_2 \end{bmatrix} \begin{bmatrix} x(t)-x(t-\tau(t)) \\ x(t)+x(t-\tau(t))-2\nu_1(t) \\ x(t-\tau(t))-x(t-\tau) \\ x(t-\tau(t))+x(t-\tau)-2\nu_2(t) \end{bmatrix}.$$

By Lemma 2.8, we have

$$\begin{aligned} &\int_{t-\tau}^{t} x^T(\theta)Vx(\theta)d\theta \\ &\quad \le -(\tau-\tau(t))\nu_2^T(t)V\nu_2(t) - 3(\tau-\tau(t))\big[\nu_2(t)-\omega_2(t)\big]^T V\big[\nu_2(t)-\omega_2(t)\big] \\ &\qquad -\tau(t)\nu_1^T(t)V\nu_1(t) - 3\tau(t)\big[\nu_1(t)-\omega_1(t)\big]^T V\big[\nu_1(t)-\omega_1(t)\big]. \end{aligned} \tag{3.14}$$

Applying Lemma 2.9, we have

$$\begin{aligned} &-\frac{1}{2}\int_{t-\tau}^{t}\int_{t-\tau}^{\theta} \dot{x}^T(s)Z\dot{x}(s)dsd\theta \\ &\quad = -\frac{1}{2}\Big[\int_{t-\tau(t)}^{t}\int_{t-\tau}^{t-\tau(t)} + \int_{t-\tau(t)}^{t}\int_{t-\tau(t)}^{\theta} + \int_{t-\tau}^{t-\tau(t)}\int_{t-\tau}^{\theta}\Big]\dot{x}^T(s)Z\dot{x}(s)dsd\theta \\ &\quad \le -\big[x(t-\tau(t))-x(t-\tau)\big]^T\left[\frac{\tau(t)Z}{2\tau}\right]\big[x(t-\tau(t))-x(t-\tau)\big] \\ &\qquad -\big[\nu_1(t)-x(t-\tau(t))\big]^T Z\big[\nu_1(t)-x(t-\tau(t))\big] \\ &\qquad -\big[\nu_2(t)-x(t-\tau)\big]^T Z\big[\nu_2(t)-x(t-\tau)\big] \\ &\qquad -\left[\frac{1}{2}x(t-\tau(t))+\nu_2(t)-\frac{3}{2}\omega_2(t)\right]^T 2Z\left[\frac{1}{2}x(t-\tau(t))+\nu_2(t)-\frac{3}{2}\omega_2(t)\right] \\ &\qquad -\left[\frac{x(t)}{2}+\nu_1(t)-\frac{3}{2}\omega_1(t)\right]^T 2Z\left[\frac{x(t)}{2}+\nu_1(t)-\frac{3}{2}\omega_1(t)\right] \end{aligned} \tag{3.15}$$

and

$$\begin{aligned} &-\frac{1}{2}\int_{t-\tau}^{t}\int_{\theta}^{t} \dot{x}^T(s)W\dot{x}(s)dsd\theta \\ &\quad = -\frac{1}{2}\Big[\int_{t-\tau(t)}^{t}\int_{\theta}^{t} + \int_{t-\tau}^{t-\tau(t)}\int_{t-\tau(t)}^{t} + \int_{t-\tau}^{t-\tau(t)}\int_{\theta}^{t-\tau(t)}\Big]\dot{x}^T(s)W\dot{x}(s)dsd\theta \end{aligned}$$

$$
\begin{aligned}
&\leq -\left[x(t)-x(t-\tau(t))\right]^T\left[\frac{(\tau-\tau(t))W}{2\tau}\right]\left[x(t)-x(t-\tau(t))\right]\\
&\quad -\left[x(t)-\nu_1(t)\right]^T W\left[x(t)-\nu_1(t)\right]-\left[x(t-\tau(t))-\nu_2(t)\right]^T W\left[x(t-\tau(t))-\nu_2(t)\right]\\
&\quad -\left[\frac{x(t)}{2}+\nu_1(t)-\frac{3}{2}\omega_1(t)\right]^T 2W\left[\frac{x(t)}{2}+\nu_1(t)-\frac{3}{2}\omega_1(t)\right]\\
&\quad -\left[\frac{1}{2}x(t-\tau(t))+\nu_2(t)-\frac{3}{2}\omega_2(t)\right]^T 2W\left[\frac{1}{2}x(t-\tau(t))+\nu_2(t)-\frac{3}{2}\omega_2(t)\right]. \qquad (3.16)
\end{aligned}
$$

By Lemma 2.10, we get

$$
-\int_{t-\tau_1}^{t} h^T(x(s))L_3h(x(s))ds \leq -\int_{t-\tau_1(t)}^{t} h^T(x(s))dsL_3\int_{t-\tau_1(t)}^{t} h(x(s))ds. \qquad (3.17)
$$

It follows from **(H1)** that $\left[f_i(x_i(t))-F_i^-x_i(t)\right]\left[f_i(x_i(t))-F_i^+x_i(t)\right]\leq 0$ for every $i=1,2,\ldots,n$, which are equivalent to

$$
\begin{bmatrix} x(t)\\ f(x(t))\end{bmatrix}^T\begin{bmatrix} F_i^-F_i^+e_ie_i^T & -\dfrac{F_i^-+F_i^+}{2}e_ie_i^T\\ * & e_ie_i^T\end{bmatrix}\begin{bmatrix} x(t)\\ f(x(t))\end{bmatrix}\leq 0
$$

for every $i=1,2,\ldots,n$.
Define $D_1=\operatorname{diag}\{d_1,d_2,\ldots,d_n\}>0$, then

$$
\sum_{i=1}^{n} y_i\begin{bmatrix} x(t)\\ f(x(t))\end{bmatrix}^T\begin{bmatrix} F_i^-F_i^+e_ie_i^T & -\dfrac{F_i^-+F_i^+}{2}e_ie_i^T\\ * & e_ie_i^T\end{bmatrix}\begin{bmatrix} x(t)\\ f(x(t))\end{bmatrix}\leq 0,
$$

which is equivalent to

$$
\begin{bmatrix} x(t)\\ f(x(t))\end{bmatrix}^T\begin{bmatrix} -F_1D_1 & F_2D_1\\ * & -D_1\end{bmatrix}\begin{bmatrix} x(t)\\ f(x(t))\end{bmatrix}\geq 0. \qquad (3.18)
$$

Similarly, from **(H2)** and **(H3)**, define $D_2=\operatorname{diag}\{\tilde{d}_1,\tilde{d}_2,\ldots,\tilde{d}_n\}>0$, $D_3=\operatorname{diag}\{\hat{d}_1,\hat{d}_2,\ldots,\hat{d}_n\}>0$ we have

$$
\begin{bmatrix} x(t))\\ g(x(t))\end{bmatrix}^T\begin{bmatrix} -G_1D_2 & G_2D_2\\ * & -D_2\end{bmatrix}\begin{bmatrix} x(t)\\ g(x(t))\end{bmatrix}\geq 0, \qquad (3.19)
$$

$$
\begin{bmatrix} x(t)\\ h(x(t))\end{bmatrix}^T\begin{bmatrix} -H_1D_3 & H_2D_3\\ * & -D_3\end{bmatrix}\begin{bmatrix} x(t)\\ h(x(t))\end{bmatrix}\geq 0. \qquad (3.20)
$$

Consider the equation

$$
\begin{aligned}
0 \;=\; & -\dot{x}(t)-Ax(t)+W_0f(x(t))+W_1g(x(t-\tau(t)))\\
& +W_2\int_{t-\tau_1(t)}^{t} h(x(s))ds+BKx(t).
\end{aligned}
$$

Multiplying both sides with $2x^T(t)N_1$ and $2\dot{x}^T(t)N_2$, respectively, we obtain

$$\begin{aligned} 0 &= -2x^T(t)N_1\dot{x}(t) - 2x^T(t)N_1Ax(t) + 2x^T(t)N_1W_0f(x(t)) \\ &\quad +2x^T(t)N_1W_1g(x(t-\tau(t))) + 2x^T(t)N_1W_2\int_{t-\tau_1(t)}^{t} h(x(s))ds \\ &\quad +2x^T(t)N_1Kx(t), \end{aligned} \tag{3.21}$$

$$\begin{aligned} 0 &= -2\dot{x}^T(t)N_2\dot{x}(t) - 2\dot{x}^T(t)N_2Ax(t) + 2\dot{x}^T(t)N_2^TW_0f(x(t)) \\ &\quad +2\dot{x}^T(t)N_2W_1g(x(t-\tau(t))) + 2\dot{x}^T(t)N_2W_2\int_{t-\tau_1(t)}^{t} h(x(s))ds \\ &\quad +2\dot{x}^T(t)N_2Kx(t). \end{aligned} \tag{3.22}$$

Adding the right-hand sides of (3.21) - (3.22) to $\dot{V}(t,x_t)$, we get

$$\dot{v}(t,x_t) \leq \xi^T(t)\psi(\tau(t))\xi(t),$$

where

$$\psi(\tau(t)) = \tau(t)\Xi_1 + (\tau - \tau(t))\Xi_2 + \Xi_3, \tag{3.23}$$

where $\Xi_i(i = 1,2,3)$ are given in Theorem 3.1. Noting that the $\psi(\tau(t))$ is convex combination about $\tau(t)$, with (3.23), (3.1) and (3.2). Thus system (2.3) with (2.4) - (2.6) is asymptotically stable. We let

$$\begin{aligned} L(t,x(t),x(t-\tau(t)),&\int_{t-\tau_1(t)}^{t} x(s)ds,u(t)) \\ &\leq x^T(t)Z_1x(t) + x^T(t-\tau(t))Z_2x(t-\tau(t)) \\ &\quad + \left(\int_{t-\tau_1(t)}^{t} x^T(s)ds\right)Z_3\left(\int_{t-\tau_1(t)}^{t} x(s)ds\right) + u^T(t)Z_4u(t). \end{aligned} \tag{3.24}$$

From (3.4) - (3.24), we obtain

$$\dot{V}(t,x_t) \leq \xi^T(t)\psi\xi(t) - L(t,x(t),x(t-\tau(t)),\int_{t-\tau_1(t)}^{t} x(s)ds,u(t)). \tag{3.25}$$

To find the upper bound of the cost function (2.7), we consider the derived condition (3.25) and $V(t,x_t) > 0$, we have

$$\dot{V}(t,x_t) \leq \xi^T(t)\psi\xi(t) - L(t,x(t),x(t-\tau(t)),\int_{t-\tau_1(t)}^{t} x(s)ds,u(t)). \tag{3.26}$$

Integrating both sides of (3.26) from 0 to t, we obtain

$$\int_0^t L(t,x(t),x(t-\tau(t)),\int_{t-\tau_1(t)}^{t} x(s)ds,u(t))dt \leq V(0,x_0) - V(t,x_t) \leq V(0,x_0),$$

because of $V(t,x_t) > 0$. Hence, letting $t \to \infty$, we finally obtain that

$$J = \int_0^\infty L(t,x(t),x(t-\tau(t)),\int_{t-\tau_1(t)}^{t} x(s)ds,u(t))dt \leq V(0,x_0) \leq \lambda_2\|\phi\|_c^2 = J^*.$$

This completes the proof of the theorem. ■

Based on Theorem 3.1, the feedback controller design, ensuring the asymptotic stability of the neural network with mixed time-varying delays is explained.

Theorem 3.2. *The neural network system (2.3) with the quadratic cost function (2.7) is asymptotically stabilized if there exist symmetric positive definite matrices $P \in \mathbb{R}^{3n\times 3n}$, M, $Q, R, S, U, V, Z, W, L_1, L_2, L_3, N_1, N_2 \in \mathbb{R}^{n\times n}$, positive diagonal matrices $D_1, D_2, D_3 \in \mathbb{R}^{n\times n}$ and B is an appropriately dimensioned matrix such that the following LMIs hold:*

$$\begin{aligned} \tilde{G}_j &> 0, \ (j=1,2), \\ \begin{bmatrix} \tilde{G}_1 - J_1 & -J_3 \\ * & \tilde{G}_2 \end{bmatrix} &\geq 0, \begin{bmatrix} \tilde{G}_1 & -J_4 \\ * & \tilde{G}_2 - J_2 \end{bmatrix} \geq 0, \\ \tau\bar{\Xi}_2 + \bar{\Xi}_3 &< 0, \qquad (3.27) \\ \tau\bar{\Xi}_1 + \bar{\Xi}_3 &< 0, \qquad (3.28) \end{aligned}$$

where

$$\begin{aligned} \tilde{G}_1 &= \begin{bmatrix} 2S+U & 0 \\ * & 6S+3S \end{bmatrix}, \\ \tilde{G}_2 &= \begin{bmatrix} U & 0 \\ * & 3U \end{bmatrix}, \\ \bar{\Xi}_1 &= -\frac{1}{\tau}\Pi_9^T \begin{bmatrix} 0 & J_3 \\ * & J_2 \end{bmatrix} \Pi_9 - e_5^T V e_5 - 3\Pi_{10}^T V \Pi_{10} - \frac{1}{2\tau}\Pi_{18}^T Z \Pi_{18}, \\ \bar{\Xi}_2 &= -\frac{1}{\tau}\Pi_9^T \begin{bmatrix} J_1 & J_4 \\ * & 0 \end{bmatrix} \Pi_9 - e_6^T V e_6 - 3\Pi_{11}^T V \Pi_{11} - \frac{1}{2\tau}\Pi_{19}^T Z \Pi_{19}, \\ \bar{\Xi}_3 &= 2\Pi_1^T P \Pi_2 + e_1^T M e_1 - (1-\mu) e_2^T M e_2 + e_1^T Q e_1 - e_4^T Q e_4 + \tau^2 e_3^T R e_3 - \\ &\quad \Pi_3^T R \Pi_3 - 3\Pi_4^T R \Pi_4 - 5\Pi_5^T R \Pi_5 + \tau^2 e_3^T S e_3 - 4\Pi_6^T S \Pi_6 - 4\Pi_7^T S \Pi_7 \\ &\quad + \Pi_8^T \begin{bmatrix} 2S & 0 \\ * & 6S \end{bmatrix} \Pi_8 - \Pi_9^T \begin{bmatrix} \tilde{G}_1 & 0 \\ * & \tilde{G}_2 \end{bmatrix} \Pi_9 + \tau^2 e_3^T U e_3 + \tau e_1^T V e_1 + \frac{\tau^2}{4} e_3^T Z e_3 \\ &\quad - \Pi_{12}^T Z \Pi_{12} - \Pi_{13}^T Z \Pi_{13} - 2\Pi_{14}^T Z \Pi_{14} - 2\Pi_{15}^T Z \Pi_{15} + \frac{\tau^2}{4} e_3^T W e_3 - \Pi_{16}^T W \Pi_{16} \\ &\quad - \Pi_{17}^T W \Pi_{17} - 2\Pi_{14}^T W \Pi_{14} - 2\Pi_{15}^T W \Pi_{15} + e_{11}^T L_1 e_{11} - e_{12}^T L_1 e_{12} + e_{13}^T L_2 e_{13} \\ &\quad - (1-\mu) e_{14}^T L_2 e_{14} + \tau_1 e_{15}^T L_3 e_{15} - e_{16}^T L_3 e_{16} - 2e_1^T N_1 e_3 - 2e_1^T N_1 A e_1 \\ &\quad + 2e_1^T N_1 W_0 e_{11} + 2e_1^T N_1 W_1 e_{14} + 2e_1^T N_1 W_2 e_{16} + 2\beta_1 e_1^T B e_1 - 2e_3^T N_2 e_3 \\ &\quad - 2e_3^T N_2 A e_1 + 2e_3^T N_2 W_0 e_{11} + 2e_3^T N_2 W_1 e_{14} + 2e_3^T N_2 W_2 e_{16} + 2\beta_2 e_3^T B e_1 \\ &\quad + \Pi_{20}^T \begin{bmatrix} -F_1 D_1 & F_2 D_1 \\ * & -D_1 \end{bmatrix} \Pi_{20} + \Pi_{21}^T \begin{bmatrix} -G_1 D_2 & G_2 D_2 \\ * & -D_2 \end{bmatrix} \Pi_{21} \\ &\quad + \Pi_{22}^T \begin{bmatrix} -H_1 D_3 & H_2 D_3 \\ * & -D_3 \end{bmatrix} \Pi_{22}. \end{aligned}$$

Meanwhile, the designed controller gains are given in the following:

$$K = Q^{-1} B. \qquad (3.29)$$

Proof. Denote

$$N_1 = \beta_1 Q, \quad N_2 = \beta_2 Q. \qquad (3.30)$$

Similarly to Theorem 3.1, the LMIs (3.27)-(3.28) can be achieved. This completes the proof. ∎

4. Numerical examples

In this section, we present two examples to illustrate the effectiveness and the reduced conservatism of our proposed methods.

Example 4.1. We consider the neural networks (2.3) with $\tau = 1.2$, $\tau_1 = 1.3$, $\mu = 0.9$, $\beta_1 = 0.9$, $\beta_2 = 0.7$,

$$\begin{aligned}
A &= \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \quad W_0 = \begin{bmatrix} 2 & -0.1 \\ -5 & 1.5 \end{bmatrix}, W_1 = \begin{bmatrix} -1.5 & -0.1 \\ -0.2 & -1 \end{bmatrix}, W_2 = \begin{bmatrix} 0.6 & 0.15 \\ -1.8 & -0.12 \end{bmatrix}, \\
Z_1 &= \begin{bmatrix} 0.2000 & 0.0003 \\ 0.0003 & 0.1996 \end{bmatrix}, Z_2 = \begin{bmatrix} 0.2000 & 0.0003 \\ 0.0003 & 0.1996 \end{bmatrix}, Z_3 = \begin{bmatrix} 0.2000 & 0.0003 \\ 0.0003 & 0.1996 \end{bmatrix}, \\
Z_4 &= \begin{bmatrix} 0.0031 & 0.0002 \\ 0.0002 & 0.0039 \end{bmatrix}, F_1 = G_1 = H_1 \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, F_2 = G_2 = H_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \\
\tau(t) &= 0.2 + \frac{e^t}{1+e^t}, \tau_1(t) = 1.2|\cos t|, \\
\phi(t) &= [-0.2, 0.2]^T, \text{and} \quad f_i(x_i) = g_i(x_i) \;=\; h_i(x_i) \;=\; \tanh(x_i).
\end{aligned}$$

LMIs of (3.27), (3.28) in Theorem 3.2 are solved. We obtain

$$\begin{aligned}
P &= \begin{bmatrix} 1.5873 & -0.0456 & -0.0000 & -0.0000 & -0.0000 & -0.0000 \\ -0.0456 & 1.6638 & -0.0000 & 0.0000 & -0.0000 & 0.0000 \\ -0.0000 & -0.0000 & 0.0340 & -0.0003 & 0.0111 & -0.0006 \\ -0.0000 & 0.0000 & -0.0003 & 0.0313 & -0.0007 & 0.0121 \\ -0.0000 & -0.0000 & 0.0111 & -0.0007 & 0.0067 & -0.0016 \\ -0.0000 & 0.0000 & -0.0006 & 0.0121 & -0.0016 & 0.0113 \end{bmatrix}, \\
M &= \begin{bmatrix} 0.2644 & 0.0005 \\ 0.0005 & 0.2471 \end{bmatrix}, Q = \begin{bmatrix} 0.2831 & 0.0022 \\ 0.0022 & 0.3323 \end{bmatrix}, \\
R &= \begin{bmatrix} 0.00000792 & -0.00000445 \\ -0.00000445 & 0.00002151 \end{bmatrix}, S = \begin{bmatrix} 0.000002331 & -0.000000858 \\ -0.000000858 & 0.000004917 \end{bmatrix}, \\
U &= \begin{bmatrix} 0.000002331 & -0.000000857 \\ -0.000000857 & 0.000004917 \end{bmatrix}, V = \begin{bmatrix} 0.0567 & -0.0001 \\ -0.0001 & 0.0600 \end{bmatrix}, \\
Z &= \begin{bmatrix} 0.00001735 & -0.00000685 \\ -0.00000685 & 0.00003802 \end{bmatrix}, W = \begin{bmatrix} 0.00001735 & -0.00000685 \\ -0.00000685 & 0.00003802 \end{bmatrix}, \\
L_1 &= \begin{bmatrix} 0.2115 & 0.0021 \\ 0.0021 & 0.2187 \end{bmatrix}, L_2 = \begin{bmatrix} 0.0007607 & -0.0000197 \\ -0.0000197 & 0.0008268 \end{bmatrix}, \\
L_3 &= \begin{bmatrix} 0.2886 & 0.0000 \\ 0.0000 & 0.2880 \end{bmatrix}, B = \begin{bmatrix} -2.2676 & 0.0651 \\ 0.0651 & -2.3770 \end{bmatrix}, \\
D_1 &= \begin{bmatrix} 0.4211 & 0 \\ 0 & 0.4211 \end{bmatrix}, D_2 = \begin{bmatrix} 0.0019 & 0 \\ 0 & 0.0019 \end{bmatrix}, \\
D_3 &= \begin{bmatrix} 0.5605 & 0 \\ 0 & 0.5605 \end{bmatrix}, N_1 = \begin{bmatrix} 0.0015 & -0.0002 \\ -0.0002 & 0.0022 \end{bmatrix}, \\
N_2 &= \begin{bmatrix} 0.00003768 & -0.00001650 \\ -0.00001650 & 0.00008745 \end{bmatrix}.
\end{aligned}$$

The state feedback control is obtained by

$$u(t) = Q^{-1}Bx(t) = \begin{bmatrix} -8.0125 & 0.2844 \\ 0.2477 & -7.1542 \end{bmatrix} x(t), \quad t \geq 0. \tag{4.1}$$

We take the initial condition $\phi(t) = \begin{bmatrix} 0 \\ \cos t \end{bmatrix}, \quad \|\phi\|_c = 1.$

Then, the upper bound on the cost function value is

$$J^* = 6.1725.$$

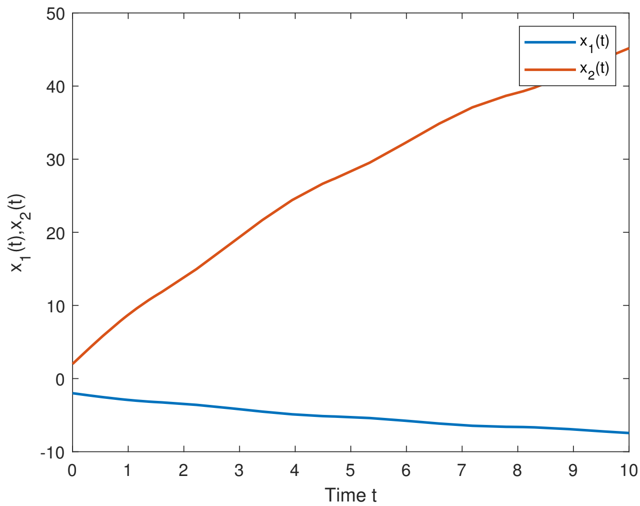

FIGURE 1. The trajectories of $x_1(t)$ and $x_2(t)$ without feedback control (4.1) in Example 4.1.

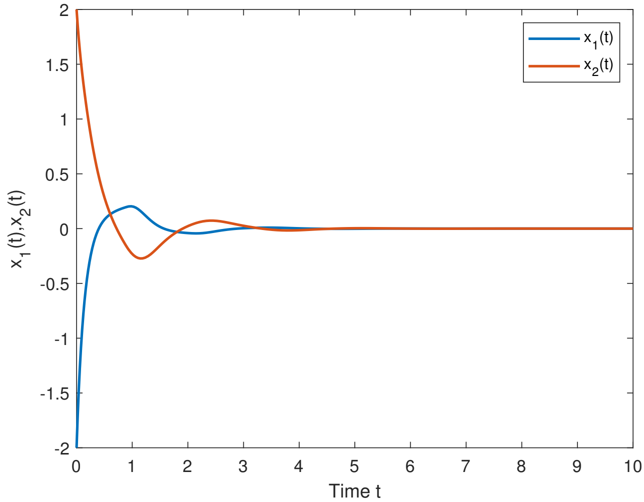

FIGURE 2. The trajectories of $x_1(t)$ and $x_2(t)$ with feedback control (4.1) in Example 4.1.

Figure 1 demonstrates the trajectories of solution $x_1(t)$ and $x_2(t)$ of neural networks with various activation functions and mixed time-varying delays without feedback control ($u(t) = 0$). Figure 2 illustrates the trajectories of solution $x_1(t)$ and $x_2(t)$ of neural networks with various activation functions and mixed time-varying delays with feedback control

$$u(t) = \begin{bmatrix} -8.0125 & 0.2844 \\ 0.2477 & -7.1542 \end{bmatrix} x(t).$$

Example 4.2. We consider the neural networks (2.3) with $\tau = 0.8$, $\tau_1 = 1.3$, $\mu = 0.9$, $\beta_1 = 0.7$, $\beta_2 = 0.9$,

$$\begin{aligned}
A &= \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \quad W_0 = \begin{bmatrix} 1.188 & 0.09 \\ 0.09 & 1.188 \end{bmatrix}, W_1 = \begin{bmatrix} 0.09 & 0.14 \\ 0.05 & 0.09 \end{bmatrix}, \\
W_2 &= \begin{bmatrix} 1 & 0.2 \\ -1.8 & -0.2 \end{bmatrix}, Z_1 = \begin{bmatrix} 0.1561 & -0.0005 \\ -0.0005 & 0.1555 \end{bmatrix}, Z_2 = \begin{bmatrix} 0.1561 & -0.0005 \\ -0.0005 & 0.1555 \end{bmatrix}, \\
Z_3 &= \begin{bmatrix} 0.1561 & -0.0005 \\ -0.0005 & 0.1555 \end{bmatrix}, Z_4 = \begin{bmatrix} 0.0024 & -0.0001 \\ -0.0001 & 0.0028 \end{bmatrix}, F_1 = \begin{bmatrix} -0.04 & 0 \\ 0 & -0.04 \end{bmatrix}, \\
G_1 &= \begin{bmatrix} -0.16 & 0 \\ 0 & -0.16 \end{bmatrix}, H_1 = \begin{bmatrix} -0.04 & 0 \\ 0 & -0.04 \end{bmatrix}, F_2 = G_2 = H_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \\
\tau(t) &= 0.2 + 0.2 \sin 10t, \ \tau_1(t) = 1.2|\sin t|, \ \phi(t) = [-0.2, 0.2]^T, \\
f_i(x_i) &= 0.2 \tanh(x_i), \ g_i(x_i) = 0.2(|x_i + 1| - |x_i - 1|) \text{ and } h_i(x_i) = \tanh(x_i).
\end{aligned}$$

LMIs of (3.27), (3.28) in Theorem 3.2 are solved. We obtain

$$P = \begin{bmatrix} 1.5301 & 0.0548 & -0.0008 & 0.0021 & 0.0045 & -0.0076 \\ 0.0548 & 1.6175 & 0.0013 & -0.0035 & -0.0073 & 0.0122 \\ -0.0008 & 0.0013 & 0.0477 & -0.0026 & 0.0216 & 0.0008 \\ 0.0021 & -0.0035 & -0.0026 & 0.0459 & 0.0003 & 0.0183 \\ 0.0045 & -0.0073 & 0.0216 & 0.0003 & 0.1626 & -0.0060 \\ -0.0076 & 0.0122 & 0.0008 & 0.0183 & -0.0060 & 0.1689 \end{bmatrix},$$

$$M = \begin{bmatrix} 0.2013 & -0.0012 \\ -0.0012 & 0.1892 \end{bmatrix}, Q = \begin{bmatrix} 0.2097 & 0.0039 \\ 0.0039 & 0.2428 \end{bmatrix},$$

$$R = \begin{bmatrix} 0.0002278 & -0.0000194 \\ -0.0000194 & 0.0002500 \end{bmatrix}, S = \begin{bmatrix} 0.0046 & -0.0070 \\ -0.0070 & 0.0115 \end{bmatrix},$$

$$U = \begin{bmatrix} 0.0049 & -0.0076 \\ -0.0076 & 0.0125 \end{bmatrix}, V = \begin{bmatrix} 0.0642 & 0.0113 \\ 0.0113 & 0.0539 \end{bmatrix},$$

$$Z = \begin{bmatrix} 0.0073 & -0.0089 \\ -0.0089 & 0.0162 \end{bmatrix}, W = \begin{bmatrix} 0.0064 & -0.0076 \\ -0.0076 & 0.0140 \end{bmatrix},$$

$$L_1 = \begin{bmatrix} 0.2155 & 0.0127 \\ 0.0127 & 0.1997 \end{bmatrix}, L_2 = \begin{bmatrix} 0.0002446 & 0.0001083 \\ 0.0001083 & 0.0003451 \end{bmatrix},$$

$$L_3 = \begin{bmatrix} 0.3541 & 0.0068 \\ 0.0068 & 0.3039 \end{bmatrix}, B = \begin{bmatrix} -1.7014 & -0.0590 \\ -0.0589 & -1.7997 \end{bmatrix},$$

$$D_1 = \begin{bmatrix} 0.4634 & 0 \\ 0 & 0.4634 \end{bmatrix}, D_2 = \begin{bmatrix} 0.0009109 & 0 \\ 0 & 0.0009109 \end{bmatrix},$$

$$D_3 = \begin{bmatrix} 0.5885 & 0 \\ 0 & 0.5885 \end{bmatrix}, N_1 = \begin{bmatrix} 0.0170 & -0.0137 \\ -0.0137 & 0.0319 \end{bmatrix},$$

$$N_2 = \begin{bmatrix} 0.0145 & -0.0212 \\ -0.0212 & 0.0356 \end{bmatrix}.$$

The state feedback control is obtained by

$$u(t) = Q^{-1}Bx(t) = \begin{bmatrix} -8.1112 & -0.1447 \\ -0.1134 & -7.4096 \end{bmatrix} x(t), \quad t \geq 0. \tag{4.2}$$

We take the initial condition $\phi(t) = \begin{bmatrix} 0 \\ \cos t \end{bmatrix}, \quad \|\phi\|_c = 1.$

Then, the upper bound on the cost function value is

$$J^* = 3.5152.$$

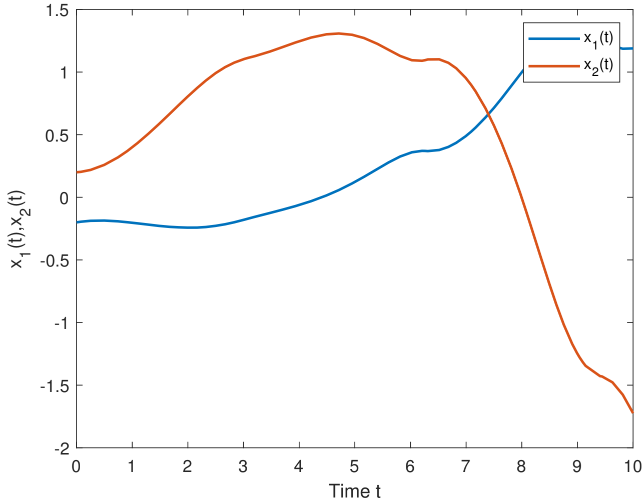

FIGURE 3. The trajectories of $x_1(t)$ and $x_2(t)$ without feedback control (4.2) in Example 4.2.

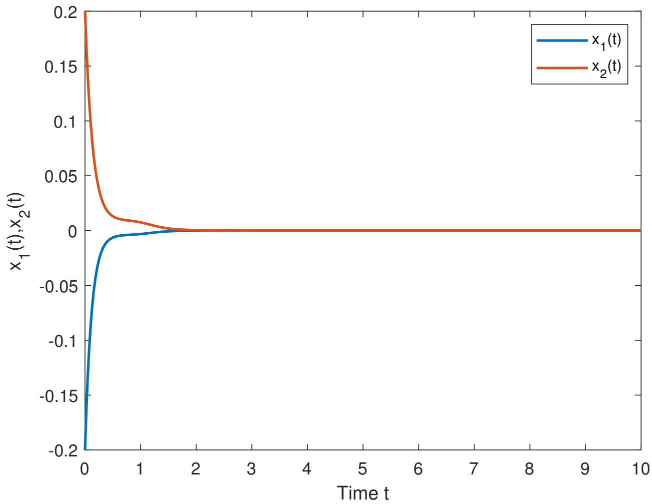

FIGURE 4. The trajectories of $x_1(t)$ and $x_2(t)$ with feedback control (4.2) in Example 4.2.

Figure 3 demonstrates the trajectories of solution $x_1(t)$ and $x_2(t)$ of neural networks with various activation functions and mixed time-varying delays without feedback control ($u(t) = 0$). Figure 4 illustrates the trajectories of solution $x_1(t)$ and $x_2(t)$ of neural networks with various activation functions and mixed time-varying delays with feedback control

$$u(t) = \begin{bmatrix} -8.1112 & -0.1447 \\ -0.1134 & -7.4096 \end{bmatrix} x(t).$$

5. Conclusions

In this paper, we have investigated the problem of guaranteed cost control for asymptotic stability of neural network with discrete and distributed time-varying delays. A Lyapunov-Krasovskii functional includes double integral term and triple integral term, using Wirtinger-based integral inequality, Jensen's integral inequality and to extended reciprocally convex inequality, new delay-dependent sufficient conditions for the existence of guaranteed cost feedback control for the system are given in terms of linear matrix inequalities. Finally, numerical examples are given to demonstrate the effectiveness of our results. Our goal in the future is to apply the nonlinear quadratic cost functions to other systems or networks that arise in other areas of science.

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References

[1] J. Cheng, J.H. Park, H.R. Karimi, H. Shen, A flexible terminal approach to sampled-data exponentially synchronization of Markovian neural networks with time-varying delayed signals, IEEE Trans. Cybern. 48(8)(2017) 2232–2244.

[2] T. Botmart, W. Weera, Guaranteed cost control for exponential synchronization of cellular neural networks with mixed time-varying delays via hybrid feedback control, J. Appl. Math. 2013 (2013), Article ID 175796 12 pages.

[3] J. Cao, J. Wang, Global asymptotic stability of a general class of recurrent neural networks with time-varying delays, IEEE Trans. Circuits Syst. I 50(1)(2013) 34–44.

[4] W. Cheng, X. Zhu, Y. Deng, A delay composition approach to stability analysis of neural networks with time-varying delay, ICICTA. 1(2010) 69–72.

[5] L. Chen, J. Sun, Global stability of an SI epidemic model with feedback controls, Appl. Math. Lett. 28(2014) 53–55.

[6] Y.H. Fan, L.L. Wang, Global asymptotical stability of a Logistic model with feedback control, Nonlinear Anal. 11(4)(2010) 2686–2697.

[7] Z. Li, M. Han, F. Chen, Influence of feedback controls on an autonomous Lotka-Volterra competitive system with infinite delays, Nonlinear Anal. 14(1)(2013) 402–413.

[8] P. Balasubramaniam, V. Vembarasan, Synchronization of recurrent neural networks with mixed time-delays via output coupling with delayed feedback, Nonlinear Dyn. 70(1)(2012) 677–691.

[9] P. Park, W.I. Lee, S.Y. Lee, Auxiliary function-based integral inequalities for quadratic functions and their applications to time-delay systems, J. Franklin. Inst. 352(14)(2015) 1378–1396.

[10] M.V. Thuan, V.N. Phat, Optimal guaranteed cost control of linear systems with mixed interval time-varying delayed state and control, J. Optim. Theory. Appl. 152(2)(2012) 394–412.

[11] P. Niamsup, K. Ratchagit, V.N. Phat, Novel criteria for finite-time stabilization and guaranteed cost control of delayed neural networks, Neurocomputing 160(2015) 281–286.

[12] K. Shi, H. Zhu, S. Zhong, Y. Zeng, Y. Zhang, W. Wang, Stability analysis of neutral type neural networks with mixed time - varying delays using triple-integral and delay-partitioning methods, ISA Transactions 58(2015) 85–95.

[13] B. Zhang, J. Lam, S. Xu, Stability analysis of distributed delay neural networks based on relaxed Lyapunov-Krasovskii functionals, IEEE Trans. Neural Netw. Learn. Syst. 26(7)(2015) 1480–1492.

[14] C. Briat, Convergence and equivalence results for the Jensens inequality-Application to time-delay and sampled-data systems, IEEE Trans. Autom. Control 56(7)(2011) 1660–1665.

[15] A. Seuret, F. Gouaisbaut, Wirtinger-based integral inequality: Application to time-delay systems, Automatica 49(9)(2013) 2860–2866.

[16] A. Seuret, F. Gouaisbaut, E. Fridman, Stability of systems with fast-varying delay using improved Wirtingers inequality, 52nd IEEE CDC. (2013) 946–951.

[17] O.M. Kwon, M.J. Park, J.H. Park, S.M. Lee, E.J. Cha, On less conservative stability criteria for neural networks with time-varying delays utilizing Wirtinger-based integral inequality, Math. Probl. Eng. 5(2014) 1–13.

[18] H.B. Zeng, Y. He, M. Wu, S.P. Xiao, Stability analysis of generalized neural networks with time-varying delays via a new integral inequality, Neurocomputing 161(2015) 148–54.

[19] K. Gu, V.L. Kharitonov, J. Chen, Stability of time-delay systems, Birkhuser, Berlin, 2003.

[20] N. Zhao, C. Lin, B. Chen, Q.G. Wang, A new double integral inequality and application to stability test for time-delay systems, Appl. Math. Lett. 65(2017) 26–31.

[21] H. Shao, H. Li, L. Shao, Improved delay-dependent stability result for neural networks with time-varying delays, ISA transactions 80(2018) 35–42.

[22] G. Zhang, T. Wang, T. Li, S. Fei, Multiple integral Lyapunov approach to mixed-delay-dependent stability of neutral neural networks, Neurocomputing 275(2018) 1782–1792.