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DERIVATIVE-FREE RMIL CONJUGATE GRADIENT METHOD FOR CONVEX CONSTRAINED EQUATIONS

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Abstract An efficient method for solving large scale unconstrained optimization problems is the conjugate method. Based on the conjugate gradient algorithm proposed by Rivaie, Mamat, June and Mohd (Applied Mathematics and Computation 218.22 (2012): 11323-11332), we propose a spectral conjugate gradient algorithm for solving nonlinear equations with convex constraints which generate sufficient descent direction at each iteration. Under the Lipschitz continuity assumption, the global convergence of the algorithm is established. Furthermore, the propose algorithm is shown to be linearly convergent under some appropriate conditions. Numerical experiments are reported to show the efficiency of the algorithm.

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1. Introduction

In the last decade, several articles have been written on the subject of iterative methods. This is due to the numerous problems encountered in the fields of science and engineering resulting in the appearance of nonlinear equations in vast applications. For instance, in [1] the subproblem in generalized proximal algorithms with Bergman distances. Also, in real-world applications such as Nash equilibrium problem in economics [2] and the signal processing problem in [3], it can be seen that both problems need to be reformulated into a nonlinear system of equations. It is therefore essential to solve these problems of nonlinear equations arising in these fields by developing numerous algorithms.

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Let C be a nonempty closed and convex subset of R^n and $F : R^n \to R^n$ be a monotone mapping. That is,

$$(x-y)^T(F(x)-F(y)) \geq 0, \quad \forall x, y \in R^n. \tag{1.1}$$

The focus of this work is on the nonlinear equation:

$$F(x) = 0, \quad x \in C. \tag{1.2}$$

Several iterative algorithms have been proposed for solving the nonlinear problem (1.2). A few includes the trust-region [4], the Levenberg-Marquardt method [5], the TPRP method [6] and the Gauss-Newton methods [7, 8]. However, the methods mentioned have appear to be typically unsuitable for handling large scale nonlinear equations because at each iteration, computation and storage of matrix is required. Nevertheless, one of the preferable method for solving this problem is the conjugate gradient (CG) method- The CG-method is a popular iterative method developed with the sole aim of solving large-scale unconstrained optimization problems. For an excellent survey on the CG-methods, see [9].

Following the well known projection scheme of Solodov and Svaiter in [10], the CG-method have been extended by many aurthors to solve (1.2). One among many of such extentions is the method of Cheng et.al in [11], where they extended the PRP method [12] to solve unconstrained monotone equations. Recently, the spectral gradient projection (SP) method [13] was extended to solve monotone nonlinear convex constrained equations. Numerical experiment indicates that the proposed method is suitable for large scale problems. Thus, CG-methods for solving unconstrained optimization problems have been extended by various authors in solving convex constrained monotone nonlinear equations. For more related articles, we refer reader to [14–17, 17, 18, 25, 25, 25, 26, 26–34] and references therein).

Motivated by the results of Rivaie et al. [35], we propose a derivative-free spectral gradient-type iterative projection method for solving (1.2). The global convergence of the method is proved under some conditions. Furthermore, the linearly convergent rate of the proposed method is proved under some assumptions.

The remaining part of this paper is presented as follows: In section 2, we introduce our algorithm and the method for unconstrained optimization problems posed in [35]. We establish the global convergence of the method in section 3. We report the results of the numerical experiments conducted on benchmark test problems in section 4. Finally, we end up the paper with the conclusion in section 5.

2. Algorithm

We begin this section by presenting our proposed algorithm for solving (1.2). We assume that the readers are familiar with the conjugate gradient method. Motivated by the RMIL conjugate gradient algorithm proposed by Rivaie et al. [35] for solving large-scale unconstrained optimization problems, we propose an efficient derivative-free algorithm for solving nonlinear monotone equations with convex constraints (1.2) by using the projection technique in Solodov and Svaiter [10]. Firstly, we define the search direction as follows:

$$d_k = \begin{cases} -v_k F(x_k) + \beta_k^{ERMIL} d_{k-1}, & \text{if } k > 0, \\ -F(x_k), & \text{if } k = 0, \end{cases} \tag{2.1}$$

where

$$\beta_k^{ERMIL} = \frac{F(x_k)^T (F(x_k) - F(x_{k-1}))}{\|d_{k-1}\|^2}. \tag{2.2}$$

For convenience, we refer to (2.1) and (2.2) as MRMIL algorithm. We note that If F is the gradient of a real-valued function $f : R^n \to R$, then the sufficient descent condition

$$d_k^T F(x_k) \le -c\|F(x_k)\|^2, \tag{2.3}$$

where c is a positive constant means that d_k is a direction of sufficient descent f at x_k. We obtain v_k to satisfy (2.3). In the following, We abbreviate $F(x_k)$ as F_k.

For $k = 0$, (2.3) obviously holds. For $k \in N$, we have

$$d_k^T F_k = -\left(v_k - \frac{\|y_{k-1}\|}{\|d_{k-1}\|} \right) \|F_k\|^2.$$

To satisfy (2.3), it on;y need that

$$v_k \ge c + \frac{\|y_{k-1}\|}{\|d_{k-1}\|}.$$

Without loss of generality, in this paper, we choose v_k as

$$v_k = c + \frac{\|y_{k-1}\|}{\|d_{k-1}\|}. \tag{2.4}$$

Next, we recall the projection operator, which is defined as a mapping $\mathrm{P}_C : R^n \to C$, where C is a non empty closed convex set such that

$$P_C(x) = \arg\min\{\|x - y\| \; |y \in C\}. \tag{2.5}$$

Throughout this article, we will denote $\|\cdot\|$, to be the Euclidean norm. A well known characterization of the projection operator is its nonexpansive property. That is, for any $x, y \in R^n$,

$$\|P_C(x) - P_C(y)\| \le \|x - y\|.$$

Consequently,

$$\|P_C(x) - y\| \le \|x - y\|, \; \forall y \in C. \tag{2.6}$$

In the remainder of this paper, we always assume that F satisfies the following assumptions

Assumption 2.1. The mapping $F : R^n \to R^n$ is *Lipschiz* continuous, that is there exists a positive L such that

$$\|F(x) - F(y)\| \le L\|x - y\|, \quad \forall x, y \in R^n \tag{2.7}$$

Assumption 2.2. Let C^* be a solution set, for any solution $x^* \in C^*$, there exist a nonnegative constant γ satisfying

$$\gamma dist(x, C^*) \le \|F(x)\|^2, \quad \forall x \in N(\gamma, x^*), \tag{2.8}$$

where $dist(x, C^*)$ is the distance from x to C^* and $N(x^*, C) := \{x \in R^n | \|x - x^*\| \le \gamma\}$.

We state the steps of the algorithm as follow

Algorithm 2.3. RMIL
Input. Set an initial point $x_0 \in R^n$, the positive constants: $Tol > 0$, $\varpi \in (0,2)$, $\rho \in (0,1)$, $\kappa > 0$, $\sigma > 0$, Set $k = 0$.

Step 0. If $\|F_k\| \leq Tol$, stop. Otherwise, generate the search direction by

$$d_k = \begin{cases} -v_k F(x_k) + \beta_k^{ERMIL} d_{k-1}, & \text{if } k > 0, \\ -F(x_k), & \text{if } k = 0, \end{cases} \tag{2.9}$$

Step 1. Let $t_k = \max\{\kappa\rho^i | i = 0, 1, 2, \cdots\}$, we set $z_k = x_k + t_k d_k$, to satisfy

$$-F(z_k)^T d_k \geq \sigma t_k \|d_k\|^2. \tag{2.10}$$

Step 2. If $z_k \in C$ and $\|F(z_k)\| = 0$, stop. Otherwise, compute the next iterate by

$$x_{k+1} = P_C[x_k - \varpi \xi_k F(z_k)], \tag{2.11}$$

where

$$\xi_k = \frac{F(z_k)^T (x_k - z_k)}{\|F(z_k)\|^2}$$

Step 3. Finally we set $k = k + 1$ and return to step 1.

Lemma 2.4. *Let d_k be a search direction generated by Algorithm 2.3 then, d_k always satisfies (2.3).*

Proof. The proof follows from (2.4). ∎

3. Convergence Analysis

In order to establish the convergence of Algorithm 2.3, we need the following lemmas.

Lemma 3.1. *Let $\{d_k\}$ and $\{x_k\}$ be two sequences generated by Algorithm 2.3. Then, there exists a step size t_k satisfying the line search (2.10) for all $k \geq 0$*

Proof. For any $i \geq 0$, suppose (2.10) does not hold for the iterate k_0−th, then we have

$$-\langle F(x_{k_0} + \kappa\rho^i d_{k_0}), d_{k_0} \rangle < \sigma\kappa\rho^i \|d_{k_0}\|^2.$$

Thus, by the continuity of F and with $0 < \rho < 1$, it follows that by letting $i \to \infty$, we have

$$-F(x_{k_0})^T d_{k_0} \leq 0,$$

which contradicts (2.3).

∎

Lemma 3.2. *Suppose that Assumption 2.1 holds. Let the sequences $\{x_k\}$ and $\{z_k\}$ be generated by Algorithm 2.3, then*

$$t_k \geq \max\left\{\kappa, \frac{\rho c \|F_k\|^2}{(L + \sigma)\|d_k\|^2}\right\}. \tag{3.1}$$

Proof. From the line search (2.10), if $t_k = \kappa$, then $t_k^* = \frac{t_k}{\rho}$ does not satisfy (2.10), that is

$$-F(x_k + \frac{t_k}{\rho} d_k)^T d_k < \sigma \frac{t_k}{\rho} \cdot \|d_k\|^2.$$

It follows from (2.3) and (2.7) that

$$\begin{aligned} c\|F_k\|^2 &= -F_k^T d_k \\ &= (F(x_k + \frac{t_k}{\rho} d_k) - F_k)^T d_k - F(x_k + \frac{t_k}{\rho} d_k)^T d_k \\ &\leq \frac{t_k}{\rho}(L+\sigma)\|d_k\|^2. \end{aligned}$$

This gives the desired inequality (3.1). ■

Lemma 3.3. *Suppose that Assumption 2.1 holds. Let* $\{x_k\}$ *and* $\{z_k\}$ *be sequences generated by Algorithm 2.3, then for any* $x^* \in C^*$ *the inequality*

$$\|x_{k+1} - x^*\|^2 \leq \|x_k - x^*\|^2 - \varpi(2-\varpi)\frac{\sigma^2\|x_k - z_k\|^4}{\|G(z_k)\|^2}. \tag{3.2}$$

holds. In addition, $\{x_k\}$ *is bounded and*

$$\sum_{k=0}^{\infty} \|x_k - z_k\|^4 < +\infty. \tag{3.3}$$

Proof. First, we begin by using the monotonicity of the mapping F. Thus, for any solution $x^* \in C^*$,

$$F(z_k)^T(x_k - x^*) \geq F(z_k)^T(x_k - z_k).$$

The above inequality together with (2.10) gives

$$F(x_k + t_k d_k)^T(x_k - z_k) \geq \sigma t_k^2\|d_k\|^2 \geq 0. \tag{3.4}$$

We have the following from (2.6) and (3.4),

$$\begin{aligned} \|x_{k+1} - x^*\|^2 &= \|P_C(x_k - \varpi\xi_k F(x_k + t_k d_k)) - x^*\|^2 \leq \|x_k - \varpi\xi_k F(x_k + t_k d_k) - x^*\|^2 \\ &= \|x_k - x^*\|^2 - 2\varpi\xi_k F(x_k + t_k d_k)^T(x_k - x^*) + \|\varpi\xi_k F(x_k + t_k d_k)\|^2 \\ &\leq \|x_k - x^*\|^2 - 2\varpi\xi_k F(x_k + t_k d_k)^T(x_k - z_k) + \|\varpi\xi_k F(x_k + t_k d_k)\|^2 \\ &= \|x_k - x^*\|^2 - \varpi(2-\varpi)\left(\frac{G(z_k)^T(x_k - z_k)}{\|G(z_k)\|}\right)^2 \\ &\leq \|x_k - x^*\|^2 - \varpi(2-\varpi)\frac{\sigma^2\|x_k - z_k\|^4}{\|G(z_k)\|^2} \end{aligned} \tag{3.5}$$

Thus, the sequence $\{\|x_k - x^*\|\}$ is a decreasing sequence, which implies that $\{x_k\}$ is bounded. That is

$$\|x_K\| \leq \varsigma \quad \forall k \geq 0. \tag{3.6}$$

Furthermore, using the continuity of F we know that there exists a constant $K_1 > 0$ such that

$$\|F(x_k)\| \leq I_1, \quad \forall k \geq 0.$$

Since $(F(x_k) - F(z_k))^T(x_k - z_k) \geq 0$, by Cauchy-Schwarz inequality, we have

$$\|F(x_k)\|\|x_k - z_k\| \geq F(x_k)^T(x_k - z_k) \geq F(z_k)^T(x_k - z_k) \geq \sigma\|x_k - z_k\|^2.$$

From the line search, the last inequality can be implied. So we have

$$\sigma\|x_k - z_k\| \leq \|F(x_k)\| \leq I_1$$

which implies that $\{z_k\}$ is bounded. By continuity of F, we know that there exists a constant $K_2 > 0$, such that

$$\|F(z_k)\| \leq I_2, \quad \forall k \geq 0.$$

the above combined with (3.5) yields

$$\varpi(2-\varpi)\frac{\sigma^2}{I_2^2}\|x_k - z_k\|^4 \leq \|x_k - x^*\|^2 - \|x_{k+1} - x^*\|^2. \tag{3.7}$$

Now, by taking the summation of (3.7), for $k \geq 0$, we have

$$\varpi(2-\varpi)\frac{\sigma^2}{I_2^2}\sum_{k=0}^{\infty}\|x_k - z_k\|^4 \leq \sum_{k=0}^{\infty}(\|x_k - x^*\|^2 - \|x_{k+1} - x^*\|^2) \leq \|x_0 - x^*\|^2 < \infty. \tag{3.8}$$

(3.8) implies that

$$\lim_{k\to\infty}\|x_k - z_k\| = 0. \tag{3.9}$$

The proof is complete. ■

Theorem 3.4. *Suppose that Assumption 2.1 hold and let $\{x_k\}$ be the sequence generated by Algorithm 2.3. Then, we have*

$$\liminf_{k\to\infty}\|F_k\| = 0. \tag{3.10}$$

Proof. Suppose (3.10) is not valid, that is, there exist a constant say $r > 0$ such that $r \leq \|\|F_k\|$, $k \geq 0$. Then this along with (2.3) implies that

$$\|d_k\| \geq cr, \quad \forall k \geq 0. \tag{3.11}$$

Since $\{\|F_k\|\}$ and $\{\|F(z_k)\|\}$ are bounded, it follows from (2.1)-(2.4) that for all $k \geq 1$,

$$\begin{aligned}
\|d_k\| &\leq c\|F_k\| + \|F_k\| \cdot \frac{\|y_{k-1}\|}{\|d_{k-1}\|} + \|F_k\| \cdot \frac{\|y_{k-1}\|}{\|d_{k-1}\|^2}\|d_{k-1}\| \\
&= c\|F_k\| + 2\|F_k\|\frac{\|y_{k-1}\|}{\|d_{k-1}\|} \\
&\leq c\|F_k\| + 2L\|F_k\|\frac{\|x_k - x_{k-1}\|}{\|d_{k-1}\|} \\
&\leq cI_1 + \frac{4I_1L\varsigma}{cr} \triangleq \Gamma.
\end{aligned}$$

Note that, by Cauchy Schwarz inequality, the first inequality is easily obtained. Similarly, from (2.7) and (3.11), the second inequality follows. Now, from (3.1), we have

$$t_k\|d_k\| \geq \max\left\{\kappa, \frac{\rho c\|F_k\|^2}{(L+\sigma)\|d_k\|^2}\right\}\|d_k\|$$
$$\geq \max\left\{\kappa c r, \frac{\rho c r^2}{(L+\sigma)\Gamma}\right\} > 0,$$

which contradicts (3.9). Hence (3.10) is valid. ∎

Theorem 3.5. *Let x_k be the sequence generated by Algorithm 2.3 under Assumption 2.1 − 2.2. Then the sequence $dist\{x_k, C^*\}$ $Q-$linearly converges to zero.*

Proof. Lets set $\mu_k = \arg\min\{\|x_k - h\| \quad |h \in C^*\}$. This implies that

$$\|x_k - t_k\| = dist(x_k, C^*).$$

From (3.2), for $\mu_k \in C^*$ we obtain

$$\begin{aligned} d(x_{k+1}, C^*)^2 &\leq \|x_{k+1} - t_k\|^2 \\ &\leq dist(x_k, C^*)^2 - \sigma^2\|t_k d_k\|^4 \\ &\leq dist(x_k, C^*)^2 - \sigma^2 c^4 t_k^4\|F_k\|^4 \\ &\leq dist(x_k, C^*)^2 - \sigma^2\gamma^2 c^4 t_k^4 d(x_k, C^*)^2 \\ &= (1 - \sigma^2\gamma^2 c^4 t_k^4) d(x_k, C^*)^2, \end{aligned}$$

Note that, from the inequality in Assumption 2.2, we obtain the fourth inequality. Let the parameter $\frac{1}{\gamma\sigma} \geq c^2$, then, $1 - \sigma^2\gamma^2 c^4 t_k^4 \in (0, 1)$ holds. Finally, we see that $d(x_k, C^*)$ $Q-$linearly converges to zero. ∎

4. Numerical Experiments

An insight of the proposed algorithm is presented in this section. We test the computational performance of Algorithm 2.3 with existing method in literature using some benchmark test problems. Precisely, we compare our algorithm with the PDY algorithm [36] designed for solving same problem (1.2). The numerical experiments are carried out on a set of seven different problems with dimension ranging from $n = 5000$ to 100,000 and initial points set as follow:

$$x_1 = (0.1, 0.1, \cdots, 0.1)^T, x_2 = (0.2, 0.2, \cdots, 0.2)^T, x_3 = (0.5, 0.5, \cdots, 0.5)^T, x_4 = (1.2, 1.2, \cdots, 1.2)^T,$$
$$x_5 = (1.5, 1.5, \cdots 1.5)^T, x_6 = (2, 2, \cdots, 2)^T, x_7 = rand(n, 1).$$

Throughout, we set parameters for PDY algorithm as in [36]. For Algorithm 1, the values of our parameters were set as follows: $c = 1$, $\rho = 0.5$, $\sigma = 0.001$. $\varpi = 1.8$. For each test problem, the iterative process is stopped when the inequality

$$\|F_k\| \leq 10^{-6}$$

is satisfied. Again, failure is declared after a thousand iteration. All algorithms were written in Matlab and run on a HP personal computer with system specifications as follows Intel(R) Core (TM) i3-7100U CPU 2.40GHZ, 8GB memory and Windows 10 operating system.

We give a list of the benchmark test problems used in our experiment. Note that in this article, we take the mapping F as $F(x) = (f_1(x), f_2(x), \cdots, f_n(x))^T$.

Problem 1. This problem is the Exponential function [37] with constraint set $C = R^n_+$, that is,

$$f_1(x) = e^{x_1} - 1,$$
$$f_i(x) = e^{x_i} + x_i - 1, \text{ for } i = 2, 3, ..., n.$$

Problem 2. Modified Logarithmic function [15] with constraint set $C = \{x \in R^n : \sum_{i=1}^n x_i \leq n, x_i > -1, i = 1, 2, \dots, n\}$, that is,

$$f_i(x) = \ln(x_i + 1) - \frac{x_i}{n}, \ i = 2, 3, ..., n.$$

Problem 3. The Nonsmooth Function [38] with constraint set $C = R^n_+$.

$$f_i(x) = 2x_i - \sin|x_i|, \ i = 1, 2, 3, ..., n.$$

Problem 4. The Strictly convex function [39], with constraint set $C = R^n_+$, that is,

$$f_i(x) = e^{x_i} - 1, \ i = 2, 3, \cdots, n.$$

Problem 5. Tridiagonal Exponential function [40] with constraint set $C = R^n_+$, that is,

$$f_1(x) = x_1 - e^{\cos(h(x_1+x_2))},$$
$$f_i(x) = x_i - e^{\cos(h(x_{i-1}+x_i+x_{i+1}))}, \text{ for } 2 \leq i \leq n-1,$$
$$f_n(x) = x_n - e^{\cos(h(x_{n-1}+x_n))}, \text{ where } h = \frac{1}{n+1}.$$

Problem 6. Nonsmooth function [41] with with constraint set $C = \{x \in R^n : \sum_{i=1}^n x_i \leq n, x_i \geq -1, \ 1 \leq i \leq n\}$.

$$f_i(x) = x_i - \sin|x_i - 1|, \ i = 2, 3, \cdots, n$$

Problem 7. The Trig exp function [37] with constraint set $C = R^n_+$, that is,

$$f_1(x) = 3x_1^3 + 2x_2 - 5 + \sin(x_1 - x_2)\sin(x_1 + x_2)$$
$$f_i(x) = 3x_i^3 + 2x_{i+1} - 5 + \sin(x_i - x_{i+1})\sin(x_i + x_{i+1}) + 4x_i - x_{i-1}e^{x_{i-1}-x_i} - 3 \text{ for } i = 2, 3, .$$
$$f_n(x) = x_{n-1}e^{x_{n-1}-x_n} - 4x_n - 3, \text{ where h} = \frac{1}{n+1}.$$

In order to visualize the behavior of Algorithm 1, we adopt the performance profiles proposed by Dolan and More in [42] to compare the performance among the tested methods. The performance profile seeks to find how well the solvers perform relative to the other solvers on a set of problems based on the total number of iterations, total number of function evaluations, and the running time of each method. The details of our numerical test are presented in the Appendix section. We denote by "Iter." the number of iterations, "Fval." the number of function evaluations and "Time." the CPU time in seconds.

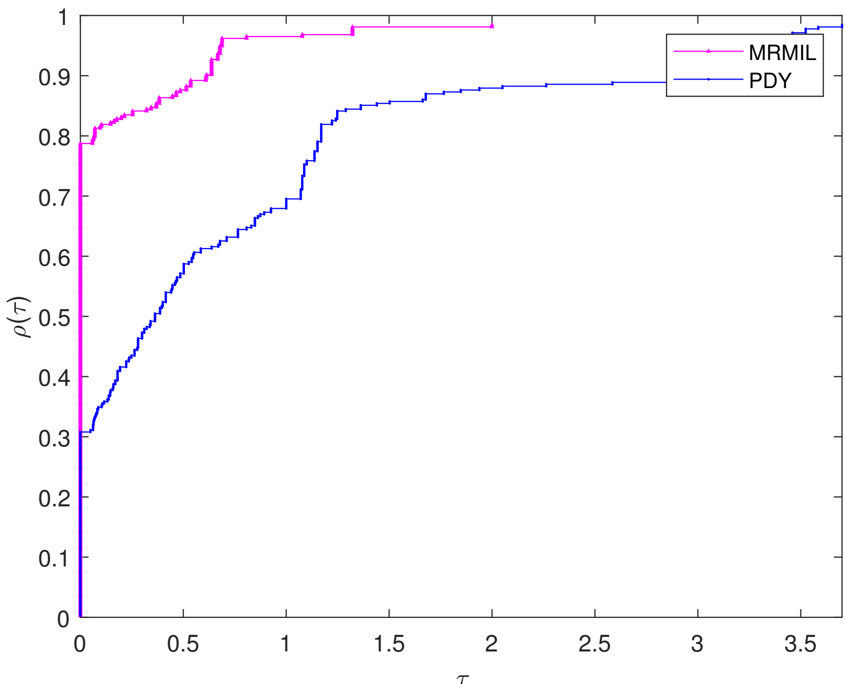

Figure 1. Performance profiles with respect to the number of iterates

The figures in this section show the performance profiles of our method versus other recent existing method. The performance of the methods are measured based on the number of iterations, the number of function F evaluations and the CPU time. It is not difficult to see that both methods solved all the test problems successfully. However, the MRMIL algorithm highly performs better on a whole based on these measures compared to PDY algorithm.

In detail, Figure 1 illustrates the performance profile of our method, where the performance index is the total number of iterations. It can be seen that the MRMIL algorithm is the best solver with probability around 79% while the probability of the compared method of solving the same problem as the best solver is around 31%. Figure 2.5 and 3 illustrates the performance profiles of the total number of function evaluation and CPU time. Similar results as Figure 1 can be derived from these figures.

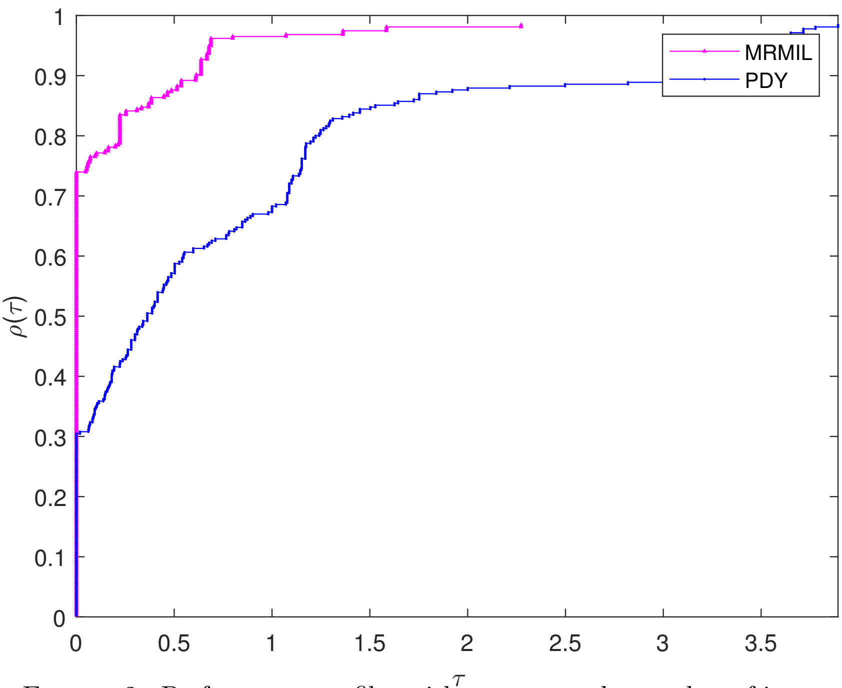

FIGURE 2. Performance profiles with respect to the number of iterates

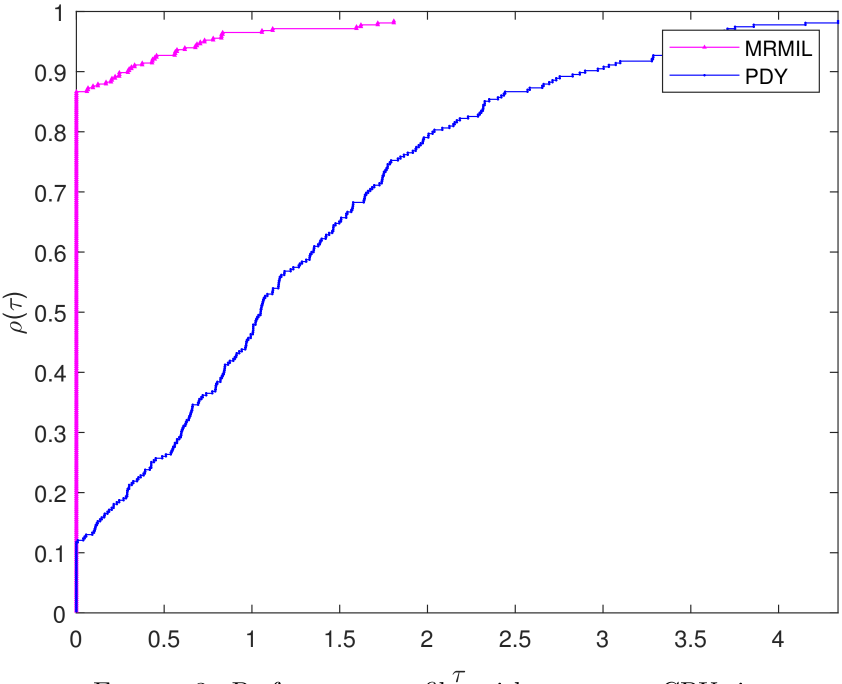

Figure 3. Performance profiles with respect to CPU time

5. Conclusion

In this article, the authors proposed a modified conjugate gradient algorithm for solving monotone nonlinear equations with convex constraints. This work can be regarded as an extension of the method in [35]. Using some technical conditions, we established the global convergence of the proposed method. We present numerical results to illustrate that our method is stable and efficient for the monotone nonlinear equations, especially for the large-scale problems with convex constraints.

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Appendix

Table 1. Numerical results for problem 1

		MRMIL				PDY			
DIM	**INP**	**ITER**	**FVAL**	**TIME**	**NORM**	**ITER**	**FVAL**	**TIME**	**NORM**
1000	x_1	5	19	0.017596	0.00E+00	16	64	0.040997	3.45E-07
	x_2	5	19	0.020114	0.00E+00	16	64	0.033587	7.03E-07
	x_3	8	32	0.014192	2.11E-07	17	68	0.011419	6.22E-07
	x_4	2	7	0.006337	0.00E+00	18	72	0.02491	4.54E-07
	x_5	2	7	0.007319	0.00E+00	18	72	0.038672	3.65E-07
	x_6	5	19	0.008786	0.00E+00	18	72	0.013774	3.80E-07
	x_7	8	32	0.006044	4.80E-07	17	68	0.019283	7.05E-07
5000	x_1	5	19	0.012849	0.00E+00	16	64	0.076967	7.61E-07
	x_2	5	19	0.011326	0.00E+00	17	68	0.07326	5.15E-07
	x_3	10	40	0.03188	1.49E-07	18	72	0.051817	4.63E-07
	x_4	2	7	0.00821	0.00E+00	19	76	0.059926	3.38E-07
	x_5	2	7	0.007023	0.00E+00	18	72	0.078845	8.12E-07
	x_6	7	27	0.020875	0.00E+00	18	72	0.072337	8.10E-07
	x_7	8	32	0.01868	7.97E-07	18	72	0.062592	5.38E-07
10000	x_1	12	48	0.048563	1.28E-08	17	68	0.097567	3.55E-07
	x_2	6	24	0.027456	2.05E-07	17	68	0.085576	7.27E-07
	x_3	8	32	0.02695	1.85E-07	18	72	0.10317	6.55E-07
	x_4	2	7	0.007457	0.00E+00	19	76	0.092351	4.77E-07
	x_5	2	7	0.012742	0.00E+00	20	80	0.13829	4.52E-07
	x_6	6	23	0.032861	0.00E+00	19	76	0.093402	5.51E-07
	x_7	9	36	0.033609	9.08E-08	18	72	0.084444	7.55E-07
50000	x_1	10	40	0.18505	2.60E-07	17	68	0.43588	7.93E-07
	x_2	8	32	0.11334	7.28E-07	18	72	0.32887	5.44E-07
	x_3	8	32	0.13543	7.35E-08	19	76	0.36732	4.86E-07
	x_4	2	7	0.033502	0.00E+00	20	80	0.41552	9.70E-07
	x_5	2	7	0.059006	0.00E+00	22	88	0.57589	8.63E-07
	x_6	6	23	0.1	0.00E+00	23	92	0.49481	8.62E-07
	x_7	9	36	0.17921	1.92E-07	19	76	0.43672	5.62E-07
100000	x_1	17	68	0.46137	5.26E-09	18	72	0.61655	3.76E-07
	x_2	17	68	0.46751	6.68E-07	18	72	0.81072	7.69E-07
	x_3	8	31	0.20832	0.00E+00	19	76	0.64764	6.88E-07
	x_4	2	7	0.080694	0.00E+00	23	92	1.0145	3.63E-07
	x_5	2	7	0.090344	0.00E+00	23	92	1.043	9.61E-07
	x_6	11	44	0.27679	8.73E-08	26	104	1.0696	3.39E-07
	x_7	9	36	0.27043	2.48E-07	20	80	0.9056	7.78E-07

Table 2. Numerical results for problem 2

		MRMIL				PDY			
DIM	**INP**	**ITER**	**FVAL**	**TIME**	**NORM**	**ITER**	**FVAL**	**TIME**	**NORM**
1000	x_1	7	23	0.004347	1.42E-08	13	51	0.077345	7.68E-07
	x_2	7	23	0.00593	1.44E-08	15	59	0.013322	3.49E-07
	x_3	8	26	0.008602	1.11E-08	16	63	0.010509	6.98E-07
	x_4	8	26	0.00346	6.52E-09	18	71	0.029102	3.52E-07
	x_5	7	23	0.005348	1.18E-08	18	71	0.014308	5.13E-07
	x_6	8	26	0.00715	5.03E-09	18	71	0.01597	8.59E-07
	x_7	16	63	0.015263	5.75E-07	17	67	0.051946	4.52E-07
5000	x_1	7	24	0.014743	5.75E-07	14	55	0.050839	5.44E-07
	x_2	7	24	0.016902	5.75E-07	15	59	0.036741	7.63E-07
	x_3	8	27	0.018674	4.90E-07	17	67	0.10101	5.12E-07
	x_4	8	27	0.016456	3.19E-07	18	71	0.073292	7.73E-07
	x_5	7	24	0.019151	4.59E-07	19	75	0.074561	3.75E-07
	x_6	8	26	0.028861	4.98E-10	19	75	0.045617	6.27E-07
	x_7	15	59	0.042091	8.23E-07	17	67	0.17854	9.89E-07
10000	x_1	9	35	0.043477	4.65E-07	14	55	0.096961	7.66E-07
	x_2	9	34	0.034886	4.65E-07	16	63	0.068215	3.55E-07
	x_3	10	38	0.048411	4.04E-07	17	67	0.097321	7.23E-07
	x_4	10	38	0.051665	2.72E-07	19	75	0.17663	3.63E-07
	x_5	9	35	0.038634	3.75E-07	19	75	0.1389	5.29E-07
	x_6	10	38	0.048532	1.63E-07	19	76	0.084686	9.51E-07
	x_7	16	63	0.066321	5.89E-07	18	71	0.18208	4.65E-07
50000	x_1	10	39	0.1595	1.04E-07	15	59	0.60225	5.78E-07
	x_2	10	39	0.13852	1.03E-07	16	63	0.38193	7.92E-07
	x_3	10	38	0.25958	9.06E-07	18	71	1.1323	5.36E-07
	x_4	10	38	0.15314	6.13E-07	21	84	0.48105	3.43E-07
	x_5	9	35	0.13068	8.30E-07	21	84	0.65056	4.72E-07
	x_6	10	38	0.15521	3.60E-07	21	84	0.49099	4.77E-07
	x_7	16	63	0.38185	8.50E-07	19	75	0.46664	3.46E-07
100000	x_1	10	39	0.34962	1.46E-07	15	59	0.79437	8.17E-07
	x_2	10	39	0.43655	1.46E-07	17	67	0.86905	3.76E-07
	x_3	11	42	0.31355	1.28E-07	18	72	0.92806	9.65E-07
	x_4	10	38	0.38263	8.68E-07	22	88	1.0076	8.28E-07
	x_5	10	39	0.28794	1.17E-07	22	88	1.542	8.18E-07
	x_6	10	38	0.2989	5.08E-07	22	88	1.3244	7.87E-07
	x_7	17	67	0.68179	5.25E-07	20	80	1.0409	5.45E-07

TABLE 3. Numerical results for problem 3

	MRMIL					PDY			
DIM	**INP**	**ITER**	**FVAL**	**TIME**	**NORM**	**ITER2**	**FVAL3**	**TIME4**	**NORM5**
1000	x_1	11	44	0.006867	9.73E-07	15	60	0.077797	4.96E-07
	x_2	12	48	0.008871	9.29E-07	16	64	0.017838	3.39E-07
	x_3	11	44	0.005343	6.21E-07	16	64	0.012914	9.24E-07
	x_4	13	52	0.008538	7.43E-07	17	68	0.010386	8.94E-07
	x_5	10	40	0.006854	7.08E-07	18	72	0.013949	3.60E-07
	x_6	14	56	0.009688	4.75E-07	18	72	0.026721	3.47E-07
	x_7					17	68	0.018595	3.94E-07
5000	x_1	13	52	0.023164	5.44E-07	16	64	0.036111	3.74E-07
	x_2	14	56	0.023803	5.19E-07	16	64	0.045275	7.58E-07
	x_3	12	48	0.024052	6.95E-07	17	68	0.060382	6.84E-07
	x_4	14	56	0.02763	4.15E-07	18	72	0.11091	6.68E-07
	x_5	11	44	0.014771	3.96E-07	18	72	0.049381	8.05E-07
	x_6	15	60	0.025569	2.66E-07	18	72	0.065425	7.46E-07
	x_7					17	68	0.06995	8.75E-07
10000	x_1	13	52	0.037055	7.69E-07	16	64	0.12557	5.28E-07
	x_2	14	56	0.034567	7.34E-07	17	68	0.092694	3.55E-07
	x_3	12	48	0.056047	9.82E-07	17	68	0.079822	9.67E-07
	x_4	14	56	0.055368	5.87E-07	18	72	0.24877	9.44E-07
	x_5	11	44	0.038703	5.60E-07	20	80	0.080159	3.38E-07
	x_6	15	60	0.037159	3.76E-07	19	76	0.084531	3.50E-07
	x_7					18	72	0.1757	4.10E-07
50000	x_1	14	56	0.23139	8.60E-07	17	68	0.37534	3.91E-07
	x_2	15	60	0.16087	8.21E-07	17	68	0.24801	7.93E-07
	x_3	14	56	0.17539	5.49E-07	18	72	0.26549	7.25E-07
	x_4	15	60	0.18294	3.28E-07	20	80	0.46666	6.42E-07
	x_5	12	48	0.15805	3.13E-07	21	84	0.32816	5.20E-07
	x_6	15	60	0.23569	8.40E-07	21	84	0.48755	3.51E-07
	x_7					18	72	0.50034	9.18E-07
100000	x_1	15	60	0.28424	6.08E-07	17	68	0.73834	5.53E-07
	x_2	16	64	0.30566	5.80E-07	18	72	0.75733	3.76E-07
	x_3	14	56	0.30675	7.77E-07	19	76	0.54971	3.40E-07
	x_4	15	60	0.46637	4.64E-07	22	88	1.353	6.92E-07
	x_5	12	48	0.26464	4.43E-07	22	88	0.69186	6.17E-07
	x_6	16	64	0.31211	2.97E-07	22	88	1.0329	5.81E-07
	x_7					20	80	1.1918	4.62E-07

Table 4. Numerical results for problem 4

		MRMIL				PDY			
DIM	INP	ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM
1000	x_1	11	44	0.005463	5.14E-07	15	60	0.008008	5.13E-07
	x_2	10	40	0.006904	7.26E-07	16	64	0.013881	3.59E-07
	x_3	2	7	0.002464	0.00E+00	16	64	0.024003	9.42E-07
	x_4	2	7	0.002327	0.00E+00	15	60	0.008609	6.44E-07
	x_5	2	7	0.00394	0.00E+00	17	68	0.016199	3.91E-07
	x_6	2	7	0.003647	0.00E+00	17	68	0.073791	7.89E-07
	x_7	10	40	0.004219	3.71E-07	17	68	0.017215	4.89E-07
5000	x_1	12	48	0.019223	5.75E-07	16	64	0.038264	3.86E-07
	x_2	11	44	0.015049	8.12E-07	16	64	0.032691	8.02E-07
	x_3	2	7	0.006792	0.00E+00	17	68	0.030878	7.00E-07
	x_4	2	7	0.006842	0.00E+00	16	64	0.026864	4.74E-07
	x_5	2	7	0.006989	0.00E+00	17	68	0.067797	8.74E-07
	x_6	2	7	0.006434	0.00E+00	19	76	0.031626	5.11E-07
	x_7	10	40	0.017007	1.66E-07	18	72	0.030529	3.71E-07
10000	x_1	12	48	0.026133	8.13E-07	16	64	0.046997	5.46E-07
	x_2	12	48	0.029647	5.74E-07	17	68	0.07771	3.76E-07
	x_3	2	7	0.008574	0.00E+00	17	68	0.0702	9.90E-07
	x_4	2	7	0.011381	0.00E+00	19	76	0.058083	3.70E-07
	x_5	2	7	0.008156	0.00E+00	18	72	0.097611	4.15E-07
	x_6	2	7	0.01211	0.00E+00	19	76	0.13803	7.22E-07
	x_7	13	52	0.065447	5.08E-07	18	72	0.075793	5.07E-07
50000	x_1	13	52	0.11185	9.09E-07	17	68	0.18421	4.04E-07
	x_2	13	52	0.17585	6.42E-07	17	68	0.19435	8.40E-07
	x_3	2	7	0.028215	0.00E+00	18	72	0.22118	7.39E-07
	x_4	2	7	0.040046	0.00E+00	20	80	0.29846	6.25E-07
	x_5	2	7	0.036865	0.00E+00	20	80	0.24516	8.13E-07
	x_6	2	7	0.034726	0.00E+00	22	88	0.45415	9.65E-07
	x_7	13	52	0.15831	3.24E-07	19	76	0.32983	6.75E-07
100000	x_1	14	56	0.26401	6.43E-07	17	68	0.56127	5.71E-07
	x_2	13	52	0.32171	9.08E-07	18	72	0.5503	3.98E-07
	x_3	2	7	0.060471	0.00E+00	19	76	0.42901	9.57E-07
	x_4	2	7	0.084635	0.00E+00	22	88	0.53544	3.99E-07
	x_5	2	7	0.081715	0.00E+00	24	96	0.95585	3.66E-07
	x_6	2	7	0.059423	0.00E+00	26	104	0.7676	3.55E-07
	x_7	13	52	0.2707	4.50E-07	19	76	0.56768	9.53E-07

TABLE 5. Numerical results for problem 5

		MRMIL				PDY			
DIM	INP	ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM
1000	x_1	28	112	0.02166	6.16E-07	18	72	0.064493	4.82E-07
	x_2	28	112	0.036568	5.92E-07	18	72	0.017572	4.64E-07
	x_3	28	112	0.021776	5.22E-07	18	72	0.025552	4.08E-07
	x_4	27	108	0.017528	7.14E-07	17	68	0.027679	8.34E-07
	x_5	27	108	0.01623	5.73E-07	17	68	0.02813	6.69E-07
	x_6	26	104	0.048145	6.76E-07	17	68	0.013746	3.94E-07
	x_7	28	112	0.019109	5.26E-07	18	72	0.016604	4.11E-07
5000	x_1	29	116	0.10404	6.90E-07	19	76	0.15354	3.58E-07
	x_2	29	116	0.077534	6.63E-07	19	76	0.1347	3.44E-07
	x_3	29	116	0.075795	5.84E-07	18	72	0.061833	9.14E-07
	x_4	28	112	0.077236	8.00E-07	18	72	0.14442	6.26E-07
	x_5	28	112	0.086837	6.42E-07	18	72	0.058368	5.02E-07
	x_6	27	108	0.074944	7.57E-07	17	68	0.080337	8.83E-07
	x_7	29	116	0.088769	5.90E-07	18	72	0.060071	9.21E-07
10000	x_1	29	116	0.13172	9.75E-07	21	84	0.13537	4.00E-07
	x_2	29	116	0.13083	9.38E-07	21	84	0.13596	3.85E-07
	x_3	29	116	0.18213	8.26E-07	20	80	0.2194	5.83E-07
	x_4	29	116	0.12873	5.66E-07	18	72	0.14363	8.85E-07
	x_5	28	112	0.15234	9.08E-07	18	72	0.16376	7.10E-07
	x_6	28	112	0.15848	5.35E-07	18	72	0.099046	4.19E-07
	x_7	29	116	0.13706	8.34E-07	20	80	0.20036	5.88E-07
50000	x_1	31	124	0.64489	5.45E-07	24	96	0.73376	7.08E-07
	x_2	31	124	0.70682	5.24E-07	24	96	0.81236	6.81E-07
	x_3	30	120	0.55822	9.24E-07	23	92	0.6838	7.26E-07
	x_4	30	120	0.53198	6.32E-07	21	84	0.57411	5.18E-07
	x_5	30	120	0.54466	5.07E-07	21	84	0.66594	4.16E-07
	x_6	29	116	0.53353	5.98E-07	18	72	0.47458	9.36E-07
	x_7	30	120	0.53253	9.32E-07	23	92	0.78547	7.33E-07
100000	x_1	31	124	1.2364	7.71E-07	29	116	3.4129	5.93E-07
	x_2	31	124	1.5374	7.42E-07	28	112	2.232	6.09E-07
	x_3	31	124	1.3392	6.53E-07	26	104	1.9924	6.39E-07
	x_4	30	120	1.2903	8.94E-07	23	92	1.6393	7.03E-07
	x_5	30	120	1.3408	7.18E-07	22	88	1.4593	3.66E-07
	x_6	29	116	1.3172	8.46E-07	20	80	1.5262	5.97E-07
	x_7	31	124	1.3756	6.59E-07	26	104	2.0768	6.44E-07

Table 6. Numerical results for problem 6

		MRMIL				PDY			
DIM	INP	ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM
1000	x_1	8	32	0.007545	2.09E-07	17	68	0.047472	6.92E-07
	x_2	8	32	0.007651	1.30E-07	17	68	0.01164	4.34E-07
	x_3	7	28	0.0044	4.82E-07	5	20	0.035276	4.50E-08
	x_4	9	36	0.006288	7.10E-08	18	72	0.031358	8.82E-07
	x_5	9	36	0.004769	2.98E-07	19	76	0.016064	8.09E-07
	x_6	9	35	0.005865	4.30E-07	18	71	0.019573	5.23E-07
	x_7	15	60	0.011421	2.47E-07	19	76	0.034046	4.32E-07
5000	x_1	8	32	0.10747	4.68E-07	18	72	0.060315	5.59E-07
	x_2	8	32	0.013586	2.90E-07	17	68	0.043677	9.70E-07
	x_3	8	32	0.02763	6.88E-08	5	20	0.020451	1.01E-07
	x_4	9	36	0.013456	1.59E-07	19	76	0.067458	7.14E-07
	x_5	9	36	0.021637	6.67E-07	20	80	0.048031	6.56E-07
	x_6	9	35	0.01803	9.62E-07	19	75	0.072431	4.22E-07
	x_7	17	68	0.067433	2.62E-07	19	76	0.072684	9.09E-07
10000	x_1	8	32	0.035531	6.62E-07	18	72	0.17816	7.90E-07
	x_2	8	32	0.024332	4.10E-07	18	72	0.12132	4.95E-07
	x_3	8	32	0.025852	9.73E-08	5	20	0.017535	1.42E-07
	x_4	9	36	0.022198	2.24E-07	20	80	0.14969	3.66E-07
	x_5	9	36	0.023586	9.43E-07	20	80	0.20198	9.28E-07
	x_6	10	39	0.027458	8.69E-08	21	84	0.09774	4.36E-07
	x_7	15	60	0.061535	7.77E-07	20	80	0.15572	4.75E-07
50000	x_1	9	36	0.091915	9.46E-08	19	76	0.30923	6.42E-07
	x_2	8	32	0.086014	9.17E-07	19	76	0.33924	4.02E-07
	x_3	8	32	0.10021	2.18E-07	5	20	0.073014	3.18E-07
	x_4	9	36	0.10806	5.02E-07	21	84	0.41239	8.23E-07
	x_5	10	40	0.10572	1.35E-07	21	84	0.57454	7.14E-07
	x_6	10	39	0.17226	1.94E-07	21	84	0.3827	9.75E-07
	x_7	18	72	0.2558	6.43E-07	21	84	0.93238	3.82E-07
100000	x_1	9	36	0.26898	1.34E-07	20	80	0.79946	7.45E-07
	x_2	9	36	0.17345	8.28E-08	19	76	1.0298	5.69E-07
	x_3	8	32	0.25039	3.08E-07	5	20	0.14119	4.50E-07
	x_4	9	36	0.21676	7.10E-07	22	88	1.0177	4.22E-07
	x_5	10	40	0.27836	1.91E-07	22	88	0.81176	7.50E-07
	x_6	10	39	0.19144	2.75E-07	22	88	0.93483	5.00E-07
	x_7	20	80	0.54314	3.45E-07	20	80	0.73771	6.67E-07

TABLE 7. Numerical results for problem 7

		MRMIL				PDY			
DIM	**INP**	**ITER**	**FVAL**	**TIME**	**NORM**	**ITER**	**FVAL**	**TIME**	**NORM**
1000	x_1	25	100	0.073028	4.63E-07	36	144	0.20315	6.34E-07
	x_2	25	100	0.11462	9.07E-07	35	140	0.2928	9.13E-07
	x_3	22	88	0.059525	9.01E-07	35	140	0.18604	7.34E-07
	x_4	25	100	0.085671	4.88E-07	33	132	0.16095	2.30E-07
	x_5	25	100	0.060418	1.67E-07	31	124	0.13438	8.06E-07
	x_6	25	100	0.080534	8.31E-07	24	96	0.10379	9.72E-07
	x_7	26	104	0.088063	5.63E-07	29	116	0.1692	3.15E-07
5000	x_1	28	112	0.35075	7.48E-07	34	136	0.71146	8.36E-07
	x_2	24	96	0.38934	8.05E-07	34	136	0.69158	7.93E-07
	x_3	25	100	0.30709	7.27E-07	34	136	0.63571	6.18E-07
	x_4	25	100	0.337	5.12E-07	31	124	0.66455	3.90E-07
	x_5	25	100	0.39557	5.84E-07	30	120	0.59363	8.11E-07
	x_6	23	92	0.27766	5.03E-07	24	96	0.54085	7.51E-07
	x_7	28	112	0.50805	6.90E-07	25	100	0.79827	2.93E-07
10000	x_1	30	120	0.71688	6.74E-07	34	136	1.8057	6.78E-07
	x_2	30	120	0.66154	7.68E-07	34	136	1.3939	6.42E-07
	x_3	25	100	0.55277	5.63E-07	33	132	1.3301	7.57E-07
	x_4	29	113	0.7388	9.13E-07	30	120	1.3051	3.94E-07
	x_5	25	100	0.57022	7.39E-07	30	120	1.1445	5.57E-07
	x_6	25	100	0.56951	8.67E-07	24	96	0.8758	7.21E-07
	x_7	29	116	0.72454	6.65E-07	25	100	0.89229	4.07E-07
50000	x_1	28	112	2.8081	8.07E-07	34	136	7.9299	6.35E-07
	x_2	30	120	2.9855	7.96E-07	33	132	6.6438	6.12E-07
	x_3	26	104	2.6057	9.25E-07	32	128	7.4126	7.22E-07
	x_4	5	17	0.40473	NaN	24	96	5.4526	3.36E-07
	x_5	7	25	0.6025	NaN	29	116	6.8103	5.83E-07
	x_6	28	112	2.9107	4.55E-07	31	124	6.0871	7.91E-07
	x_7	29	116	3.0354	2.75E-07	27	108	5.43	3.65E-07
100000	x_1	30	119	6.2617	4.81E-07	33	132	19.5575	8.00E-07
	x_2	28	112	5.7996	8.27E-07	33	132	17.3005	7.49E-07
	x_3	29	116	6.1519	8.53E-07	40	160	21.0229	9.75E-07
	x_4	5	17	0.83758	NaN	30	120	12.1478	9.85E-07
	x_5	26	104	5.5499	7.56E-07	28	112	10.8844	9.46E-07
	x_6	33	131	7.1703	6.89E-07	26	104	9.8098	9.05E-07
	x_7	31	124	6.7745	5.21E-07	27	108	9.8646	4.03E-07

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