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A MODIFIED DYNAMIC EQUATION OF EVASION DIFFERENTIAL GAME PROBLEM IN A HILBERT SPACE

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Abstract We study a modified dynamic equation of evasion differential game problem of one-pursuer-one-evader in a Hilbert space. Control functions of the players are subject to integral constraints. Sufficient conditions for possibility of evasion from the pursuer were obtained. In addition, we estimate the distance that the evader can maintain from the pursuer on some interval of time. An illustrative example is presented to demonstrate the efficiency of the results obtained.

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1. Introduction

Evasion differential game problem involve finding sufficient conditions for avoidance of contact of atleast one dynamic object, evader, from countably many other dynamic objects, pursuers, whose goal through the game is to establish contact with the evader. This class of differential game problem have been extensively studied and fundamental results have been obtained (see e.g.[2], [3] and some references therein).
Among the works dedicated to differential games of several players, equation of motions

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described by

$$\begin{cases} \dot{x}_j(t) = a(t)u_j(t), & x_j(0) = x_j^0, \; j = 1, 2, \dots \\ \dot{y}(t) = a(t)v(t), & y(0) = y^0, \end{cases} \tag{1.1}$$

with integral or geometric constraints on players control parameters, where $a(t)$ is a scalar function defined on some intervals, were investigated in [9], [8] and [6]. Ibragimov et. al.[8] obtained sufficient conditions that guarantees evasion in a two dimensional space with $a(t) = 1$ and integral constraints on players control functions. Also in the space $\mathbb{R}^2$, Ibragimov and Salleh [6] obtained sufficient conditions for avoidance of contacts of the evader from all the pursuer where $a(t) = 1$ and players control functions were subject to coordinate wise integral constraints. Recently, Ibragimov et. al. [9] also studied evasion differential game problem of countably many pursuer and evaders with $a(t) = 1$ and obtained sufficient condition of evasion of at least one of the evader from pursuer. Alias et. al [1] investigated evasion differential game problem of countably many pursuer and countably many evaders in the Hilbert space $\mathbb{R}^n$ with $a(t) = 1$ and integral constraints imposed on players control functions. They proved that evasion is possible under the assumption that the total resource of evaders exceeds (or equals) that of the pursuer and initial positions of all the evaders are not limit points for initial positions of the pursuer.

In [4] and [5], pursuit-evasion differential game described by (1.1) was studied in the Hilbert space $\mathbb{R}^n$ with geometric and integral constraints on control functions of the players respectively, where $a(t) = \theta - t$, and θ is the duration of the game. In both papers, sufficient conditions for completion of pursuit as well as value of the game were obtained.

Ibragimov and Satimov [7] considered differential game problem of many pursuer and many evaders described by (1.1) on a nonempty convex subset of R^n where all players are confined within the convex set. In the paper, $a(t)$ is a scalar measurable function satisfying some conditions and the control functions of players are subjected to integral constraints. It was proven that pursuit can be completed if the total resources of the pursuer is greater than that of the evaders.

The work in [10] dealt with the case when all players are endowed with equal dynamic possibilities with geometric constraints, where $a(t) = 1$. The pursuit problem was solved with the assumption that the evader's initial position must lie in the convex hull of that of the pursuer, else evasion is possible. The results in [10] was later adopted in developing an efficient method of resolving functions for a linear group pursuit problem in [11].

Motivated by the work in [8], we consider evasion differential game problem of one-pursuer-one-evader in n-dimensional space with motion of players governed by (1.1), where $a(t)$ is an arbitrary nonnegative measurable scalar function and the control functions of players are subject to integral constraint. We solve the game by presenting explicit strategy for the evader which guarantees evasion. We find sufficient conditions that guarantees avoidance of contact of the evader from the pursuer by presenting an explicit strategy for the evader. We also estimated the distance of the evader from the pursuer on some time interval.

2. Statement of the Problem

Let the motions of the pursuer P and the evader E be described by (1.1) with $j = 1$, where $x(t)$, x_0, $u(t)$, $y(t)$, y_0, $v(t) \in \mathbb{R}^n$; $u(t) = (u_1(t), u_2(t), ..., u_n(t))$ and $v(t) = (v_1(t), v_2(t), ..., v_n(t))$ are control functions of the pursuer P and evader E respectively,

$a(t)$ is a nonnegative measurable scalar function for all $t \geq 0$. The moment players admissible controls $u(\cdot), v(\cdot)$ are chosen, the corresponding motion, solutions of equations (1.1) is given by

$$x(t) = x^0 + \int_{t_0}^{t} a(s)u(s)ds, \ y(t) = y^0 + \int_{t_0}^{t} a(s)v(s)ds.$$

Denote $A(t_0, t) = \int_{t_0}^{t} a^2(s)ds$.

Definition 2.1. A function $u(\cdot) = (u_1(\cdot), u_2(\cdot), \cdots, u_n(\cdot))$ with Borel measurable coordinates $u_k, \ k \in I$, satisfying the inequality

$$\int_0^\infty \| u(t) \|^2 dt \leq \rho^2, \tag{2.1}$$

where ρ is a given positive numbers, is called admissible control of the pursuer.

Definition 2.2. A function $v(\cdot) = (v_1(\cdot), v_2(\cdot), \cdots, v_n(\cdot))$ with Borel measurable coordinates $v_k, \ k \in I$, satisfying the inequality

$$\int_0^\infty \| v(t) \|^2 dt \leq \sigma^2, \tag{2.2}$$

where σ is a given positive number, is called admissible control of the evader.

Definition 2.3. A function

$$V(x, y, u), \quad V : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n,$$

is called strategy of the evader if:

(1) for any admissible controls of the pursuer $u = u(t), \ t \geq 0$, the system (1.1) with $j = 1$ has a unique solution at $v = V(x, y, u), \ t \geq 0$;

(2) the inequality

$$\int_0^\infty \|V(x, y, u)\|^2 dt \leq \sigma^2$$

holds.

Definition 2.4. Evasion is said to be possible in the game described by (1.1), (2.1) and (2.2) with initial positions $x^0, y^0 \in \mathbb{R}^n$, if there exists a strategy V of the evader such that for all admissible controls of the pursuer $u(\cdot)$, the relation $x(t) \neq y(t)$ holds, for all $t \in [0, \infty)$.

Research Problem:

Find necessary and sufficient conditions that guarantee evasion in the differential game problem described by (1.1), (2.1) and (2.2).

3. Main Results

In this section, we present the main results of the research work.

For a game of one-evader- one-pursuer, we begin by dropping the index j in the game problem described by (1) and the corresponding motion of the players takes the form

$$\begin{aligned} \dot{x}(t) &= a(t)u(t), \quad x(0) = x_0, \\ \dot{y}(t) &= a(t)v(t), \quad y(0) = y_0, \end{aligned} \tag{3.1}$$

with players control functions $u(t), v(t)$ satisfying (2.1) and (2.2). We now state the following theorem

Theorem 3.1. *If $\rho \leq \sigma$, then evasion is possible in the game described by (3.1), (2.1) and (2.2).*

The proof will be presented in two parts;
i. Reduction;
ii. Solution to the evasion problem (which involves construction of the evader's strategy, admissibility of the strategy and estimation of the distance guaranteed by the strategy.

i. Reduction
Here, we reduce the game with $\rho \leq \sigma$ to the game with $\rho < \sigma$ and show that evasion is possible as follows;
Let $d = \|x_0 - y_0\|$ and $r = \frac{d}{3}$. For $\rho = \sigma$, we set

$$v(t) = 0$$

for all $t \geq 0$.
Consequently, we have
(a) $y(t) = y_0$ for all $t \geq 0$
(b) $\|x(\tau) - x_0\| = r$, for some $\tau > 0$ (such τ may or may not exists) or $\|x(t) - x_0\| < r$ for all $t \geq 0$.
We first show that at time τ, the total energy resources $\rho(\tau)$ of the pursuer is less than that of the evader $\sigma(\tau)$, where

$$\begin{aligned} \rho(\tau) &= \left(\rho^2 - \int_0^\tau \|u(s)\|^2 ds\right)^{\frac{1}{2}} \\ \sigma(\tau) &= \left(\sigma^2 - \int_0^\tau \|v(s)\|^2 ds\right)^{\frac{1}{2}}. \end{aligned} \tag{3.2}$$

Considering the first case of (b), we have

$$\begin{aligned} r &= \|x(\tau) - x_0\| \\ &= \left\| \int_0^\tau a(s)u(s)ds \right\| \\ &\leq \int_0^\tau |a(s)| \|u(s)\| ds \\ &\leq \left(\int_0^\tau a^2(s)ds \right)^{\frac{1}{2}} \left(\int_0^\tau \|u(s)\|^2 ds \right)^{\frac{1}{2}} \\ &\leq A^{\frac{1}{2}}(0,\tau) \left(\int_0^\tau \|u(s)\|^2 ds \right)^{\frac{1}{2}}, \end{aligned}$$

thus

$$\int_0^\tau \|u(s)\|^2 ds \geq \frac{r^2}{A(0,\tau)}. \tag{3.3}$$

Using the inequality (3.3), we obtain

$$\begin{aligned} \rho^2(\tau) = \rho^2 &- \int_0^\tau \|u(s)\|^2 ds \\ &\leq \rho^2 - \frac{r^2}{A(0,\tau)} \\ &< \rho^2 \\ &= \sigma^2(\tau). \end{aligned} \tag{3.4}$$

That is, at time τ, $\rho^2(\tau) < \sigma^2(\tau)$.
The last equality in (3.4) follows from (3.2), using the evader's strategy $v(t) = 0,\ t \geq 0$.
We now show that evasion is possible in the interval $0 \leq t \leq \tau$ as follows

$$\begin{aligned} \|y(t) - x(t)\| &= \left\| y(t) - x_0 + \int_0^t a(s)(v(s) - u(s))ds \right\| \\ &= \left\| y_0 - x_0 - \int_0^t a(s)u(s))ds \right\| \\ &\geq \|y_0 - x_0\| - \int_0^t |a(s)| \|u(s)\| ds \\ &\geq \|y_0 - x_0\| - \int_0^\tau |a(s)| \|u(s)\| ds \\ &\geq d - A^{\frac{1}{2}}(0,\tau) \left(\int_0^\tau \|u(s)\|^2 ds \right)^{\frac{1}{2}} \\ &\geq d - r \\ &= 3r - r = 2r > 0. \end{aligned}$$

Hence, $y(t) \neq x(t)$ for all $t \in [0, \tau]$.
Considering the second case of (b), that is, $\|x(t) - x_0\| < r$ for all $t \geq 0$, evasion is also

shown as follows

$$\begin{aligned}\|y(t)-x(t)\| &= \|y(t)-x_0+x_0-x(t)\| \\ &\geq \|y_0-x_0\|-\|x_0-x(t)\| \\ &\geq d-A^{\frac{1}{2}}(0,\tau)\left(\int_0^\tau \|u(s)\|^2 ds\right)^{\frac{1}{2}} \\ &\geq d-r \\ &= 3r-r=2r>0.\end{aligned}$$

Therefore, $\|y(t)-x(t)\|>0$, for all $t\geq 0$.

ii. Construction of the evader's strategy
Next, we consider possibility of evasion in the interval $[\tau,\infty)$ with both player's (that is, pursuer and evader) initial positions as $x(\tau), y(\tau)$ respectively.
With the assumption of the theorem, we construct the evader's strategy (interval wise) as follows

$$V(t)=\begin{cases}(0,0,0,\ldots,0) & ,0\leq t\leq\tau_1 \\ ((\pm(\alpha a(t)+|u_1(t)|),(\alpha a(t)+|u_2(t)|)),\ldots,(\alpha a(t)+|u_n(t)|)) & ,\tau_1<t\leq\tau_1^{'} \\ (0,(\alpha a(t)+|u_2(t)|),(\alpha a(t)+|u_3(t)|),\ldots,(\alpha a(t)+|u_n(t)|)) & ,\tau_1^{'}<t\leq\tau_2^{'} \\ (0,\|u(t)\|,\|u(t)\|,\|u(t)\|,\|u(t)\|,\ldots,\|u(t)\|) & ,t>\tau_2^{'},\end{cases} \tag{3.5}$$

where $\tau_2^{'}=\tau_1^{'}+\frac{a_1}{\alpha}$, $a_1\leq\alpha A(\tau_1,t)$, $0<a_1<\min\{1,d\}$ and $\|y(\tau_1)-x(\tau_1)\|=a_1$. That is, τ_1 is the first time that the pursuer comes to a_1 distance of the evader for all $t<\tau_1$.
In the construction of the strategy,
$V_k(t)=\pm(\alpha a(t)+|u_k(t)|)$, $t_s\leq t\leq t_{s+1}$ means $V_k(t)=+(\alpha a(t)+|u_k(t)|)$, if $x_k(t_s)\leq y_k(t_s)$ and $V_k(t)=-(\alpha a(t)+|u_k(t)|)$, if $x_k(t_s)>y_k(t_s)$ for all $k\in\{1,2,3,\cdots,n\}$

iii. Admissibility of the strategy
We now show that the strategy (3.5) is admissible, that is, it satisfies (2.2). Let α satisfy $n\alpha^2A+n\rho^2\leq\sigma^2(1-2\alpha\sqrt{A})$, where $A=A(\tau_1,\tau_1^{'}+\frac{a_1}{\alpha})$.

$$\begin{aligned}\int_0^\infty \|V(s)\|^2 ds &= \left(\int_0^{\tau_1}+\int_{\tau_1}^{\tau_1^{'}}+\int_{\tau_1^{'}}^{\tau_1^{'}+\frac{a_1}{\alpha}}+\int_{\tau_1^{'}+\frac{a_1}{\alpha}}^\infty\right)\|V(s)\|^2 ds \\ &= \left(\int_{\tau_1}^{\tau_1^{'}}+\int_{\tau_1^{'}}^{\tau_1^{'}+\frac{a_1}{\alpha}}+\int_{\tau_1^{'}+\frac{a_1}{\alpha}}^\infty\right)\|V(s)\|^2 ds \\ &= \int_{\tau_1}^{\tau_1^{'}}\Sigma_{k=1}^n(a(s)\alpha+|u_k(s)|)^2 ds \\ &\quad+\int_{\tau_1^{'}}^{\tau_1^{'}+\frac{a_1}{\alpha}}\Sigma_{k=2}^n(a(s)\alpha+|u_k(s)|)^2 ds+\int_{\tau_1^{'}+\frac{a_1}{\alpha}}^\infty(n-1)\|u(s)\|^2 ds\end{aligned} \tag{3.6}$$

The first term on the right hand side of (3.6) yields

$$\begin{aligned}\int_{\tau_1}^{\tau_1^{'}} \Sigma_{k=1}^{n}(a(s)\alpha+|u_k(s)|)^2 ds &= \int_{\tau_1}^{\tau_1^{'}} \Sigma_{k=1}^{n}(a^2(s)\alpha^2+|u_k(s)|^2+2\alpha a(s)|u_k(s)|)ds \\ &= \int_{\tau_1}^{\tau_1^{'}} \Sigma_{k=1}^{n} a^2(s)\alpha^2 ds + \int_{\tau_1}^{\tau_1^{'}} \Sigma_{k=1}^{n}|u_k(s)|^2 ds \\ &\quad + 2\alpha \int_{\tau_1}^{\tau_1^{'}} a(s)\Sigma_{k=1}^{n}|u_k(s)|ds \\ &\leq n\alpha^2 A(\tau_1, \tau_1^{'}) + 2\alpha \int_{\tau_1}^{\tau_1^{'}} a(s)\sum_{k=1}^{n}|u_k(s)|ds \\ &\quad + \int_{\tau_1}^{\tau_1^{'}} \|u(s)\|^2 ds.\end{aligned} \tag{3.7}$$

The second term yields

$$\begin{aligned}\int_{\tau_1^{'}}^{\tau_1^{'}+\frac{a_1}{\alpha}} \sum_{k=2}^{n}(a(s)\alpha+|u_k(s)|)^2 ds &= \int_{\tau_1^{'}}^{\tau_1^{'}+\frac{a_1}{\alpha}} \sum_{k=2}^{n}(a^2(s)\alpha^2+|u_k(s)|^2+2\alpha a(s)|u_k(s)|)ds \\ &\leq \int_{\tau_1^{'}}^{\tau_1^{'}+\frac{a_1}{\alpha}} \sum_{k=2}^{n} a^2(s)\alpha^2 ds + \int_{\tau_1^{'}}^{\tau_1^{'}+\frac{a_1}{\alpha}} \sum_{k=2}^{n}|u_k(s)|^2 ds \\ &\quad + 2\alpha \int_{\tau_1^{'}}^{\tau_1^{'}+\frac{a_1}{\alpha}} a(s)\sum_{k=2}^{n}|u_k(s)|ds \\ &\leq (n-1)\alpha^2 A\left(\tau_1^{'}, \tau_1^{'}+\frac{a_1}{\alpha}\right) \\ &\quad + 2\alpha \int_{\tau_1^{'}}^{\tau_1^{'}+\frac{a_1}{\alpha}} a(s)\sum_{k=2}^{n}|u_k(s)|ds \\ &\quad + \int_{\tau_1^{'}}^{\tau_1^{'}+\frac{a_1}{\alpha}} \|u(s)\|^2 ds.\end{aligned} \tag{3.8}$$

Hence, (3.6) takes the form

$$\begin{aligned}\int_0^\infty \|V(s)\|^2 ds &\le n\alpha^2 A(\tau_1,\tau_1^{'}) + 2\alpha\int_{\tau_1}^{\tau_1^{'}} a(s)\Sigma_{k=1}^n|u_k(s)|ds + \int_{\tau_1}^{\tau_1^{'}}\|u(s)\|^2 ds\\ &+ (n-1)\alpha^2 A\left(\tau_1^{'},\tau_1^{'}+\frac{a_1}{\alpha}\right) + 2\alpha\int_{\tau_1^{'}}^{\tau_1^{'}+\frac{a_1}{\alpha}} a(s)\Sigma_{k=2}^n|u_k(s)|ds\\ &+ \int_{\tau_1^{'}}^{\tau_1^{'}+\frac{a_1}{\alpha}}\|u(s)\|^2 ds + (n-1)\int_{\tau_1^{'}+\frac{a_1}{\alpha}}^\infty \|u(s)\|^2 ds\\ &\le n\alpha^2\left(A(\tau_1,\tau_1^{'}) + A\left(\tau_1^{'},\tau_1^{'}+\frac{a_1}{\alpha}\right)\right) + 2\alpha\int_{\tau_1}^{\tau_1^{'}+\frac{a_1}{\alpha}} a(s)\|u(s)\|^2 ds\\ &+ n\int_{\tau_1}^\infty \|u(s)\|^2 ds\\ &\le n\alpha^2 A\left(\tau_1,\tau_1^{'}+\frac{a_1}{\alpha}\right) + 2\alpha\int_{\tau_1}^{\tau_1^{'}+\frac{a_1}{\alpha}} a(s)\|u(s)\|^2 ds + n\rho^2\\ &\le n\alpha^2 A\left(\tau_1,\tau_1^{'}+\frac{a_1}{\alpha}\right) + 2\alpha\rho^2 A^{\frac{1}{2}}\left(\tau_1,\tau_1^{'}+\frac{a_1}{\alpha}\right) + n\rho^2\\ &\le n\alpha^2 A\left(\tau_1,\tau_1^{'}+\frac{a_1}{\alpha}\right) + 2\alpha\sigma^2 A^{\frac{1}{2}}\left(\tau_1,\tau_1^{'}+\frac{a_1}{\alpha}\right) + n\rho^2\\ &\le \sigma^2.\end{aligned} \tag{3.9}$$

Hence, $\int_0^\infty \|V(s)\|^2 ds \le \sigma^2$. That is, the strategy (3.5) is admissible. It should be noted that the last inequality in (3.9) follows from the condition on α.

iv. Solution to the evasion problem
We now solve the evasion problem (1.1), (2.1) and (2.2) as follows;
Observe that for each $k \in \{2,3,\dots,n\}$, we have

$$\begin{aligned}y_k(\tau_1^{'}) - x_k(\tau_1^{'}) &= y_k(\tau_1) - x_k(\tau_1) + \int_{\tau_1}^{\tau_1^{'}} a(s)(v_k(s)-u_k(s))ds\\ &\ge -a_1 + \int_{\tau_1}^{\tau_1^{'}} a(s)(\alpha a(s) + |u_k(s)| - u_k(s))ds\\ &\ge -a_1 + \alpha\int_{\tau_1}^{\tau_1^{'}} a^2(s)ds\\ &= -a_1 + \alpha A(\tau_1,\tau_1^{'})\\ &\ge 0\end{aligned} \tag{3.10}$$

The last inequality follows from (3.5). This implies that at time $\tau_1^{'}$, the pursuer can not be above the hyperplane $y = \Sigma_{k=2}^n y_k(\tau_1^{'})$.
Also, for $t \in (\tau_1^{'}, \tau_1^{'} + \frac{a_1}{\alpha}]$ and $k \in \{2,3,\dots,n\}$ we have

$$\begin{aligned} y_k(t) - x_k(t) &= y_k(\tau_1^{'}) - x_k(\tau_1^{'}) + \int_{\tau_1^{'}}^{t} a(s)((a(s)\alpha - |u_k(s)|) - u_k(s))ds \\ &\geq \alpha A(\tau_1^{'}, t) > 0. \end{aligned}$$

Let $t > \tau_1^{'} + \frac{a_1}{\alpha}$. Then using the last inequality at $t = \tau_1^{'} + \frac{a_1}{\alpha}$, we obtain

$$\begin{aligned} y_k(t) - x_k(t) &= y_k(\tau_1^{'} + \frac{a_1}{\alpha}) - x_k(\tau_1^{'} + \frac{a_1}{\alpha}) + \int_{\tau_1^{'}+\frac{a_1}{\alpha}}^{t} a(s)(\|u(s)\| - u_k(s))ds \\ &\geq y_k(\tau_1^{'} + \frac{a_1}{\alpha}) - x_k(\tau_1^{'} + \frac{a_1}{\alpha}) \\ &> \alpha A(\tau_1^{'}, \tau_1^{'} + \frac{a_1}{\alpha}) > 0. \end{aligned} \tag{3.11}$$

Hence, $y(t) \neq x(t)$ for all $t \geq 0$.

v. Estimation of the distance $\|y(t) - x(t)\|,\ \tau_1 < t \leq \tau_1^{'}$

By Cauchy Schwartz inequality,

$$\left\| \int_{\tau_1}^{t} a(s)V(s)ds \right\| \leq \left(\int_{\tau_1}^{t} a^2(s)ds \right)^{\frac{1}{2}} \left(\int_{\tau_1}^{t} \|V(s)\|^2 ds \right)^{\frac{1}{2}} \leq \sigma A^{\frac{1}{2}}(\tau_1, t)$$

Therefore $\left\| \int_{\tau_1}^{t} a(s)V(s)ds \right\| \leq \sigma A^{\frac{1}{2}}(\tau_1, t)$.
Similarly,

$$\left\| \int_{\tau_1}^{t} a(s)u(s)ds \right\| \leq \rho A^{\frac{1}{2}}(\tau_1, t) \leq \sigma A^{\frac{1}{2}}(\tau_1, t).$$

Consequently

$$\begin{aligned} \|y(t) - x(t)\| &= \left\| y(\tau_1) - x(\tau_1) + \int_{\tau_1}^{t} a(s)V(s)ds - \int_{\tau_1}^{t} a(s)u(s)ds \right\| \\ &\geq \|y(\tau_1) - x(\tau_1)\| - \left\| \int_{\tau_1}^{t} a(s)V(s)ds \right\| - \left\| \int_{\tau_1}^{t} a(s)u(s)ds \right\| \\ &\geq a_1 - 2\sigma A^{\frac{1}{2}}(\tau_1, t). \end{aligned} \tag{3.12}$$

The last inequality of (3.12) holds if $\int_{\tau_1}^{t} \|u(s)\|^2 ds \leq \rho^2$ and $\int_{\tau_1}^{t} \|v(s)\|^2 ds \leq \sigma^2$.

Also, using the argument that $\|y(t)-x(t)\| = \|((y_1(t)-x_1(t)),(y_2(t)-x_2(t)),\dots,(y_n(t)-x_n(t)))\|$, we have

$$\begin{aligned}
\|y(t)-x(t)\| &= \left(|y_1(t)-x_1(t)|^2+|y_2(t)-x_2(t)|^2+\ldots+|y_n(t)-x_n(t)|^2\right)^{\frac{1}{2}}\\
&\geq \left(|y_1(t)-x_1(t)|^2\right)^{\frac{1}{2}}\\
&= |y_1(t)-x_1(t)|\\
&= |y_1(\tau_1)-x_1(\tau_1) \pm \int_{\tau_1}^{t} a(s)(\alpha a(s)+|u_1(s)|)ds - \int_{\tau_1}^{t} a(s)u_1(s)ds|\\
&\geq |y_1(\tau_1)-x_1(\tau_1)| + \int_{\tau_1}^{t} a(s)(\alpha a(s)+|u_1(s)|)ds - \int_{\tau_1}^{t} a(s)|u_1(s)|ds\\
&= |y_1(\tau_1)-x_1(\tau_1)| + \alpha\int_{\tau_1}^{t} a^2(s)ds\\
&\geq \alpha\int_{\tau_1}^{t} a^2(s)ds = \alpha A(\tau_1,t).
\end{aligned} \tag{3.13}$$

From the two inequalities (3.12), (3.13), to estimate the distance $\|y(t)-x(t)\|,\ \tau_1 \leq t \leq \tau_1'$, we let

$$g(t) = \max\{a_1 - 2\sigma A^{\frac{1}{2}}(\tau_1,t), \alpha A(\tau_1,t)\},$$

so that $\|y(t)-x(t)\| \geq g(t)$.
Note that at boundaries of the interval, that is, at $t=\tau_1,\ g(t)=a_1$ and at $t=\tau_1',\ g(t) = \max\{a_1 - 2\sigma A^{\frac{1}{2}}(\tau_1,\tau_1'), \alpha A(\tau_1,\tau_1'))\}$.

It can be shown that the minimizer $t^* \in [\tau_1,\tau_1']$ of the function $g(t)$ can be obtain from

$$A(\tau_1,t^*) = \frac{a_1^2}{(\sigma+\sqrt{\sigma^2+a_1\alpha})^2}. \tag{3.14}$$

Using the inequality $a_1\alpha \leq \sigma^2$ and (3.14), we now estimate the distance as follows

$$\begin{aligned}
\|y(t)-x(t)\| &\geq \max\{a_1 - 2\sigma A^{\frac{1}{2}}(\tau_1,t), \alpha A(\tau_1,t)\}\\
&\geq \alpha A(\tau_1,t)\\
&\geq \frac{a_1^2\alpha}{(\sigma+\sqrt{\sigma^2+a_1\alpha})^2}\\
&\geq \frac{a_1^2\alpha}{(\sigma+\sqrt{\sigma^2+\sigma^2})^2}\\
&\geq \frac{a_1^2\alpha}{9\sigma^2}.
\end{aligned} \tag{3.15}$$

Hence, $\|y(t)-x(t)\| \geq \frac{a_1^2\alpha}{9\sigma^2}$. That is, the value $\frac{a_1^2\alpha}{9\sigma^2}$ is the smallest distance the evader can keep from the pursuer on the time interval $\tau_1 < t \leq \tau_1'$. ■

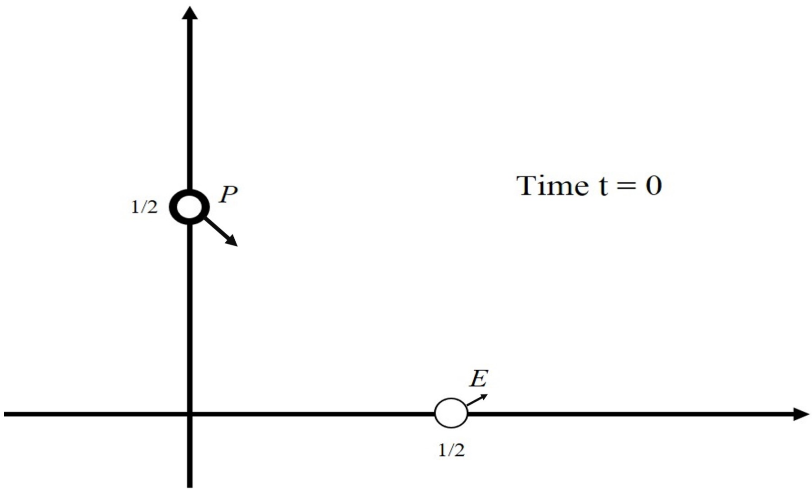

Figure 3.1 Initial positions of the pursuer P and evader E.

3.1. Illustrative example

Consider a simple motion evasion problem of one Pursuer P one Evader E in the space $\mathbb{R}^2$ governed by the equations

$$\begin{cases} \dot{x}(t) = u(t), & x(0) = x^0 \\ \dot{y}(t) = v(t), & y(0) = y^0, \end{cases}$$

where all the variables are defined as in section 2 with control functions $u(\cdot)$ and $v(\cdot)$ satisfying the inequalities

$$\int_0^\infty \| u(t) \|^2 dt \leq \rho^2,$$
$$\int_0^\infty \| v(t) \|^2 dt \leq \sigma^2.$$

Given the values $\rho^2 = \frac{1}{2}$ and $\sigma^2 = \frac{10}{\sqrt{\sqrt{2}}}$ which satisfies the hypothesis of the theorem, and initial positions of the both players as in Figure 3.1 below, then for any admissible control law $u(\cdot) = (u_1(\cdot), u_2(\cdot))$ of the pursuer, if the evader adopt the admissible strategy $V(t) = (v_1(t), v_2(t))$ where

$$(v_1(t), v_2(t)) = \begin{cases} ((\pm(\alpha + |u_1(t)|), (\alpha + |u_2(t)|))) & , 0 \leq t \leq 5\sqrt{2} \\ (0, (\alpha + |u_2(t)|)) & , 5\sqrt{2} < t \leq 10\sqrt{2} \\ (0, \|u(t)\|) & , t > 10\sqrt{2}, \end{cases} \quad (3.16)$$

with $\alpha = 0.1$ and $a_1 = \frac{\sqrt{2}}{2}$, then the conclusion of Theorem 1 follows.

That is, using similar argument in (3.10) and (3.11) with $n = 2$, then the strategy (3.16) guarantees $y(t) \neq x(t)$ for all $t \geq 0$. Based on the given values, the smallest distance the pursuer can get to the evader for all time is approximately 6.607×10^{-4}.
Lastly, the admissibility of the strategy (3.16) for all $t \in [0, \infty)$ follows from

$$\int_0^\infty \|V(s)\|^2 ds \leq 2\alpha^2 A(0, 5\sqrt{2}) + 2\alpha \int_0^{5\sqrt{2}} \sum_{k=1}^{2} |u_k(s)| ds + \int_0^{5\sqrt{2}} \|u(s)\|^2 ds$$
$$\leq 2(0.1)^2 A(0, 10\sqrt{2}) + 2(0.1) \left(\frac{10}{\sqrt{\sqrt{2}}} \right) A^{\frac{1}{2}}(0, 10\sqrt{2}) + 1$$
$$\leq \frac{10}{\sqrt{\sqrt{2}}} = \sigma^2.$$

4. Conclusion

We have studied a more general evasion differential game problem in the Hilbert space $\mathbb{R}^n$ with integral constraints, where the scalar function $a(t)$ is such that it is identically non-zero for all time $t \geq 0$. We have proved that if the total energy resources of the pursuer is less than that of the evader, then evasion is guaranteed through out the game. An estimate of the smallest distance the pursuer can get to the evader is obtained. Lastly, an example is provided in $\mathbb{R}^2$ with $a(t) = 1$ to illustrate the efficiency of the results obtained. The problem studied in this paper with countably many pursuers P_j, $j = 1, 2, \cdots$ is an open problem.

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