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EXTEND SUZUKI'S MAPPING IN HADAMARD SPACES

Duangkamon Kitkuan[1], **Kanikar Muangchoo**[2*]

[1] *Department of Mathematics, Faculty of Science and Technology, Rambhai Barni Rajabhat University, Chanthaburi 22000, Thailand, e-mail : or_duangkamon@hotmail.com*

[2] *Faculty of Science and Technology, Rajamangala University of Technology Phra Nakhon (RMUTP), 1381 Pracharat 1 Road, Wongsawang, Bang Sue, Bangkok 10800, Thailand, e-mail : kanikar.m@rmutp.ac.th*

Abstract In this paper, we prove some strong and Δ-convergence theorems for Suzuki generalized nonexpansive mappings in the setting of Hadamard spaces (or CAT(0) spaces) by using the iteration process in [12] and [13]. We also give an example to show the efficiency of the proposed process.

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1. Introduction

Banach contraction theorem is well-known shrinking theory and the iterative process have been developed to estimate fixed points of different types of mappings. Some of the well-known iterative processes are those of Mann [3], Ishikawa [4], Agarwal [5], Noor [6], Abbas [7], SP [8], Picard Mann [9], Picard-S [10], Thakur [11] and so on([22–30]).

In [12] and [13] introduced the iterative scheme for nonexpansive mappings in a uniformly convex Banach spaces:

$$\begin{cases} z_n = (1 - s_n)u_n + s_n \mathscr{T} u_n, \\ w_n = (1 - t_n)z_n + t_n \mathscr{T} z_n, \\ x_{n+1} = (1 - \nu_n)\mathscr{T} w_n + \nu_n \mathscr{T} w_n, \quad \forall n \in \mathbb{N}, \end{cases} \tag{1.1}$$

where $\{s_n\}$, $\{t_n\}$ and $\{\nu_n\}$ are sequence in $(0, 1)$.

On the other hand, we know that every Banach space is a CAT(0) space. For details about CAT(0) spaces, please see [14]. Some results are restored here for the CAT(0) space $\mathscr{X}$.

If $\mathscr{T} q = q$, then a point q is called a fixed point of a mapping $\mathscr{T}$ and $F(\mathscr{T})$ represents the set of all fixed points of the mapping $\mathscr{T}$. Let $\mathscr{C}$ be a nonempty subset of a CAT(0)

*Corresponding author.

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space $\mathscr{X}$. A mapping $\mathscr{T}:\mathscr{C}\to\mathscr{C}$ is called a contraction if there exists $\beta\in(0,1)$ such that

$$d(\mathscr{T}w,\mathscr{T}z)\le\beta d(w,z),\quad\forall\, w,z\in\mathscr{C}.$$

A mapping $\mathscr{T}:\mathscr{C}\to\mathscr{C}$ is called nonexpansive if

$$d(\mathscr{T}w,\mathscr{T}z)\le d(w,z),\quad\forall\, w,z\in\mathscr{C}.$$

Suzuki [18] introduced a new condition on a mapping, called condition $(\mathscr{C})$, which is weaker than nonexpansiveness. A mapping $\mathscr{T}:\mathscr{C}\to\mathscr{C}$ is said to satisfy condition $(\mathscr{C})$ if for all $w,z\in\mathscr{C}$, we have

$$\frac{1}{2}d(w,\mathscr{T}w)\le d(w,z)\quad\text{implies}\quad d(\mathscr{T}w,\mathscr{T}z)\le d(w,z). \tag{1.2}$$

The mapping satisfying condition A mapping $\mathscr{T}:\mathscr{C}\to\mathscr{C}$ is said to satisfy condition $(\mathscr{C})$ is called a Suzuki generalized nonexpansive mapping. The following is an example of a Suzuki generalized nonexpansive mapping which is not nonexpansive.

Motivated by the above, we prove some strong and Δ-convergence theorems results using iterative scheme (1.1) for Suzuki generalized nonexpansive mappings in the setting of CAT(0) space is given by

$$\begin{cases} z_n=(1-s_n)u_n\bigoplus s_n\mathscr{T}u_n,\\ w_n=(1-t_n)z_n\bigoplus t_n\mathscr{T}z_n,\\ x_{n+1}=(1-\nu_n)\mathscr{T}z_n\bigoplus \nu_n\mathscr{T}w_n,\quad\forall\, n\in\mathbb{N},\end{cases} \tag{1.3}$$

where $\{s_n\}$, $\{t_n\}$ and $\{\nu_n\}$ are sequence in $(0,1)$.

2. Preliminaries

Lemma 2.1 (Dhompongsa et al. [15]). *Let $\mathscr{X}$ be a CAT(0) space, $y,w,z\in\mathscr{X}$ and $\nu\in[0,1]$. Then*

(i) $d(\nu y\bigoplus(1-\nu)w,z)\le\nu d(y,z)+(1-\nu)d(w,z)$.

(ii) $d^2(\nu y\bigoplus(1-\nu)w,z)\le\nu d^2(y,z)+(1-\nu)d^2(w,z)-\nu(1-\nu)d^2(y,w)$.

Lemma 2.2 (Laokul et al. [17]). *Let x be a point in a $CAT(0)$ space $(\mathscr{X},d)$ and $\{\nu_n\}$ be a sequence in a closed interval $[a,b]$ for some $a,b\in(0,1)$. Assume that $\{w_n\}$ and $\{z_n\}$ be two sequences in $\mathscr{X}$ such that $\limsup_{n\to\infty}d(w_n,q)\le\alpha$, $\limsup_{n\to\infty}d(z_n,q)\le\alpha$ and $\lim_{n\to\infty}d(((1-\nu_n)w_n\oplus\nu_n z_n),q)=\alpha$ for some $\alpha\ge0$. Then $\lim_{n\to\infty}d(w_n,z_n)=0$.*

Proposition 2.3 (Suzuki [18]). *Let $\mathscr{X}$ be a CAT(0) space, $\mathscr{C}$ be a nonempty subset of $\mathscr{X}$ and $\mathscr{T}:\mathscr{C}\to\mathscr{C}$ be any mapping. Then:*

(i) *If $\mathscr{T}$ is nonexpansive then $\mathscr{T}$ is a Suzuki generalized nonexpansive mapping.*

(ii) *If $\mathscr{T}$ is a Suzuki generalized nonexpansive mapping and has a fixed point, then $\mathscr{T}$ is a quasi-nonexpansive mapping.*

(iii) *If $\mathscr{T}$ is a Suzuki generalized nonexpansive mapping, then $d(w,\mathscr{T}z)\le 3d(\mathscr{T}w,w)+d(w,z)$ for all $w,z\in\mathscr{C}$.*

Lemma 2.4 (Suzuki [18]). *Let $\mathscr{X}$ be a CAT(0) space and $\mathscr{C}$ be a weakly compact convex subset of $\mathscr{X}$. Let $\mathscr{T}$ be a mapping on $\mathscr{C}$. Assume that $\mathscr{T}$ is a Suzuki generalized nonexpansive mapping. Then $\mathscr{T}$ has a fixed point.*

Let $\mathscr{X}$ be a CAT(0) space, $\mathscr{C}$ be a nonempty closed convex subset of $\mathscr{X}$, and let $\{u_n\}$ be a bounded sequence in $\mathscr{X}$. $u \in \mathscr{X}$, we set

$$r(u, \{u_n\}) = \limsup_{n\to\infty} d(u, u_n).$$

The asymptotic radius of $\{u_n\}$ relative to $\mathscr{C}$ is given by

$$r(\mathscr{C}, \{u_n\}) = \inf\{r(u, \{u_n\}) : u \in \mathscr{C}\},$$

and the asymptotic center of $\{u_n\}$ relative to $\mathscr{C}$ is the set

$$\mathscr{A}(\mathscr{C}, \{u_n\}) = \{u \in \mathscr{C} : r(u, \{u_n\}) = r(\mathscr{C}, \{u_n\})\}.$$

Lemma 2.5 (Kirk and Panyanak[16]). *Every bounded sequence in a complete CAT(0) space always has a Δ-convergent subsequence.*

3. Main results

Theorem 3.1. *Let $\mathscr{X}$ be a Hadamard space, $\mathscr{C}$ be a nonempty closed convex subset of $\mathscr{X}$, and $\mathscr{T} : \mathscr{C} \to \mathscr{C}$ be a Suzuki generalized nonexpansive mapping with $F(\mathscr{T}) \neq \emptyset$. For arbitrarily chosen $u_0 \in \mathscr{C}$, let the sequence $\{u_n\}$ be generated by (1.3) then $\lim_{n\to\infty} d(u_n, q)$ exists for any $q \in F(\mathscr{T})$.*

Proof. Let $q \in F(\mathscr{T})$ and $p \in \mathscr{C}$. Since $\mathscr{T}$ is a Suzuki generalized nonexpansive mapping, we obtain

$$\frac{1}{2}d(q, \mathscr{T}q) = 0 \leq d(q, p) \quad \text{implies that} \quad d(\mathscr{T}q, \mathscr{T}p) \leq d(q, p).$$

Using Lemma 2.1(i) and Proposition 2.3(ii), we have

$$\begin{aligned} d(z_n, q) &= d(((1 - s_n)u_n \bigoplus s_n\mathscr{T}u_n), q) \\ &\leq (1 - s_n)d(u_n, q) + s_n d(\mathscr{T}u_n, q) \\ &\leq (1 - s_n)d(u_n, q) + s_n d(u_n, q) \\ &= d(u_n, q). \end{aligned} \tag{3.1}$$

Using (3.1), we obtain

$$\begin{aligned} d(w_n, q) &= d(((1 - t_n)z_n \bigoplus t_n\mathscr{T}z_n), q) \\ &\leq (1 - t_n)d(z_n, q) + t_n d(\mathscr{T}z_n, q) \\ &\leq (1 - t_n)d(z_n, q) + t_n d(z_n, q) \\ &\leq d(z_n, q) \\ &\leq d(u_n, q). \end{aligned} \tag{3.2}$$

It follows that

$$\begin{aligned} d(x_{n+1},q) &= d(((1-\nu_n)\mathscr{T}z_n \bigoplus \nu_n\mathscr{T}w_n),q) \\ &\le (1-\nu_n)d(\mathscr{T}z_n,q)+\nu_n d(\mathscr{T}w_n,q) \\ &\le (1-\nu_n)d(z_n,q)+\nu_n d(w_n,q) \\ &\le (1-\nu_n)d(u_n,q)+\nu_n d(u_n,q) \\ &= d(u_n,q). \end{aligned} \tag{3.3}$$

This implies that $\{d(u_n,q)\}$ is bounded and non-increasing for all $p \in F(\mathscr{T})$. Therefore, $\lim_{n\to\infty} d(u_n,q)$ exists. ∎

Theorem 3.2. *Let $\mathscr{X}$, $\mathscr{C}$, $\mathscr{T}$ and $\{u_n\}$ satisfy the hypotheses of Theorem 3.1, where $\{s_n\}$, $\{t_n\}$, $\{\nu_n\}$ are sequences of real numbers in $[a,b]$ for some a,b with $0<a\le b<1$. Then $F(\mathscr{T})\neq\emptyset$ if and only if $\{u_n\}$ is bounded and $\lim_{n\to\infty} d(u_n,\mathscr{T}u_n)=0$.*

Proof. Assume $F(\mathscr{T})\neq\emptyset$ and let $q\in F(\mathscr{T})$. Using Theorem 3.1, $\lim_{n\to\infty} d(u_n,q)$ exists and $\{u_n\}$ is bounded. Let

$$\lim_{n\to\infty} d(u_n,q)=\alpha. \tag{3.4}$$

From (3.1), (3.2) and (3.4), we get

$$\limsup_{n\to\infty} d(z_n,q)\le \limsup_{n\to\infty} d(u_n,q)\le\alpha \tag{3.5}$$

and

$$\limsup_{n\to\infty} d(w_n,q)\le \limsup_{n\to\infty} d(u_n,q)\le\alpha. \tag{3.6}$$

Using Proposition 2.3(ii), we obtain

$$d(\mathscr{T}u_n,q)=d(\mathscr{T}u_n,\mathscr{T}q)\le d(u_n,q)\Rightarrow \limsup_{n\to\infty} d(\mathscr{T}u_n,q)\le\limsup_{n\to\infty} d(u_n,q)\le\alpha. \tag{3.7}$$

In the same way,

$$d(\mathscr{T}z_n,q)=d(\mathscr{T}z_n,\mathscr{T}q)\le d(u_n,q)\Rightarrow \limsup_{n\to\infty} d(\mathscr{T}z_n,q)\le\limsup_{n\to\infty} d(u_n,q)\le\alpha \tag{3.8}$$

and

$$d(\mathscr{T}w_n,q)=d(\mathscr{T}w_n,\mathscr{T}q)\le d(u_n,q)\Rightarrow \limsup_{n\to\infty} d(\mathscr{T}w_n,q)\le\limsup_{n\to\infty} d(u_n,q)\le\alpha. \tag{3.9}$$

Again,

$$\lim_{n\to\infty} d(x_{n+1},q)=\lim_{n\to\infty} d(((1-\nu_n)\mathscr{T}z_n\bigoplus\nu_n\mathscr{T}w_n),q)=\alpha. \tag{3.10}$$

From (3.8)–(3.10) and using Lemma 2.2, we get

$$\lim_{n\to\infty} d(\mathscr{T}z_n,\mathscr{T}w_n)=0. \tag{3.11}$$

On the other hand,

$$\begin{aligned} d(x_{n+1}, q) &= d(((1-\nu_n)\mathscr{T} z_n \oplus \nu_n \mathscr{T} w_n), q) \\ &\leq (1-\nu_n) d(\mathscr{T} z_n, q) + \nu_n d(\mathscr{T} w_n, q) \\ &\leq (1-\nu_n) d(z_n, q) + \nu_n d(w_n, q) \\ &= d(z_n, q) - \nu_n d(z_n, q) + \nu_n d(w_n, q). \end{aligned} \tag{3.12}$$

This implies that

$$\frac{d(x_{n+1}, q) - d(z_n, q)}{\nu_n} \leq d(w_n, q) - d(z_n, q). \tag{3.13}$$

Then,

$$d(x_{n+1}, q) - d(z_n, q) \leq \frac{d(x_{n+1}, q) - d(z_n, q)}{\nu_n} \leq d(w_n, q) - d(z_n, q) \tag{3.14}$$

implies that

$$d(x_{n+1}, q) \leq d(w_n, q). \tag{3.15}$$

Hence,

$$\alpha \leq \liminf_{n\to\infty} d(w_n, q). \tag{3.16}$$

From (3.6) and (3.16), we get

$$\begin{aligned} \alpha &= d(w_n, q) \\ &= d(((1-t_n) z_n \bigoplus t_n \mathscr{T} z_n), q) \end{aligned} \tag{3.17}$$

From (3.5), (3.8), (3.17) and using Lemma 2.2, we get

$$\lim_{n\to\infty} d(z_n, \mathscr{T} z_n) = 0. \tag{3.18}$$

From (3.1) and (3.2), we get

$$d(w_n, q) \leq d(z_n, q) \leq d(u_n, q). \tag{3.19}$$

This gives

$$\lim_{n\to\infty} d(z_n, q) = \alpha. \tag{3.20}$$

Using Lemma 2.1(ii),

$$\begin{aligned} d(z_n, q)^2 &= d(((1-s_n) u_n \bigoplus s_n \mathscr{T} u_n), q)^2 \\ &\leq (1-s_n) d(u_n, q)^2 + s_n d(\mathscr{T} u_n, q)^2 - s_n(1-s_n) d(u_n, \mathscr{T} u_n)^2 \\ &\leq (1-s_n) d(u_n, q)^2 + s_n d(u_n, q)^2 - s_n(1-s_n) d(u_n, \mathscr{T} u_n)^2 \\ &= d(u_n, q)^2 - s_n(1-s_n) d(u_n, \mathscr{T} u_n)^2. \end{aligned} \tag{3.21}$$

So,

$$d(u_n, \mathscr{T} u_n)^2 = \frac{1}{s_n(1-s_n)} (d(u_n, q)^2 - d(z_n, q)^2). \tag{3.22}$$

Using (3.4) and (3.20), $\limsup_{n\to\infty} d(\mathscr{T} u_n, u_n) \leq 0$ and hence,

$$\lim_{n\to\infty} d(u_n, \mathscr{T} u_n) = 0. \tag{3.23}$$

Conversely, assume $\{u_n\}$ is bounded and $\lim_{n\to\infty} d(u_n, \mathscr{T}u_n) = 0$. Let $q \in \mathscr{A}(\mathscr{C}, \{u_n\})$. By Proposition 2.3(iii), we obtain

$$\begin{aligned} r(\mathscr{T}q, \{u_n\}) &= \limsup_{n\to\infty} d(u_n, \mathscr{T}q) \\ &\leq \limsup_{n\to\infty}(3d(\mathscr{T}u_n, u_n) + d(u_n, q)) \\ &\leq \limsup_{n\to\infty} d(u_n, q) \\ &= r(q, \{u_n\}). \end{aligned} \tag{3.24}$$

This implies that $\mathscr{T}q \in \mathscr{A}(\mathscr{C}, \{u_n\})$. Since $\mathscr{X}$ is uniformly convex, $\mathscr{A}(\mathscr{C}, \{u_n\})$ is a singleton and hence we have $\mathscr{T}q = q$. Thus $F(\mathscr{T}) \neq \emptyset$. ■

Theorem 3.3. *Let $\mathscr{C}$, $\mathscr{X}$, $\mathscr{T}$ and $\{u_n\}$ be as in Theorem 3.2 with $F(\mathscr{T}) \neq \emptyset$. Then $\{u_n\}$, Δ-converges to a fixed point of $\mathscr{T}$.*

Proof. The proof of the following Δ-convergence theorem is similar to the proof of [19]. ■

Theorem 3.4. *Let $\mathscr{C}$, $\mathscr{X}$, $\mathscr{T}$ and $\{u_n\}$ be as in Theorem 3.2 such that $\mathscr{C}$ is compact subset of $\mathscr{X}$. Then $\{u_n\}$ converges strongly to a fixed point of $\mathscr{T}$.*

Proof. Using Lemma 2.4, we have $F(\mathscr{T}) \neq \emptyset$ and Theorem 3.1 we have $\lim_{n\to\infty} d(\mathscr{T}u_n, u_n) = 0$. Since $\mathscr{C}$ is compact, there exists a subsequence $\{u_{n_l}\}$ of $\{u_n\}$ such that $\{u_{n_l}\}$ converges strongly to p for some $q \in \mathscr{C}$. From Proposition 2.3(iii), we have

$$d(u_{n_l}, \mathscr{T}q) \leq 3d(\mathscr{T}u_{n_l}, u_{n_l}) + d(u_{n_l}, q), \quad \forall\, n \geq 1.$$

Tanking $l \to \infty$, we obtain $\mathscr{T}q = q$, *i.e.*, $q \in F(\mathscr{T})$. From Theorem 3.1, $\lim_{n\to} d(u_n, q)$ exists for every $q \in F(\mathscr{T})$ and hence the $\{u_n\}$ converge strongly to q. ■

A mapping $\mathscr{T} : \mathscr{C} \to \mathscr{C}$ is said to satisfy Condition (I) [20] if there exists a non-decreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(\alpha) > 0$ for all $\alpha > 0$ such that

$$d(u, \mathscr{T}u) \geq f(d(u, F(\mathscr{T})), \quad \forall\, u \in \mathscr{C}.$$

Theorem 3.5. *Let $\mathscr{C}$, $\mathscr{X}$, $\mathscr{T}$ and $\{u_n\}$ be as in Theorem 3.2 with $F(\mathscr{T}) \neq \emptyset$. If $\mathscr{T}$ satisfies condition (I), then $\{u_n\}$ converges strongly to a fixed point of $\mathscr{T}$.*

Proof. Using Theorem 3.1, we have $\lim_{n\to\infty} d(u_n, q)$ exists for all $q \in F(\mathscr{T})$ and $\lim_{n\to\infty} d(u_n, F(\mathscr{T}))$ exists. Suppose $\lim_{n\to\infty} d(u_n, q) = \alpha$ for some $\alpha \geq 0$. If $\alpha = 0$ then the result follows. Assume $\alpha > 0$, from the hypothesis and condition (I),

$$f(d(u_n, F(\mathscr{T}))) \leq d(\mathscr{T}u_n, u_n). \tag{3.25}$$

Since $F(T) \neq \emptyset$, using Theorem 3.2, we have $\lim_{n\to\infty} d(\mathscr{T}u_n, u_n) = 0$. Thus (3.25) implies that

$$\lim_{n\to\infty} f(d(u_n, F(\mathscr{T}))) = 0. \tag{3.26}$$

Since f is a nondecreasing function, from (3.26) we have $lim_{n\to\infty} d(u_n, F(\mathscr{T})) = 0$. Hence, we have a subsequence $\{u_{n_l}\}$ of $\{u_n\}$ and a sequence $\{w_l\} \subset F(\mathscr{T})$ such that

$$d(u_{n_l}, w_l) < \frac{1}{2^l}, \quad \forall\, l \geq 1.$$

Using (3.4), we obtain

$$d(u_{n_{l+1}}, w_l) \leq d(u_{n_l}, w_l) < \frac{1}{2^l}, \quad \forall\, l \geq 1.$$

Thus,

$$\begin{aligned} d(w_{l+1}, w_l) &\leq d(w_{l+1}, u_{l+1}) + d(u_{l+1}, w_l) \\ &\leq \frac{1}{2^{l+1}} + \frac{1}{2^l} \\ &< \frac{1}{2^{l-1}} \\ &\to 0 \quad \textbf{as} \quad l \to \infty. \end{aligned}$$

This shows that $\{w_l\}$ is a Cauchy sequence in $F(\mathscr{T})$ and it converges to a point q. Since $F(\mathscr{T})$ is closed, $q \in F(\mathscr{T})$ and then $\{u_{n_l}\}$ converges strongly to q. Since $lim_{n\to\infty} d(u_n, q)$ exists, we have $u_n \to q \in F(\mathscr{T})$. ∎

4. Numerical illustrations

Define a mapping $\mathscr{T} : [4, 5] \to [4, 5]$ by

$$\mathscr{T}w = \begin{cases} 9 - w, & \text{if } w \in \left[4, \frac{37}{9}\right), \\ \frac{w + 40}{9}, & \text{if } w \in \left[\frac{37}{9}, 5\right]. \end{cases}$$

Next, we show that $\mathscr{T}$ is a Suzuki generalized nonexpansive mapping but not nonexpansive.

Take $w = \frac{411}{100}$ and $z = \frac{37}{9}$, then

$$\begin{aligned} d(\mathscr{T}w, \mathscr{T}z) &= |\mathscr{T}w - \mathscr{T}z| \\ &= \left|9 - \frac{409}{100} - \frac{397}{81}\right| \\ &= \frac{71}{8100} \\ &> \frac{1}{900} \\ &= |w - z| \\ &= d(w, z). \end{aligned}$$

Thus, $\mathscr{T}$ is not a nonexpansive mapping.
Now, we verify that $\mathscr{T}$ is a Suzuki generalized nonexpansive mapping.
Case I. Let $w \in \left[4, \frac{37}{9}\right)$, then $\frac{1}{2}d(w, \mathscr{T}w) = \frac{9-2w}{2} \in \left(\frac{7}{18}, \frac{1}{2}\right]$. For $\frac{1}{2}d(w, \mathscr{T}w) \leq d(w, z)$ we must have $\frac{9-2w}{2} \leq z - w$, *i.e.*, $\frac{9}{2} \leq z$, hence $z \in \left[\frac{9}{2}, 5\right]$. We have

$$d(\mathscr{T}w, \mathscr{T}z) = \left|\frac{z + 40}{9} - (9 - w)\right| = \left|\frac{z + 9w - 41}{9}\right| < \frac{1}{9}$$

and

$$d(w,z) = |w - z| = \left|\frac{37}{9} - \frac{9}{2}\right| = \frac{7}{18} > \frac{1}{9}.$$

Thus, $\frac{1}{2}d(w, \mathscr{T}w) \leq d(w,z) \Rightarrow d(\mathscr{T}w, \mathscr{T}z) \leq d(w,z)$.
Case II. Let $w \in [\frac{37}{9}, 5]$, then $\frac{1}{2}d(w, \mathscr{T}w) = \frac{1}{2}|\frac{w+40}{9} - w| = \frac{40-8w}{18} \in [0, \frac{64}{18}]$. For $\frac{1}{2}d(w, \mathscr{T}w) \leq d(w,z)$, we must have $\frac{40-8w}{18} \leq |z - w|$, which gives two possibilities:
(a). Let $w < z$, then $\frac{40-8w}{18} \leq z - w \Rightarrow z \leq \frac{40+10w}{18} \Rightarrow z \in [\frac{730}{162}, 5] \subset [\frac{37}{9}, 5]$. So

$$d(\mathscr{T}w, \mathscr{T}z) = \left|\frac{w+40}{9} - \frac{z+40}{9}\right| = \frac{1}{9}d(w,z) \leq d(w,z).$$

Thus, $\frac{1}{2}d(w, \mathscr{T}w) \leq d(w,z) \Rightarrow d(\mathscr{T}w, \mathscr{T}z) \leq d(w,z)$.
(b) . L e t $w > z$, then $\frac{40-8w}{18} \leq z - w \Rightarrow z \leq w - \frac{40-8w}{18} = \frac{26w-40}{18} \Rightarrow z \in [\frac{602}{162}, 5]$. Since $z \in [4, 5]$. So $z \leq \frac{26w-40}{18} \Rightarrow w \in [\frac{112}{26}, 5]$ the case is $w \in [\frac{112}{26}, 5]$ and $z \in [4, 5]$.
Now, $w \in [\frac{112}{26}, 5]$ and $\in [\frac{37}{9}, 5]$ is already included in (a). So, let $w \in [\frac{112}{26}, 5]$ and $z \in [4, \frac{37}{9})$ then

$$d(\mathscr{T}w, \mathscr{T}z) = \left|\frac{w+40}{9} - (9 - z)\right| = \left|\frac{w+9z-41}{9}\right|.$$

For convenience, first we consider $w \in [\frac{112}{26}, \frac{39}{8}]$ and $z \in [4, \frac{37}{9})$, then $d(\mathscr{T}w, \mathscr{T}z) \leq \frac{3}{72}$ and $d(w,z) > \frac{23}{117}$. Thus, $d(\mathscr{T}w, \mathscr{T}z) \leq d(w,z)$.
Next consider $w \in [\frac{39}{8}, 5]$ and $z \in [4, \frac{37}{9})$, then $d(\mathscr{T}w, \mathscr{T}z) \leq \frac{1}{9}$ and $d(x,y) > \frac{55}{72}$. Thus, $d(\mathscr{T}w, \mathscr{T}z) \leq d(w,z)$. Hence, $\frac{1}{2}d(w, \mathscr{T}w) \leq d(w,z) \Rightarrow d(\mathscr{T}w, \mathscr{T}z) \leq d(w,z)$.
Therefore, $\mathscr{T}$ is a Suzuki generalized nonexpansive mapping. With help of Matlab Program Software, we obtain the comparison Table 1 and Figure 1 for various iterative schemes with control sequences $\nu_n = 0.65, t_n = 0.35, s_n = 0.95$ and initial guess $u_1 = 4.2$.

TABLE 1. Results comparison

n	Algorithm (1.3)	Thakur	S
1	4.2000	4.2000	4.2000
2	4.9890	4.9627	4.9291
3	4.9998	4.9983	4.9937
4	5.0000	4.9999	4.9994
5	5.0000	5.0000	5.0000
6	5.0000	5.0000	5.0000
7	5.0000	5.0000	5.0000

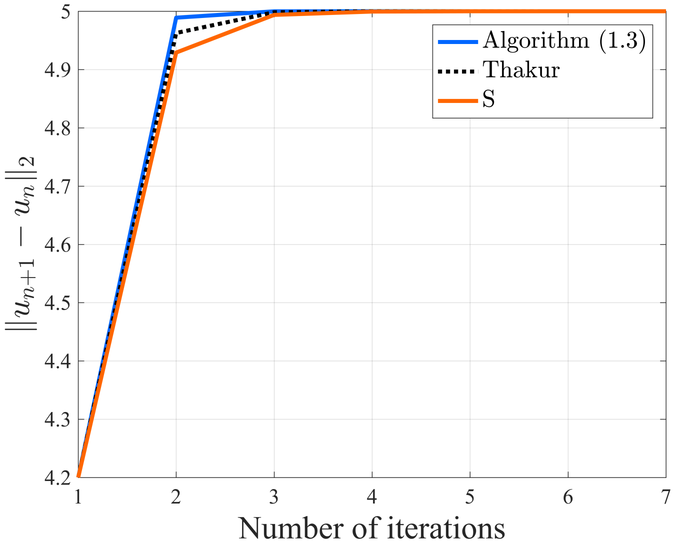

FIGURE 1. Graph of results comparison

5. Conclusions

The extension of the linear version of the fixed point results to nonlinear domains has its own significance. We extend a linear version of convergence results to the fixed point of a mapping satisfying Suzuki generalized nonexpansive mappings for iteration process [12] and [13] to nonlinear Hadamard spaces.

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References

[1] Z. Opial, Weak convergence of the sequence of successive approximations for nonexpansive mappings. Bull. Am. Math. Soc. 73(1967) 591–597.

[2] E. Picard, Memoire sur la theorie des equations aux d'erives partielles et la methode des approximations successives. J de Math Pures Appl. 231(1890) 145–210.

[3] W.R. Mann, Mean value methods in iteration, Proc. Am. Math. Soc. 4(1953) 506–510.

[4] S. Ishikawa, Fixed points by a new iteration method, Proc. Amer. Math. Soc. 44(1974) 147–150.

[5] R.P. Agarwal, D. O'Regan, D.R. Sahu,*Iterative construction of fixed points of nearly asymptotically nonexpansive mappings*, J. Nonlinear Convex Anal. 8(2007) 61–79.

[6] M.A. Noor, New approximation schemes for general variational inequalities, J. Math. Anal. Appl. 251(2000) 217–229.

[7] M. Abbas, T. Nazir, A new faster iteration process applied to constrained minimization and feasibility problems, Mater. Vesn. 66(2014) 223–234.

[8] W. Phuengrattana, S. Suantai, On the rate of convergence of Mann, Ishikawa, Noor and SP-iterations for continuous functions on an arbitrary interval, J. Comput. Appl. Math. 235(2011) 3006–3014.

[9] S.H. Khan, A Picard-Mann hybrid iterative process, Fixed Point Theory Appl. 2013 (2013), ArticleID 69.

[10] F. Gursoy, V. Karakaya, A Picard-S hybrid type iteration method for solving a differential equation with retarded argument (2014) arXiv:1403.2546v2.

[11] D. Thakur, B.S. Thakur, M. Postolache, New iteration scheme for numerical reckoning fixed points of non-expansive mappings J. Inequal. Appl. 2014 (2014) 328.

[12] V.K. Sahu, H.K. Pathak, R. Tiwari, Convergence theorems for new iteration scheme and comparison results, Aligarh Bull. Math. 35(2016) 19–42.

[13] Thakur, D. Thakur, M. Postolache, New iteration scheme for approximating fixed point of non-expansive mappings, Filomat 30(2016) 2711–2720.

[14] M. Bridson, A. Heaflinger, Metric Space of Non-positive Curvature, Springer, Berlin, 1999.

[15] S. Dhompongsa, B. Panyanak, On Δ-convergence theorems in CAT(0) spaces, Comput. Math. Appl. 56(10)(2008) 2572–2579.

[16] W.A. Kirk, B. Panyanak, A concept of convergence in geodesic spaces, Nonlinear Anal. 68(12)(2008) 3689–3696.

[17] T. Laokul, B. Panyanak, Approximating fixed points of nonexpansive mappings in CAT(0) spaces, Int. Journal of Math. Analysis 3(27)(2009) 1305 – 1315.

[18] T. Suzuki, Fixed point theorems and convergence theorems for some generalized nonexpansive mappings, J.Math. Anal. Appl. 340(2008) 1088–1095.

[19] M. Basarir, A. Sahin, On the strong and Δ-convergence of S-iteration process for generalized nonexpansive mappings on CAT(0) space, Thai J. Math. 12(2014) 549–559.

[20] H.F. Senter, W.G. Dotson, Approximating fixed points of nonexpansive mappings Proc. Am. Math. Soc. 44(1974) 375–380.

[21] N. Pakkaranang, P. Kumam, Y.J. Cho, P. Saipara, A. Padcharoen, C. Khaofong, Strong convergence of modified viscosity implicit approximation methods for asymptotically nonexpansive mappings in complete CAT(0) spaces, J. Math. Comput. Sci. 17(3)(2017) 345–354.

[22] N. Pakkaranang, P. Kumam, P. Cholamjiak, R. Supalatulatorn, P. Chaipunya, Proximal point algorithms involving fixed point iteration for nonexpansive mappings in CAT(κ) spaces, Carpathian Journal of Mathematics 34(2)(2018) 229–237.

[23] W. Kumam, N. Pakkaranang, P. Kumam, P. Cholamjiak, Convergence analysis of modified Picard-S hybrid iterative algorithms for total asymptotically nonexpansive mappings in Hadamard spaces, International Journal of Computer Mathematics 97(2020) 175–188.

[24] P. Thounthong, N. Pakkaranang, Y.J. Cho, W. Kumam, P. Kumam, The numerical reckoning of modified proximal point methods for minimization problems in Non-positive curvature metric spaces, International Journal of Computer Mathematics 97(2020) 245–262.

[25] N. Wairojjana, N. Pakkaranang, I. Uddin, P. Kumam, A.M. Awwal, Modified proximal point algorithms involving convex combination technique for solving minimization problems with convergence analysis, Optimization (2019); https://doi.org/10.1080/02331934.2019.1657115.

[26] N. Pakkaranang, P. Kewdee, P. Kumam, The modified multi step iteration process for pairwise generalized nonexpansive mapping in CAT(0) spaces, Econometrics for Financial Applications. ECONVN 2018. Studies in Computational Intelligence 760(2018) 381–393.

[27] D. Kitkuan, A. Padcharoen, Strong convergence of a modified SP-iteration process for generalized asymptotically quasi-nonexpansive mappings in CAT (0) spaces, J. Nonlinear Sci. Appl. 9(2016) 2126–2135.

[28] N. Akkasriworn, D. Kitkuan, A. Padcharoen, Convergence theorems for generalized I-asymptotically nonexpansive mappings in a Hadamard spaces, Commun. Korean Math. Soc. 31(3)(2016) 483–493.

[29] W. Kumam, D. Kitkuan, A. Padcharoen, P. Kumam, Proximal point algorithm for nonlinear multivalued type mappings in Hadamard spaces, Mathematical Methods in the Applied Sciences 42(17)(2019) 5758–5768.

[30] C. Klangpraphan, B. Panyanak, Fixed point theorems for some generalized multi-valued nonexpansive mappings in Hadamard spaces, Thai J. Math. 17(2)(2019) 543–555.
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