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A MODIFIED PARALLEL HYBRID SUBGRADIENT EXTRAGRADIENT METHOD OF VARIATIONAL INEQUALITY PROBLEMS

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Abstract In this paper, we modify the parallel hybrid subgradient extragradient methods for finding common solutions of variational inequality problems in Hilbert spaces for a class of Lipzchitz continuous that the Lipschitz constant is unknown. We then prove the strong convergence theorems under some suitable conditions. Finally, we give an example in Euclidean spaces by applying our main theorem to solve signal recovery.

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1. Introduction

The variational inequality problem (VIP) can mathematically be formulated as the problem of finding a point $x^* \in C$ such that

$$\langle A(x^*), x - x^* \rangle \geq 0, \forall x \in C \tag{1.1}$$

where H is a real Hilbert space with the inner product $\langle ., . \rangle$ and the induced norm $\|.\|$, C is a nonempty closed convex subset of H and $A : H \to H$ is a nonlinear operator. The set of solutions of VIP (1.1) is denoted by $VI(A, C)$. The VIP was introduced and studied by Hartman and Stampacchia in 1996 [1] . Using the projection technigue, it is well know that VI(C,A) is equivalent to the following fixed point equation (see [2]), $x = P_C(x - \lambda Ax), \lambda > 0$ and $r_\lambda(x) := x - P_C(x - \lambda Ax) = 0$. Many algorithms which based on projections over closed convex sets have been proposed for solving VIP (1.1). In 1976, Korelevich [3] proposed the projection method which is called the extragradient method, for solving saddle point problems. The VIP was solved for Lipschitz continuous

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and monotone (even, pseudomonotone) mappings A. The extragradient method is defined as follows:

$$\begin{cases} y_n = P_C(x_n - \lambda A(x_n)), \\ x_{n+1} = P_C(x_n - \lambda A(y_n)), \end{cases} \tag{1.2}$$

where P_C is the metric projection onto C and λ is a suitable parameter. In the case of C has a simple structure, then the projections onto it can be discovered easily, the extragradient method is computable and very useful. However, we have to solve two distance optimization problems in the extragradient method to obtain the next approximation x_{n+1} over each iteration that is we have to use the projection onto C into two times. Later on, Censor et al. [4] proposed the following algorithm, which is called the subgradient exttagradient method, for VIP (1.1) in Hilbert spaces,

$$\begin{cases} y_n = P_C(x_n - \lambda A(x_n)), \\ x_{n+1} = P_{T_n}(x_n - \lambda A(y_n)), \end{cases} \tag{1.3}$$

where T_n is a half-space whose bounding hyperplane is supported on C at y_n, i.e.,

$$T_n = \{v \in H : \langle (x_n - \lambda A(x_n)) - y_n, v - y_n\rangle \leq 0\}.$$

Censor et al. [4] proved that the sequence $\{x_n\}$ generated by (1.3) converge weakly to a solution of the VIP. Moreover, in order to obtain the strong convergence of iterative sequences, Censor et al. [5] proposed the following algorithm which combines the subgradient extragradient method and hybrid (outer approximation) method,

$$\begin{cases} y_n = P_C(x_n - \lambda A(x_n)), \\ z_n = \alpha_n x_n + (1-\alpha_n) P_{T_n} x_n, \\ C_n = \{z \in H : \|z_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in H : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0. \end{cases} \tag{1.4}$$

In 2015, Gibali [6] introduced a self-adaptive subgradient extragradient method by adopting Armijo-like searches and obtained convergence result for VI(A,C) in R^n under the assumption of pseudo-monotonicity and continuity of A (A is pseudo-monotone if for all $x, y \in H$, we have $\langle Ay, x-y\rangle \geq 0 \Rightarrow \langle Ax, x-y\rangle \geq 0$). Gibali [6] remarked that the Armijo-like searches can be viewed as a local approximation of the Lipschitz constant of A. In recent years, the extragradient method has been studied and developed a lot of attention, see, for example [7–9] and the references therein.

Very recently, Shehu and Iyiola [10] proposed the following modified viscosity approximation with adoption of Armijo-like step size rule which is called viscosity type subgradient extragradient like mothods method for a Lipschitz continuous monotone mapping that the Lipschitz constant is unknown in an infinite dimensional Hilbert space.

$$\begin{cases} y_n = P_C(x_n - \lambda_n A x_n), \lambda_n = \rho^{l_n} \\ (l_n \text{ is the smallest nonnegative integer } l \\ \quad \text{such that } \lambda_n \|Ax_n - Ay_n\| \leq \mu \|r_{\rho^l}(x_n)\|) \\ z_n = P_{T_n}(x_n - \lambda_n A y_n), \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) z_n, n \geq 1 \end{cases} \tag{1.5}$$

where $T_n := \{z \in H : \langle x_n - \lambda_n A x_n - y_n, z - y_n\rangle \leq 0\}, \rho, \mu \in (0,1)$ and $\{\alpha_n\} \subseteq (0,1)$.

Our interest in this paper is to study the problem of finding common solution to variational inequality problems (CSVIP). The CSVIP is stated as follows: Let K_i , i = 1

, . . ., N be a finite family of nonempty closed and convex subsets of H such that K := $\bigcap_{i=1}^{N} K_i \neq \emptyset$. Let A_i : H → H, i = 1,...,N be mappings. The CSVIP is to find $x^* \in$ K such that

$$\langle A_i(x^*), x - x^* \rangle \geq 0, \forall x \in K_i,\ i = 1, ..., N. \tag{1.6}$$

If $N = 1$, CSVIP (1.6) becomes VIP(1.1). The CSVIP is a generalization of many mathematical models, in the sense that, it includes many special cases [11] such as: convex feasibility problems, common linear programing problem, common minimizer problem, common saddle - point problems, Hierarchical variational inequality problems. These problems have practical applicable abilities in signal processing, network resource allocation, image processing and many other fields, for instance, see in [12], [13], [14], [15]. As a result various techniques and iterative schemes have been developed over the year to solve the CSVIP, see [16], [17], [18], [19], [20] and the references therein.

In 2012, Censor et al. [11] proposed an algorithm by solving distance optimization problem of the intersection closed convex subset $C_n^1, C_n^2, ..., C_n^N$ and W_n for finding a particular solution of the CSVIP when A_i,= 1,...,N are multi-valued mapping from H to 2^H. Choose $x_1 \in H$ and compute

$$\begin{cases} y_n^i = P_{K_i}(x_n - \lambda_n^i A_i(x_n)), \\ z_n^i = P_{K_i}(x_n - \lambda_n^i A_i(y_n^i)), \\ C_n^i = \{z \in H : \langle x_n - z_n^i, z - x_n - \gamma_n^i(z_n^i - x_n)\rangle \leq 0\}, \\ C_n = \bigcap_{i=1}^{N} C_n^i, \\ W_n = \{z \in H : \langle x_1 - x_n, z - x_n \rangle \leq 0\}, \\ x_{n+1} = P_{C_n \cap W_n} x_1. \end{cases} \tag{1.7}$$

Very recently, Anh and Hieu [21], [22] proposed a parallel monotone hybrid algorithm for finding a common fixed point of a finite family of quasi ϕ - nonexpansive mappings $\{S_i\}_{i=1}^N$ in Banach spaces. This algorithm is respected to Hilbert spaces as follows:

$$\begin{cases} x_0 \in C, \\ y_n^i = \alpha_n x_n + (1 - \alpha_n) S_i x_n, i = 1, ..., N, \\ i_n = \operatorname{argmax}\{\|y_n^i - x_n\| : i = 1, ..., N\}, \bar{y}_n := y_n^{i_n}, \\ C_{n+1} = \{v \in C_n : \|v - \bar{y}_n\| \leq \|v - x_n\|\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \end{cases} \tag{1.8}$$

where $0 < \alpha_n < 1$, $\limsup_{n\to\infty} \alpha_n < 1$. According to this algorithm, the intermediate approximations y_n^i can be found simultaneously. Then, among all y_n^i the furthest element from x_n, denoted by $\bar{y}_n$, is chosen. After that, based on this element we construct the closed convex set C_{n+1}. Finally, the next approximation x_{n+1} is defined as the projection of x_0 onto C_{n+1}.

Inspired by the previous results, we introduce the new algorithm by modifying the hybrid subgradient extragradient method combining subgradient extra-gradient method with adoption of Armijo-like step size rule and projection onto the set of intersection sets of half-spaces. We prove strong convergence theorem under some suitable conditions in Hilbert spaces to find common solution of variational inequality problems (CSVIP).

Moreover, we apply our main result to reduce noise in signal processing problems.

2. Preliminaries and lemmas

In order to prove our main result, we recall some basic definitions and lemma needed for further investigation. In a Hilbert space H, let C be a nonempty closed and convex subset of H. For every point $x \in H$, there exists a unique nearest point of C, denoted by $P_C x$, such that $\|x - P_C x\| \leq \|x - y\|$ for all $y \in C$. Such a P_C is called the metric projection from H onto C.

Definition 2.1. [23] A mapping $A : H \to H$ is said to be

(i) monotone if $\langle A(x) - A(y), x - y\rangle \geq 0$ for all $x, y \in H$;

(ii) pseudomonotone if the relation $\langle A(x) - A(y), x - y\rangle \geq 0$ implies that $\langle A(x) - A(y), x - y\rangle \leq 0$ for all $x, y \in H$;

(iii) α - inverse strongly monotone if there exists a positive constant α such that

$$\langle A(x) - A(y), x - y\rangle \geq \alpha\|A(x) - A(y)\|^2, \forall x, y \in H; \tag{2.1}$$

(iv) maximal monotone if it is monotone and its graph

$$G(A) := \{(x, A(x)) : x \in H\} \tag{2.2}$$

is not a proper subset of one of any other monotone mapping;

(v) L - Lipschitz continuous if there exists a positive constant L such that $\|A(x) - A(y)\| \leq L\|x - y\|$ for all $x, y \in H$.

Let C be a nonempty, closed and convex subset of a real Hilbert space H. It is well-known that a monotone mapping $A : H \to H$ is maximal iff, for each $(x, y) \in H \times H$ such that $\langle x - u, y - u\rangle \geq 0$ for all $(u, v) \in G(A)$, it follows that $y = A(x)$. We have the following result concerning with the convexity and closedness of the solution set $VI(A, C)$.

Lemma 2.2. [24] *Let C be a nonempty, closed convex subset of a Hilbert space H and A be a monotone, hemicontinuous mapping of C into H. Then*

$$VI(A, C) = \{u \in C : \langle v - u, A(v)\rangle \geq 0, \forall v \in C\}. \tag{2.3}$$

For every $x \in H$, the projection $P_C x$ of x onto C defined by $\|x - P_C x\| \leq \|x - y\|$ for all $x \in C$. Since C is a nonempty closed and convex subset of H, $P_C x$ exists and is unique. The projection $P_C : H \to C$ has the following characterization:

Lemma 2.3. [23] *Let $P_C : H \to C$ be the metric projection from H onto the nonempty closed convex subset C of H. Then*

(i) *P_C is 1 - inverse strongly monotone, i.e., for all $x, y \in H$,*

$$\langle P_C x - P_C y, x - y\rangle \geq \|P_C x - P_C y\|^2. \tag{2.4}$$

(ii) *For all $y \in H, x \in C$,*

$$\|x - P_C y\|^2 + \|P_C y - y\|^2 \leq \|x - y\|^2. \tag{2.5}$$

(iii) *$z = P_C x$ if and only if*

$$\langle x - z, z - y\rangle \geq 0, \forall y \in C. \tag{2.6}$$

The normal cone N_C to a set C at a point $x \in C$ defined by

$$N_C(x) = \{x^* \in H : \langle x - y, x^*\rangle \geq 0, \forall y \in C\}.$$

We have the following result.

Lemma 2.4. [25] *Let C be a nonempty closed convex subset of a Hilbert space H and let A be a monotone and hemi-continuous mapping of C into H with $D(A) = C$. Let Q be a mapping defined by:*

$$\mathcal{Q}(x) = \begin{cases} A(x) + N_C(x) & if x \in C, \\ \emptyset & if x \notin C. \end{cases} \tag{2.7}$$

Then Q is a maximal monotone and $Q^{-1}0 = VI(A, C)$.

Lemma 2.5. [26] *(Martinez-Yanes and Xu 2006) Let C be a nonempty closed and convex subset of a real Hilbert space H_1. For each $x, y \in H_1$ and $a \in \mathbb{B}$, the set*

$$D = \{v \in C : \|y - v\|^2 \leq \|x - v\|^2 + \langle z, v\rangle + a\},$$

is closed and convex.

Lemma 2.6. [27] *There exists a nonnegative integer l_n satisfying (1.5).*

3. Main results

In this section, we introduce new parallel hybrid subgradient extragradient algorithms and prove the convergence theorems of iteration sequences generated by the algorithms. Let $A_i : H \to H$ be a family mappings for all $i = 1, \ldots, N$. We assume $F := \cap_{i=1}^{N} VI(A_i, K_i) \neq \phi$. We have the following parallel algorithm.

Algorithm 3.1. *(Modified parallel hybrid subgradient extragradient method)*

Initialization: *Choose $x_0 \in H$ and take $\rho > 0, \mu \in (0, 1)$. Set $n := 0$*

Step 1. *Find N projections y_n^i on K_i in parallel*

$$y_n^i = P_{K_i}(x_n - \lambda_n^i A_i(x_n)), i = 1, \ldots, N.$$

where $\lambda_n^i = \rho^{l_n^i}$ and l_n^i is the smallest nonegative integer l^i such that

$$\rho^{l_n^i}\|A_i x_n - A_i y_n^i\| \leq \mu\|r_\rho l_n^i(x_n)\|. \tag{3.1}$$

Step 2. *Find N projections y_n^i on half-space T_n^i in parallel*

$$z_n^i = P_{T_n^i}(x_n - \lambda_n^i A_i(y_n^i)), i = 1, \ldots, N,$$

where $T_n^i = \{v \in H : \langle (x_n - \lambda_n^i A_i(x_n)) - y_n^i, v - y_n^i\rangle \leq 0\}$.

Step 3. *Construct half-spaces $C_n^i, i = 1, \ldots, N$,*

$$C_n^i = \{v \in H : \|z_n^i - v\| \leq \|x_n - v\|\}.$$

Step 4. *Construct two half-spaces C_n and Q_n,*

$$\begin{aligned} C_n &= \bigcap_{i=1}^{N} C_n^i, \\ Q_n &= \{v \in H : \langle v - x_n, x_n - x_0\rangle \geq 0\}. \end{aligned}$$

Step 5. *The next approximation x_{n+1} is defined as the projection of x_0 onto the intersection $C_n \cap Q_n$, i.e.,*

$$x_{n+1} = P_{C_n \cap Q_n}(x_0).$$

Step 6. *Set $n := n + 1$ and back to **Step 1.***

Lemma 3.2. *Suppose that $x^* \in F$ and the sequences $\{y_n^i\}, \{z_n^i\}$ generated by **Step 1.** and **Step 2.** of **Algorithm 3.1**. Then*

$$\|z_n^i - x^*\|^2 \leq \|x_n - x^*\|^2 - c(\|y_n^i - x_n\|^2 + \|z_n^i - y_n^i\|^2), \tag{3.2}$$

where $c = 1 - \mu > 0$.

Proof. Since A_i is monotone on K_i and $y_n^i \in K_i$, we obtain

$$\langle A_i(y_n^i) - A_i(x^*), y_n^i - x^* \rangle \geq 0, \forall x^* \in F.$$

This together with $x^* \in VI(A_i, K_i)$ implies that

$$\begin{aligned} \langle A_i(y_n^i), y_n^i - x^* \rangle &\geq 0 \\ \langle A_i(y_n^i), y_n^i - z_n^i \rangle + \langle A_i(y_n^i), z_n^i - x^* \rangle &\geq 0 \end{aligned} \tag{3.3}$$

So,

$$\langle A_i(y_n^i), z_n^i - x^* \rangle \geq \langle A_i(y_n^i), z_n^i - y_n^i \rangle. \tag{3.4}$$

From the characterization of the metric projection onto T_n^i, we have

$$\langle z_n^i - y_n^i, (x_n - \lambda_n^i A_i(x_n)) - y_n^i \rangle \leq 0.$$

Thus

$$\begin{aligned} & \langle z_n^i - y_n^i, (x_n - \lambda_n^i A_i(x_n)) - y_n^i \rangle \\ = & \langle z_n^i - y_n^i, (x_n - \lambda_n^i A_i(x_n)) - y_n^i \rangle + \lambda_n^i \langle z_n^i - y_n^i, A_i(x_n) - A_i(y_n) \rangle \\ \leq & \lambda_n^i \langle z_n^i - y_n^i, A_i(x_n) - A_i(y_n) \rangle. \end{aligned} \tag{3.5}$$

Let $t_n^i = x_n - \lambda_n^i A_i(y_n^i)$ and write again $z_n^i = P_{T_n^i}(t_n^i)$. From relations (2.5) and (3.4), we got

$$\begin{aligned} & \|z_n^i - x^*\|^2 \\ \leq & \|t_n^i - x^*\|^2 - \|P_{T_n^i}(t_n^i) - t_n^i\|^2 \\ = & \|x_n - \lambda_n^i A_i(y_n^i) - x^*\|^2 - \|z_n^i - (x_n - \lambda_n^i A_i(y_n^i))\|^2 \\ = & \|x_n - x^*\|^2 - \|z_n^i - x_n\|^2 + 2\lambda_n^i \langle x^* - z_n^i, A_i(y_n^i) \rangle \\ \leq & \|x_n - x^*\|^2 - \|z_n^i - x_n\|^2 + 2\lambda_n^i \langle y_n^i - z_n^i, A_i(y_n^i) \rangle. \end{aligned} \tag{3.6}$$

By the same proof of Lemma 2.6 [27] , we know that there exists a nonnegative integer l_n^i satisfying (1.5) for all $i = 1, ..., N$ and from (3.5) that

$$\begin{aligned} & \|z_n^i - x_n\|^2 - 2\lambda_n^i \langle y_n^i - z_n^i, A_i(y_n^i) \rangle \\ = & \|z_n^i - y_n^i + y_n^i - x_n\|^2 - 2\lambda_n^i \langle y_n^i - z_n^i, A_i(y_n^i) \rangle \\ = & \|z_n^i - y_n^i\|^2 + \|y_n^i - x_n\|^2 - 2\lambda_n^i \langle z_n^i - y_n^i, A_i(x_n) - A_i(y_n^i) \rangle \\ \geq & \|z_n^i - y_n^i\|^2 + \|y_n^i - x_n\|^2 - 2\lambda_n^i \|z_n^i - y_n^i\| \|A_i(x_n) - A_i(y_n^i)\| \\ \geq & \|z_n^i - y_n^i\|^2 + \|y_n^i - x_n\|^2 - 2\mu \|z_n^i - y_n^i\| \|x_n - y_n^i\| \\ \geq & \|z_n^i - y_n^i\|^2 + \|y_n^i - x_n\|^2 - \mu(\|z_n^i - y_n^i\|^2 + \|x_n - y_n^i\|^2) \\ \geq & (1 - \mu)(\|z_n^i - y_n^i\|^2 + \|x_n - y_n^i\|^2) \\ \geq & c(\|z_n^i - y_n^i\|^2 + \|x_n - y_n^i\|^2). \end{aligned} \tag{3.7}$$

From (3.6) and (3.7), we obtain inequality (3.2). The proof of Lemma 3.2 is complete. ■

Lemma 3.3. *Suppose that* $\{x_n\}, \{y_n^i\}, \{z_n^i\}$ *generated by Algorithm 3.1. Then*
(i) $F \subset C_n \cap Q_n$ *and* x_{n+1} *is well-defined for all* $n \geq 0$.
(ii) *There hold the following relations for all* $i = 1, \dots, N$.

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} \|y_n^i - x_n\| = \lim_{n\to\infty} \|z_n^i - x_n\| = 0.$$

Proof. (i) Since A_i is Lipschitz continuous, A_i is continuous. Thus, Lemma 2.1 ensure that $VI(A_i, K_i)$ is closed and convex for all $i = 1, \dots, N$. Hence, F is closed and convex. From the definitions of Q_n, we see that Q_n is closed and convex for all $n \geq 0$. From Lemma 2.2.

$$C_n^i = \{v \in H : \|z_n^i - v\| \leq \|x_n - v\|\},$$

and Lemma 2.5, we see that C_n^i is closed and convex for all $i = 1, \dots, N$ and $n \geq 0$. Hence C_n is closed and convex for all $n \geq 0$. From Lemma 3.2, we have $\|z_n^i - p\| \leq \|x_n - p\|$, so $p \in C_n^i$, for all $i = 1, \dots, N$. Thus, $F \subset C_n$ for all $n \geq 0$. We next show that $F \subset Q_n, \forall n \geq 0$, by the induction. Indeed, $F \subset Q_0$ and so $F \subset C_0 \cap Q_0$. Assume that $F \subset Q_n$ for some $n \geq 0$. From $x_{n+1} = P_{C_n \cap Q_n} x_0$ and the characterization of the metric projection (2.6), we obtain

$$\langle v - x_{n+1}, x_{n+1} - x_0 \rangle \geq 0, \forall v \in Q_n.$$

Since $F \subset Q_n$,so $\langle v - x_{n+1}, x_{n+1} - x_0 \rangle \geq 0$ for all $v \in F$. This together with the definition of Q_{n+1} implies that $F \subset Q_{n+1}$. Thus, by the induction $F \subset Q_n$ for all $n \geq 0$. Thus $F \subset Q_n \cap C_n$. Since $F \neq \emptyset$,thus $P_F x_0$ and $x_{n+1} = P_{C_n \cap Q_n} x_0$ are well defined.
(ii) We have $x_n = P_{Q_n} x_0$ and $F \subset Q_n$. For each $u \in F$, we have

$$\|x_n - x_0\| \leq \|p - x_0\|, \forall n \geq 0. \tag{3.8}$$

Thus, the sequence $\{\|x_n - x_0\|\}$ and so $\{x_n\}$ are bounded. From $x_{n+1} \in Q_n$ and $x_n = P_{Q_n} x_0$, we also obtain

$$\|x_n - x_0\| \leq \|x_{n+1} - x_0\|, \forall n \geq 0.$$

This implies that the sequence $\{\|x_n - x_0\|\}$ is nondecreasing. Therefore, there exists the lim of the sequence $\{\|x_n - x_0\|\}$. Moreover, from $x_{n+1} \in Q_n$ and $x_n = P_{Q_n} x_0$, we get

$$\|x_n - x_{n+1}\|^2 \leq \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2.$$

From this inequality, letting $n \to \infty$, we find

$$\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0. \tag{3.9}$$

By the definition of C_n and $x_{n+1} \in C_n$, we have

$$\|z_n^i - x_{n+1}\| \leq \|x_n - x_{n+1}\|. \tag{3.10}$$

From (3.10), we got

$$\lim_{n\to\infty} \|z_n^i - x_{n+1}\| = 0, \forall i = 1, \dots, N. \tag{3.11}$$

From Lemma 3.2 and the triangle inequality, for each $p \in F$, one has

$$\begin{aligned} c\|y_n^i - x_n\|^2 &\leq \|x_n - p\|^2 - \|z_n^i - p\|^2 \\ &\leq (\|x_n - p\| + \|z_n^i - p\|)\|x_n - z_n^i\|. \end{aligned} \tag{3.12}$$

From (3.11), (3.12) and the boundedness of $\{x_n\}, \{z_n^i\}$, we get

$$\lim_{n\to\infty} \|y_n^i - x_n\| = 0, i = 1, \ldots, N.$$

The proof of Lemma 3.3 is complete. ∎

Theorem 3.4. *Let $K_i, i = 1, \ldots, N$ be closed and convex subsets of a real Hilbert space H such that $K = \cap_{i=1}^N K_i \neq \varnothing$. Suppose that $A_i : H \to H$ is a monotone mapping for all $i = 1, \ldots, N$. In addition, the solution set F is nonempty. Then, the sequences $\{x_n\}, \{y_n^i\}, \{z_n^i\}$ generated by Algorithm 3.1 converge strongly to $P_F x_0$.*

Proof. By Lemma 3.3, F, C_n, Q_n are nonempty closed and convex subsets. Besides, $F \subset C_n \cap Q_n$ for all $n \geq 0$. Therefore, $P_F x_0, P_{C_n \cap Q_n} x_0$ are well-defined. From Lemma 3.2, $\{x_n\}$ is bounded. Assume that p is a weak cluster point of $\{x_n\}$ and there exists a subsequence of $\{x_n\}$ converging weakly to p. Without loss of generality, we denote this subsequence again by $\{x_n\}$ and write $x_n \rightharpoonup p$. Since $\|y_n^i - x_n\| \to 0, y_n^i \rightharpoonup p$. Now we prove that $p \in \cap_{i=1}^N VI(A_i, K_i)$. Indeed, Lemma 2.3 ensures that the mapping

$$\mathcal{Q}_i(x) = \begin{cases} A_i x + N_{K_i}(x) & if x \in K_i, \\ \emptyset & if x \notin K_i. \end{cases}$$

is maximal monotone, where $N_{K_i}(x)$ is the normal cone to K_i at $x \in K_i$. For all (x, y) in the graph of $\mathcal{Q}_i$, *i.e.*, $(x, y) \in \mathcal{G}(\mathcal{Q}_i)$, we have $y - A_i(x) \in N_{K_i}(x)$. By the definition of $N_{K_i}(x)$, we find that

$$\langle x - z, y - A_i(x)\rangle \geq 0$$

for all $z \in K_i$. Since $y_n^i \in K_i$,

$$\langle x - y_n^i, y - A_i(x)\rangle \geq 0,$$

Therefore,

$$\langle x - y_n^i, y\rangle \geq \langle x - y_n^i, A_i(x)\rangle. \tag{3.13}$$

Taking into account $y_n^i = P_{K_i}(x_n - \lambda_n^i A_i x_n)$ and Lemma 2.2(iii), we got

$$\langle x - y_n^i, y_n^i - x_n + \lambda_n^i A_i(x_n)\rangle \geq 0,$$

or

$$\langle x - y_n^i, A_i(x_n)\rangle \geq \langle x - y_n^i, \frac{x_n - y_n^i}{\lambda_n^i}\rangle. \tag{3.14}$$

Therefore, from (3.13), (3.14) and the monotoniccity of A_i, we find that

$$\begin{aligned} \langle x - y_n^i, y\rangle &\geq \langle x - y_n^i, A_i(x)\rangle \\ &= \langle x - y_n^i, A_i(x) - A_i(y_n^i)\rangle + \langle x - y_n^i, A_i(y_n^i) - A_i(x_n)\rangle + \langle x - y_n^i, A_i(x_n)\rangle \\ &\geq \langle x - y_n^i, A_i(y_n^i) - A_i(x_n)\rangle + \langle x - y_n^i, \frac{x - y_n^i}{\lambda_n^i}\rangle. \end{aligned} \tag{3.15}$$

Since $\|x_n - y_n^i\| \to 0$ and A_i is L-Lipcshitz continuous,

$$\lim_{n\to\infty} \|A_i(y_n^i) - A_i(x_n)\| = 0. \tag{3.16}$$

Passing the lim in (3.15) as $n \to \infty$ and using (3.16), $y_n^i \rightharpoonup p$, we obtain $\langle x - p, y\rangle \geq 0$ for all $(x, y) \in \mathcal{G}(\mathcal{Q}_i)$. This together with the maximal monotonicity of $\mathcal{Q}_i$ implies that

$p \in Q_i^{-1}0 = VI(A_i, K_i)$ for all $1 \leq i \leq N$. Hence, $p \in F = \cap_{i=1}^N VI(A_i, K_i)$.
Finally, we show that $x_n \to p = x^\dagger := P_F x_0$. From (16) and $x^\dagger \in F$, we have

$$\|x_n - x_0\| \leq \|x^\dagger - x_0\|, \forall n \geq 0.$$

This relation together with the lower weak semicontinuty of the norm implies that

$$\|p - x_0\| \leq \liminf_{n\to\infty} \|x_n - x_0\| \leq \limsup_{n\to\infty} \|x_n - x_0\| \leq \|x^\dagger - x_0\|.$$

By the definition of $x^\dagger, p = x^\dagger$ and $\lim_{n\to\infty} \|x_n - x_0\| \leq \|x^\dagger - x_0\|$. Thus from $x_n - x_0 \rightharpoonup x^\dagger - x_0$ and the Kadec-Klee property of H, we obtain $x_n - x_0 \to x^\dagger - x_0$, and so $x_n \to x^\dagger$.
Lemma 3.2 ensures that the sequences $\{y_n^i\}, \{z_n^i\}$ also converge strongly to $P_F x_0$. The proof of Theorem 3.4 is completed. ∎

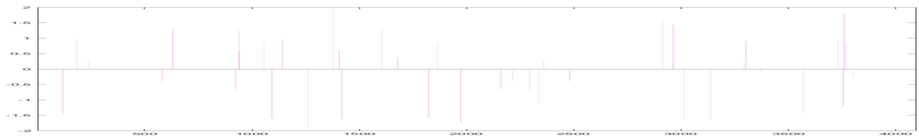
4. Application to Signal Recovering Problem.

In signal processing, compressed sensing can be modeled as the following under determinated linear equation system $b = Bx + \nu$ where x is a original signal with N components to be recovered $(x \in \mathbb{R}^N), \nu, b$ are noise and the observed signal with noisy for M components respectively $(\nu, b \in \mathbb{R}^M)$ and $B : \mathbb{R}^N \to \mathbb{R}^M (M < N)$ is a bounded linear observation operator. Finding the solutions of $b = Bx + \nu$ can be seen as solving the LASSO problem

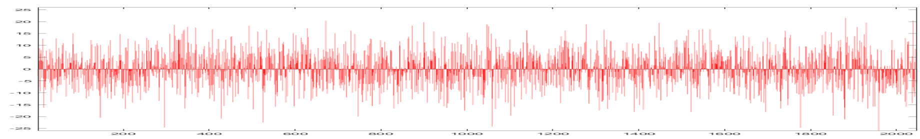
$$\min_{x\in\mathbb{R}^N} \frac{1}{2}\|b - Bx\|_2^2 \text{ subject to } \|x\|_1 \leq t, \tag{4.1}$$

where $t > 0$ is a given constant. We can apply the Algorithm 3.1 to solve the problem (4.1) by setting $A_i x = B^T(Bx - b)$ for all $i = 1, 2, ..., N$.

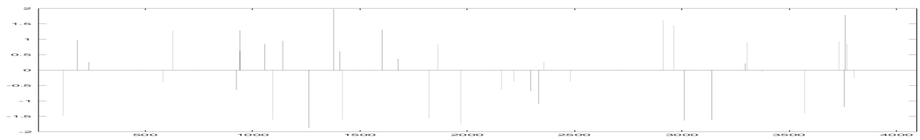
In this experiment, the original signal x with $N = 4096$ is generated by the uniform distribution in the interval $[-2, 2]$ with $m = 40$ nonzero element. The matrix B is generated by the normal distribution with mean zero and variance one. The observation b with $M = 2048$ is generated by white Gaussian noise with signal-to-noise ratio $SNR = 40$. The process is started with $t = m$ and signal initial data x_0 with $N = 4096$ are picked randomly.



Original signal



Degraded Signal



Initial Signal Data

Fig. 1 The original signal (x), Degraded Signal (b) and Proposed (x_0)

The parameters $\alpha_n, \beta_n, \gamma_n$ and μ on an implemented algorithm in solving the image deblurring is set as equation (3.1). The Cauchy error and the signal error are measured by using second norm $\|x_n - x_{n-1}\|_2$ and $\|x_n - x\|_2$ respectively. The performance of the proposed method at n^{th} iteration is measured quantitatively by the means of the signal-to-ratio (SNR), which is defined by

$$SNR(x_n) = 20log_{10}\Big(\frac{\|x\|_2}{\|x_n - x\|_2}\Big),$$

where x_n is the recovered signal at n^{th} iteration by using the proposed method. The Cauchy error, signal error and SNR quality of the proposed method for recovering the degraded signal are shown on figure 2. The Cauchy error shows that the proposed method can be applied to signal recovering problem. And, the signal error confirms the convergence of the implemented algorithm.

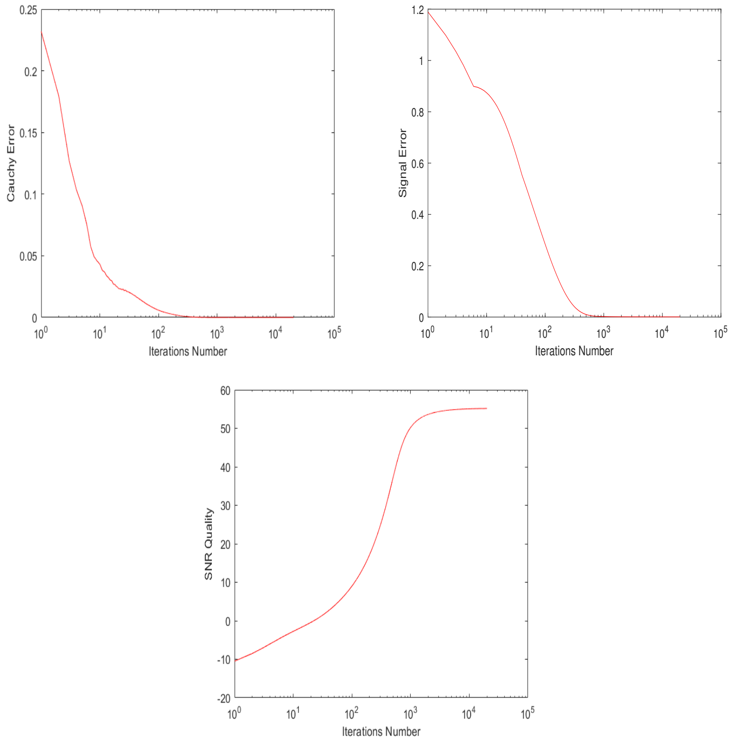
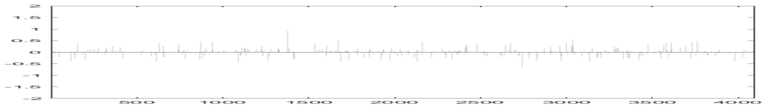
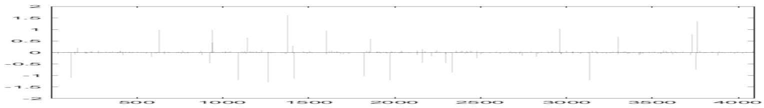


Fig. 2 Cauchy Error, Signal Error and SNR Quality of the proposed methods in recovering the obsered signal.

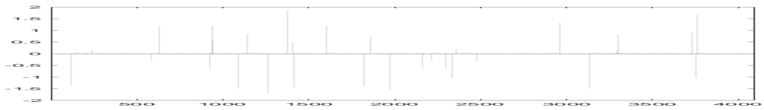
It is clearly seen that the solution of the signal recovering problem solved by the proposed algorithm get the quality improvements of observed signal. The improvement of the recovering signal based on SNR quality are also show on figure 3.



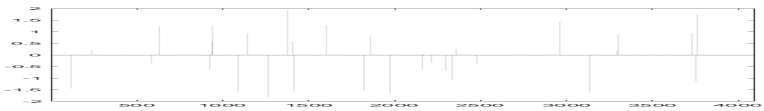
Recovering Signal with SNR = 1 (29 Iteration)



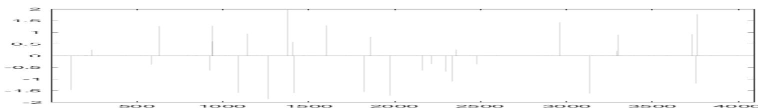
Recovering Signal with SNR = 10 (112 Iteration)



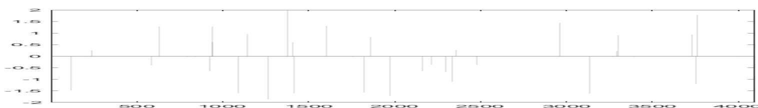
Recovering Signal with SNR = 20 (238 Iteration)



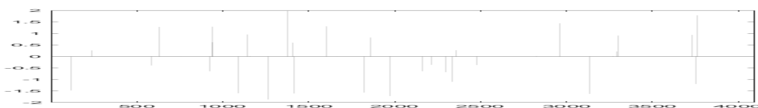
Recovering Signal with SNR = 30 (383 Iteration)



Recovering Signal with SNR = 40 (560 Iteration)



Recovering Signal with SNR = 50 (979 Iteration)



Recovering Signal with SNR = 55 (7884 Iteration)
Fig. 3 Recovering Signal based on SNR quality.

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