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MODIFIED ALMOST TYPE $\mathcal{Z}$-CONTRACTION

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Abstract In this paper, we introduce the notion of Almost type $\mathcal{Z}$-contraction and a new fixed point theorem in frame of metric spaces. We prove existence of fixed points for cyclic mappings. Also, we obtain fixed point results for weak contraction type mappings.

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1. Introduction

Banach contraction principle is widely and extensively applied in defferent branches of Mathematics and is regarded as one of cornerstones in the study of metric fixed point theory. A lot of authors studied generalizations of this principle.

Theorem 1.1. [1] *Let (X, d) be a complete metric space and Γ be a self-mapping on the set X such that $\exists \rho \in [0, 1)$,*

$$d(\Gamma\phi, \Gamma\varphi) \leq \rho d(\phi, \varphi), \quad \forall\, \phi, \varphi \in X. \tag{1.1}$$

Then, Γ has a unique fixed point in X.

In addition, Berinde [3] introduce almost contractions which exhibits new features with respect to the ones of the particular results incorporated as follows:

Theorem 1.2. [3] *Let (X, d) be a complete metric space and a self-mapping Γ on the set X be an almost contraction, that is, a mapping for which there exist $\delta \in [0, 1)$ and $\exists L \geq 0$ such that*

$$d(\Gamma\phi, \Gamma\varphi) \leq \delta d(\phi, \varphi) + L d(\varphi, \Gamma\phi), \quad \forall\, \phi, \varphi \in X. \tag{1.2}$$

Then,

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(i) $\mathrm{Fix}(\Gamma) \neq \emptyset$, *where* $\mathrm{Fix}(\Gamma) = \{\phi \in X : \Gamma\phi = \phi\}$;
(ii) *For any* $\phi_0 \in X$, *the Picard iteration* $\{\phi_n\}$ *given by* $\phi_{n+1} = \Gamma\phi_n$ *for each* $n \geq 0$ *converges to some* $\phi^* \in \mathrm{Fix}(\Gamma)$;
(iii) *The following estimate holds*

$$d(\phi_{n+i-1}, \phi^*) \leq \frac{\delta^i}{1-\delta} d(\phi_n, \phi_{n-1}), \quad \forall\, n \geq 0,\, i \geq 1.$$

Subsequently, Babu et al. [5] defined the class of mappings satisfying condition (B) as follows:

Definition 1.3. [5] Let (X, d) be a metric space and a self-mapping Γ on X is said to satisfy condition (B) if there exist a constant $\delta \in (0,1)$ and $\exists L \geq 0$ such that

$$d(\Gamma\phi, \Gamma\varphi) \leq \delta d(\phi, \varphi) + L\mathcal{Q}(\phi, \varphi), \quad \forall\, \phi, \varphi \in X, \tag{1.3}$$

where $\mathcal{Q}(\phi, \varphi) = \min\{d(\phi, \Gamma\phi), d(\varphi, \Gamma\varphi), d(\phi, \Gamma\varphi), d(\varphi, \Gamma\phi)\}$.

They proved a fixed point theorem for such mappings in complete metric spaces. They also discussed quasi-contraction, almost contraction and the class of mappings that satisfy condition (B) in detail.

Khojasteh et al. [6] originated the notion of $\mathcal{Z}$-contractions using a specific family of functions called simulation functions. Subsequently, many researchers generalized this idea in many ways (see [7–21] and proved many interesting results in the arena of fixed point theory.

Definition 1.4. [6] A mapping $\zeta : [0, \infty)^2 \to \mathbb{R}$ is called a simulation function if it satisfies the following conditions:

(ζ_1) $\zeta(0,0) = 0$;
(ζ_2) $\zeta(t, s) < s - t$ for all $t, s > 0$;
(ζ_3) if $\{t_n\}$, $\{s_n\}$ are sequences in $(0, \infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$, then $\limsup_{n\to\infty} \zeta(t_n, s_n) < 0$.

Denoted by $\mathcal{Z}$ is the set of all simulation functions.

Example 1.5. [6] The following are some examples of simulation functions.

(i) $\zeta(t, s) = \alpha s - t$ for all $t, s \in [0, \infty)$, where $\alpha \in [0, 1)$;
(ii) $\zeta(t, s) = \frac{s}{1+s} - t$ for all $t, s \in [0, \infty)$;
(iii) $\zeta(t, s) = sf(s) - t$ for all $t, s \in [0, \infty)$, where $f : [0, \infty) \to [0, 1)$ such that $\lim_{t\to\kappa} f(t) < 1$ for all $\kappa > 0$.

Definition 1.6. [6] Let (X, d) be a metric space and $\zeta \in \mathcal{Z}$. A mapping $\Gamma : X \to X$ is called a $\mathcal{Z}$-contraction with respect to ζ if

$$\zeta(d(\Gamma\phi, \Gamma\varphi), d(\phi, \varphi)) \geq 0$$

holds for all $\phi, \varphi \in X$.

Motivated and inspired by Definition 1.6, Definition 1.3 and Theorem 1.2, we define an Almost type $\mathcal{Z}$-contraction mappings in metric spaces as follows:

Definition 1.7. Let (X, d) be a metric space and $\zeta \in \mathcal{Z}$. We say that $\Gamma : X \to X$ is a modified almost type $\mathcal{Z}$-contraction if there is a constant $L \geq 0$ such that

$$\zeta(d(\Gamma\phi, \Gamma\varphi), \mathcal{P}(\phi, \varphi) + L\mathcal{Q}(\phi, \varphi)) \geq 0, \quad \forall\, \phi, \varphi \in X, \tag{1.4}$$

where

$$\mathcal{P}(\phi,\varphi)=\max\left\{d(\phi,\varphi),\frac{[1+d(\phi,\Gamma\phi)]d(\varphi,\Gamma\varphi)}{1+d(\phi,\varphi)}\right\}$$

and

$$\mathcal{Q}(\phi,\varphi)=\min\left\{d(\phi,\Gamma\phi),d(\varphi,\Gamma\varphi),d(\phi,\Gamma\varphi),d(\varphi,\Gamma\phi)\right\}.$$

Remark 1.8. If Γ is a modified almost type $\mathcal{Z}$-contraction with respect to $\zeta\in\mathcal{Z}$, then

$$d(\Gamma\phi,\Gamma\varphi)<\mathcal{P}(\phi,\varphi)+L\mathcal{Q}(\phi,\varphi),\quad \forall\,\phi,\varphi\in X. \tag{1.5}$$

2. Main Results

Lemma 2.1. *If Γ is a modified almost type $\mathcal{Z}$-contraction in and Γ has a fixed point, then the fixed point is unique.*

Proof. Let (X,d) be a metric space and $\Gamma:X\to X$ be a modified almost type $\mathcal{Z}$-contraction with respect to $\zeta\in\mathcal{Z}$. Suppose that there are two distinct fixed points $\phi^*,\varphi^*\in X$ of the mapping Γ. Then, $d(\phi^*,\varphi^*)>0$. Thus, it follows from equation (1.4) and (ζ_2) that

$$0\le\zeta(d(\Gamma\phi^*,\Gamma\varphi^*),\mathcal{P}(\phi^*,\varphi^*)+L\mathcal{Q}(\phi^*,\varphi^*)), \tag{2.1}$$

where

$$\mathcal{P}(\phi^*,\varphi^*)=\max\{d(\phi^*,\varphi^*),\frac{[1+d(\phi^*,\Gamma\phi^*)]d(\varphi^*,\Gamma\varphi^*)}{1+d(\phi^*,\varphi^*)}\}=d(\phi^*,\varphi^*)$$

and

$$\mathcal{Q}(\phi^*,\varphi^*)=\min\{d(\phi^*,\Gamma\phi^*),d(\varphi^*,\Gamma\varphi^*),d(\phi^*,\Gamma\varphi^*),d(\varphi^*,\Gamma\phi^*)\}=0.$$

This together with (2.1) shows that

$$\begin{aligned}0&\le\zeta(d(\Gamma\phi^*,\Gamma\varphi^*),\mathcal{P}(\phi^*,\varphi^*)+L\mathcal{Q}(\phi^*,\varphi^*)\\&=\zeta(d(\phi^*,\varphi^*),d(\phi^*,\varphi^*))\\&<d(\phi^*,\varphi^*)-d(\phi^*,\varphi^*)\\&=0\end{aligned} \tag{2.2}$$

which is a contradiction. Hence, the fixed point of Γ in X is unique. ∎

Theorem 2.2. *Let (X,d) be a complete metric space and $\Gamma:X\to X$ be a modified almost type $\mathcal{Z}$-contraction with respect to $\zeta\in\mathcal{Z}$. Let $\{\phi_n\}$ be a sequence of Picard of initial point at $\phi_0\in X$. Then*

$$\lim_{n\to\infty}d(\phi_n,\phi_{n+1})=0. \tag{2.3}$$

Proof. Let $\phi_0\in X$ and consider the Picard sequence $\{\phi_n=T^n\phi_0=T\phi_{n-1}\}$, $n\ge0$. If $\phi_{n_0}=\phi_{n_0+1}$ for some n_0, then ϕ_{n_0} is a fixed point of Γ. Therefore, for the rest of the proof, we assume that $d(\phi_n,\phi_{n+1})>0$ for all $n\ge0$. From equation (1.4), for all $n\ge1$, we obtain

$$0\le\zeta(d(\Gamma\phi_{n-1},\Gamma\phi_n),\mathcal{P}(\phi_{n-1},\phi_n)+L\mathcal{Q}(\phi_{n-1},\phi_n)), \tag{2.4}$$

where

$$\begin{aligned}\mathcal{P}(\phi_{n-1},\phi_n) &= \max\left\{d(\phi_{n-1},\phi_n), \frac{[1+d(\phi_{n-1},\Gamma\phi_{n-1})]d(\phi_n,\Gamma\phi_n)}{1+d(\phi_{n-1},\phi_n)}\right\} \\ &= \max\left\{d(\phi_{n-1},\phi_n), \frac{[1+d(\phi_{n-1},\phi_n)]d(\phi_n,\phi_{n+1})}{1+d(\phi_{n-1},\phi_n)}\right\} \\ &= \max\{d(\phi_{n-1},\phi_n), d(\phi_n,\phi_{n+1})\}\end{aligned} \tag{2.5}$$

and

$$\begin{aligned}\mathcal{Q}(\phi_{n-1},\phi_n) &= \min\{d(\phi_{n-1},\Gamma\phi_{n-1}), d(\phi_n,\Gamma\phi_n), d(\phi_{n-1},\Gamma\phi_n), d(\phi_n,\Gamma\phi_{n-1})\} \\ &= \min\{d(\phi_{n-1},\phi_n), d(\phi_{n-1},\phi_{n+1})\} \\ &= 0.\end{aligned} \tag{2.6}$$

This together with (2.4) shows that

$$\begin{aligned}0 &\le \zeta(d(\Gamma\phi_{n-1},\Gamma\phi_n), \mathcal{P}(\phi_{n-1},\phi_n) + L\mathcal{Q}(\phi_{n-1},\phi_n)) \\ &= \zeta(d(\phi_n,\phi_{n+1}), \max\{d(\phi_{n-1},\phi_n), d(\phi_n,\phi_{n+1})\}) \\ &< \max\{d(\phi_{n-1},\phi_n), d(\phi_n,\phi_{n+1})\} - d(\phi_n,\phi_{n+1}).\end{aligned} \tag{2.7}$$

By inequality (2.7) shows that

$$\mathcal{P}(\phi_{n-1},\phi_n) = d(\phi_{n-1},\phi_n), \quad \forall n \ge 1 \tag{2.8}$$

which implies that

$$d(\phi_n,\phi_{n+1}) < d(\phi_{n-1},\phi_n), \quad \forall n \ge 1. \tag{2.9}$$

Therefore, the sequence $\{d(\phi_n,\phi_{n+1})\}$ is decreasing, so there is some $\kappa \ge 0$ such that

$$\lim_{n\to\infty} d(\phi_{n-1},\phi_n) = \kappa.$$

If $\varphi > 0$ then since Γ is a modified almost type $\mathcal{Z}$-contraction with respect to $\zeta \in \mathcal{Z}$ and (ζ_3), we get

$$0 \le \limsup_{n\to\infty} \zeta(d(\phi_n,\phi_{n+1}), d(\phi_{n-1},\phi_n)) < 0$$

which is a contradiction. Hence, $\kappa = 0$, that is, equation (2.3) holds. ∎

Theorem 2.3. *Let (X,d) be a complete metric space and $\Gamma : X \to X$ be a modified almost type $\mathcal{Z}$-contraction with respect to $\zeta \in \mathcal{Z}$. Then the Picard sequence $\{\phi_n\}$ generated by Γ such that $\Gamma\phi_{n-1} = \phi_n$ for all $n \ge 1$ with initial value $x_0 \in X$ is a bounded sequence.*

Proof. Let $\phi_0 \in X$ and $\{\phi_n\}$ be the Picard sequence. Assume that $\{\phi_n\}$ is not bounded. Then there is a subsequence $\{\phi_{n_k}\}$ such that $n_1 = 1$ and for each $k \in \mathbb{N}, n_{k+1}$ is the minimum integer greater than n_k such that

$$d(\phi_{n_{k+1}},\phi_{n_k}) > 1$$

and

$$d(\phi_m,\phi_{n_k}) \le 1 \text{ for } n_k \le m \le n_{k+1} - 1.$$

Therefore, by the triangle inequality, we get

$$\begin{aligned}1 < d(\phi_{n_{k+1}},\phi_{n_k}) &\le d(\phi_{n_{k+1}},\phi_{n_{k+1}-1}) + d(\phi_{n_{k+1}-1},\phi_{n_k}) \\ &\le d(\phi_{n_{k+1}},\phi_{n_{k+1}-1}) + 1.\end{aligned}$$

Taking $k \to \infty$ and by using Theorem 2.2, we get

$$\lim_{k\to\infty} d(\phi_{n_{k+1}}, \phi_{n_k}) = 1. \tag{2.10}$$

Since Γ is a modified almost type $\mathcal{Z}$-contraction with respect to $\zeta \in \mathcal{Z}$, we have

$$d(\phi_{n_{k+1}}, \phi_{n_k}) \leq \mathcal{P}(\phi_{n_{k+1}-1}, \phi_{n_k-1})$$

which

$$\begin{aligned} 1 < d(\phi_{n_{k+1}}, \phi_{n_k}) &\leq \mathcal{P}(\phi_{n_{k+1}-1}, \phi_{n_k-1}) \\ &= \max\left\{ d(\phi_{n_{k+1}-1}, \phi_{n_k-1}) \frac{[1 + d(\phi_{n_{k+1}-1}, \phi_{n_{k+1}})] d(\phi_{n_k-1}, \phi_{n_k})}{1 + d(\phi_{n_{k+1}-1}, \phi_{n_k-1})} \right\}. \end{aligned}$$

Taking $k \to \infty$, we get

$$1 \leq \lim_{k\to\infty} \mathcal{P}(\phi_{n_{k+1}-1}, \phi_{n_k-1}) \leq 1,$$

that is,

$$\lim_{k\to\infty} \mathcal{P}(\phi_{n_{k+1}-1}, \phi_{n_k-1}) = 1. \tag{2.11}$$

Since

$$\begin{aligned} &\mathcal{Q}(\phi_{n_{k+1}-1}, \phi_{n_k-1}) \\ &= \min\{d(\phi_{n_{k+1}-1}, \phi_{n_{k+1}}), d(\phi_{n_k-1}, \phi_{n_k}), d(\phi_{n_{k+1}-1}, \phi_{n_k}), d(\phi_{n_k-1}, \phi_{n_{k+1}})\}. \end{aligned}$$

Taking $k \to \infty$ and using Theorem 2.2, we get

$$\lim_{k\to\infty} \mathcal{Q}(\phi_{n_{k+1}-1}, \phi_{n_k-1}) = 0. \tag{2.12}$$

By equation (1.4), we have

$$\begin{aligned} 0 &\leq \zeta(d(\Gamma\phi_{n_{k+1}-1}, \Gamma\phi_{n_k-1}), \mathcal{P}(\phi_{n_{k+1}-1}, \phi_{n_k-1}) + L\mathcal{Q}(\phi_{n_{k+1}-1}, \phi_{n_k-1})) \\ &< \mathcal{P}(\phi_{n_{k+1}-1}, \phi_{n_k-1}) + L\mathcal{Q}(\phi_{n_{k+1}-1}, \phi_{n_k-1}) - d(\phi_{n_{k+1}}, \phi_{n_k}) \end{aligned} \tag{2.13}$$

which implies that

$$d(\phi_{n_{k+1}}, \phi_{n_k}) < \mathcal{P}(\phi_{n_{k+1}-1}, \phi_{n_k-1}) + L\mathcal{Q}(\phi_{n_{k+1}-1}, \phi_{n_k-1}).$$

Moreover, by using (ζ_3), we get

$$\limsup_{n\to\infty} \zeta(d(\phi_{n_{k+1}}, \phi_{n_k}), \mathcal{P}(\phi_{n_{k+1}-1}, \phi_{n_k-1}) + L\mathcal{Q}(\phi_{n_{k+1}-1}, \phi_{n_k-1})) < 0 \tag{2.14}$$

which contradicts equation (2.13). This contradiction proves that $\{\phi_n\}$ is a bounded sequence. ∎

Theorem 2.4. *Let (X, d) be a complete metric space and $\Gamma : X \to X$ be a modified almost type $\mathcal{Z}$-contraction with respect to $\zeta \in \mathcal{Z}$. Then the Picard sequence $\{\phi_n\}$ is a Cauchy sequence.*

Proof. From Theorem 2.3, we claim that sequence $\{\phi_n\}$ is a Cauchy sequence. Consider the sequence $\{C_n\} \subset [0, \infty)$ given by

$$C_n = \sup\{d(\phi_i, \phi_j) : i, j \geq n\}, \quad n \in \mathbb{N}. \tag{2.15}$$

It is clear that $\{C_n\}$ is a positive decreasing sequence. So, there is some $C \geq 0$ such that $\lim_{n\to\infty} C_n = C$. If $C > 0$, then, by definition of C_n, for every $k \in \mathbb{N}$, n_k and m_k exist such that $m_k > n_k \geq k$ and

$$C_k - \frac{1}{k} < d(\phi_{m_k}, \phi_{n_k}) \leq C_k.$$

Hence,

$$\lim_{k\to\infty} d(\phi_{m_k}, \phi_{n_k}) = C. \tag{2.16}$$

Using equation (1.4) and the triangular inequality, we get

$$d(\phi_{m_k}, \phi_{n_k}) \leq d(\phi_{m_k}, \phi_{m_k-1}) + d(\phi_{m_k-1}, \phi_{n_k-1}) + d(\phi_{n_k-1}, \phi_{n_k})$$

and

$$d(\phi_{m_k-1}, \phi_{n_k-1}) \leq d(\phi_{m_k-1}, \phi_{m_k}) + d(\phi_{m_k}, \phi_{n_k}) + d(\phi_{n_k}, \phi_{n_k-1})$$

Taking $k \to \infty$, using Theorem 2.2 and equation (2.16), we get

$$\lim_{k\to\infty} d(\phi_{m_k-1}, \phi_{n_k-1}) = C. \tag{2.17}$$

Since Γ is a modified almost type $\mathcal{Z}$-contraction with respect to $\zeta \in \mathcal{Z}$, we can deduce that

$$d(\phi_{m_k}, \phi_{n_k}) = d(\Gamma\phi_{m_k-1}, \Gamma\phi_{n_k-1}) < \mathcal{P}(\phi_{m_k-1}, \phi_{n_k-1}) + L\mathcal{Q}(\phi_{m_k-1}, \phi_{n_k-1})$$

which

$$\mathcal{P}(\phi_{m_k-1}, \phi_{n_k-1}) = \max\left\{d(\phi_{m_k-1}, \phi_{n_k-1}), \frac{[1 + d(\phi_{m_k-1}, \phi_{m_k})]d(\phi_{n_k-1}, \phi_{n_k})}{1 + d(\phi_{m_k-1}, \phi_{n_k-1})}\right\}.$$

Taking $k \to \infty$ and using Theorem 2.2 and equation (2.16), we get

$$\lim_{k\to\infty} \mathcal{P}(\phi_{m_k-1}, \phi_{n_k-1}) = C. \tag{2.18}$$

Additionally, with the aid of equation (1.4), we have

$$\lim_{k\to\infty} \mathcal{Q}(\phi_{m_k-1}, \phi_{n_k-1}) = 0. \tag{2.19}$$

By (2.17),(2.18),(2.19) and (ζ_3), we get

$$0 \leq \limsup_{k\to\infty} \zeta(d(\phi_{m_k}, \phi_{n_k}), \mathcal{P}(\phi_{m_k-1}, \phi_{n_k-1}) + L\mathcal{Q}(\phi_{m_k-1}, \phi_{n_k-1})) < 0$$

which is a contradiction and so $C = 0$. That is, $\{\phi_n\}$ is a Cauchy sequence. ∎

Theorem 2.5. *Let (X, d) be a complete metric space and $\Gamma : X \to X$ be a modified almost type $\mathcal{Z}$-contraction with respect to $\zeta \in \mathcal{Z}$. Then the Picard sequence $\{\phi_n\}$ converges to fixed point.*

Proof. Since (X, d) is a complete metric space, there is a $\phi^* \in X$ such that $\lim_{n\to\infty} \phi_n = \phi^*$. We will show that the point ϕ^* is a fixed point of Γ. Suppose that $\Gamma\phi^* \neq \phi^*$. Then $d(\phi, \Gamma\phi^*) > 0$. By equation (1.4), ($\zeta_2$) and ($\zeta_3$), we get

$$\begin{aligned}
0 &\leq \limsup_{n\to\infty} \zeta(d(\Gamma\phi_n, \Gamma\phi^*), \mathcal{P}(\phi_n, \phi^*) + L\mathcal{Q}(\phi_n, \phi^*)) \\
&\leq \limsup_{n\to\infty} [\mathcal{P}(\phi_n, \phi^*) + L\mathcal{Q}(\phi_n, \phi^*) - d(\phi_{n+1}, \Gamma\phi^*)] \\
&= -d(\phi^*, \Gamma\phi^*)
\end{aligned} \tag{2.20}$$

which implies that $d(\phi^*, \Gamma\phi^*) = 0$, that is, ϕ^* is a fixed point of Γ. The uniqueness of the fixed point follows from Lemma 2.1. ∎

Example 2.6. Let $X = [0,3]$ be endowed with the usual metric. Then (X,d) is a complete metric space. Define a mapping $\Gamma : X \to X$ as $\Gamma\phi = 3-\phi$ for all $\phi \in X$. Then, Γ is not a $\mathcal{Z}$-contraction with respect to ζ where for all $t,s \in [0,\infty)$

$$\zeta(t,s) = \alpha s - t, \quad \alpha \in [0,1).$$

In fact, for all $\phi \neq \varphi$, we have

$$\begin{aligned}
\zeta(d(\Gamma\phi,\Gamma\varphi),d(\phi,\varphi)) &= \alpha\left|\phi-\varphi\right| - \left|3-\phi-(3-\varphi)\right| \\
&= \alpha\left|\phi-\varphi\right| - \left|\phi-\varphi\right| \\
&< \left|\phi-\varphi\right| - \left|\phi-\varphi\right| \\
&= 0.
\end{aligned}$$

Now, we show that Γ is a modified almost $\mathcal{Z}$-contraction with respect to ζ.

$$\begin{aligned}
\zeta(d(\Gamma\phi,\Gamma\varphi),\mathcal{P}(\phi,\varphi)+L\mathcal{Q}(\phi,\varphi)) &= \alpha[\left|\mathcal{P}(\phi,\varphi)+L\mathcal{Q}(\phi,\varphi)\right|] - \left|3-\phi-(3-\varphi)\right| \\
&= \alpha[\left|\mathcal{P}(\phi,\varphi)+L\mathcal{Q}(\phi,\varphi)\right|] - \left|\phi-\varphi\right|,
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{P}(\phi,\varphi) &= \max\left\{\left|\phi-\varphi\right|, \frac{[1+\left|\phi-(3-\phi)\right|]\left|\varphi-(3-\varphi)\right|}{1+\left|\phi-\varphi\right|}\right\} \\
&= \max\left\{\left|\phi-\varphi\right|, \frac{[1+\left|2\phi-3\right|]\left|2\varphi-3\right|}{1+\left|\phi-\varphi\right|}\right\}
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{Q}(\phi,\varphi) &= \min\left\{\left|\phi-(3-\phi)\right|, \left|\varphi-(3-\varphi)\right|, \left|\phi-(3-\varphi)\right|, \left|\varphi-(3-\phi)\right|\right\} \\
&= \min\left\{\left|2\phi-3\right|, \left|2\varphi-3\right|, \left|2\phi-3\right|, \left|\phi+\varphi-3\right|\right\} \\
&= \min\left\{\left|2\phi-3\right|, \left|2\varphi-3\right|, \left|\phi+\varphi-3\right|\right\}.
\end{aligned}$$

We deduce that

$$\begin{aligned}
&\zeta(d(\Gamma\phi,\Gamma\varphi),\mathcal{P}(\phi,\varphi)+L\mathcal{Q}(\phi,\varphi)) \\
&= \alpha[\max\left\{\left|\phi-\varphi\right|, \frac{[1+\left|2\phi-3\right|]\left|2\varphi-3\right|}{1+\left|\phi-\varphi\right|}\right\} \\
&+ L\min\left\{\left|2\phi-3\right|, \left|2\varphi-3\right|, \left|\phi+\varphi-3\right|\right\}] - \left|\phi-\varphi\right|.
\end{aligned}$$

Hence, we get two cases:

Case(i): If $\phi = \varphi$, then

$$\zeta(d(\Gamma\phi,\Gamma\varphi),\mathcal{P}(\phi,\varphi)+L\mathcal{Q}(\phi,\varphi)) = \alpha[1+\left|2\phi-3\right|]\left|2\phi-3\right| + L\left|2\phi-3\right|] \geq 0.$$

Case(ii): Without loss of generality, assume that $\phi > \varphi$. Then

$$\begin{aligned}
&\zeta(d(\Gamma\phi,\Gamma\varphi),\mathcal{P}(\phi,\varphi)+L\mathcal{Q}(\phi,\varphi)) \\
&= \alpha\frac{[1+\left|2\phi-3\right|]\left|2\varphi-3\right|}{1+\left|\phi-\varphi\right|} + \alpha L\left|2\varphi-3\right| - \left|\phi-\varphi\right|.
\end{aligned}$$

If we especially choose $\alpha = \frac{1}{2}$ and $L = 8$, then we get

$$\begin{aligned}
&\zeta(d(\Gamma\phi,\Gamma\varphi),\mathcal{P}(\phi,\varphi)+L\mathcal{Q}(\phi,\varphi)) \\
&= \frac{1}{2}\frac{[1+\left|2\phi-3\right|]\left|2\varphi-3\right|}{1+\left|\phi-\varphi\right|} + 4\left|2\varphi-3\right| - \left|\phi-\varphi\right|.
\end{aligned}$$

Thus, all of the conditions of Theorem 2.5 are satisfied. Hence, Γ has a unique fixed point $\phi^* = \frac{3}{2}$.

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