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AN ACCELERATED SUBGRADIENT EXTRAGRADIENT ALGORITHM FOR STRONGLY PSEUDOMONOTONE VARIATIONAL INEQUALITY PROBLEMS

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Abstract In this article, we propose an accelerated subgradient extragradient algorithm for solving variational inequality problems involving strongly pseudomonotone operator by introducing an inertial extrapolation step with time variable step size. The scheme uses a non-summable and diminishing step size without the prior knowledge of the modulus of strong monotonicity and the lipschitz constant of the underlying operator. Furthermore, we prove the strong convergence of a sequence generated by the proposed algorithm to a solution of the problem under mild assumptions. We give numerical experiments to illustrate the inertial - effect and the computational performance of our proposed algorithm in comparison with the existing state of the art algorithms.

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1. Introduction

The *Variational inequality problem* (VIP) is an approach of finding a point $p \in E$ such that

$$\langle \mathcal{A}(p), x - p\rangle \geq 0, \qquad \forall\, x \in E, \tag{1.1}$$

where E is a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$, $\mathcal{A} : \mathcal{H} \longrightarrow \mathcal{H}$ is a monotone operator and $\langle \cdot, \cdot \rangle$ denotes an inner product space. It is a fundamental problem in optimization theory which is applied in many areas of study such as transportation problems, equilibrium, economics, engineering and so on (see [3, 4, 9, 11, 17, 19–21, 23, 27, 28, 32–35, 37, 43, 44]).

There are basically two approaches to the variational inequality (VIP) problem, namely, regularization and the projection method. Based on these, many study have been carried out and a number of algorithms have been considered and proposed (see for example [10, 12–15, 25, 26, 40, 41]).

In this study we are interested in the projection method. The basic idea of the projection method comes from extending the *gradient projection method* for minimizing a function $f(x)$ subject to $x \in E$ which is given by:

$$x^{n+1} = P_E\left(x^n - \lambda_n \bigtriangledown f(x^n)\right), \qquad \forall\, n \geq 1, \tag{1.2}$$

where $\{\lambda_n\}$ is a sequence of positive real numbers satisfying a particular condition and P_E is a metric projection onto E. Replacing the gradient operator $\bigtriangledown f(x^n)$ with an operator $\mathcal{A}$ gives an extension of the method to the problem (VIP), where a sequence $\{x^n\}$ is generated by the following scheme:

$$x^{n+1} = P_E\left(x^n - \lambda_n \mathcal{A} x^n\right), \qquad \forall\, n \geq 1. \tag{1.3}$$

However, a slightly strong assumption of strong monotonicity or strong inverse monotonicity needs to be placed on the operator $\mathcal{A}$ to guarantee the convergence of the sequence generated by this scheme (see [48]). To solve this problem, Korpelevich proposed the extragradient method for solving saddle point problem in [22], which was further extented to solving VI Problems.

$$\begin{cases} z^n = P_E\left(x^n - \lambda \mathcal{A} x^n\right), \\ x^{n+1} = P_E\left(x^n - \lambda \mathcal{A} z^n\right) \qquad n \geq 1. \end{cases} \tag{1.4}$$

The method requires only the operator $\mathcal{A}$ to be monotone and L - Lipschitz continuous for the convergence of the generated sequence $\{x^n\}$ with $\lambda \in (0, 1/L)$.

Actually, the extragradient method needs to compute two projections onto the set E in each iteration, which is going to be difficult in a situation where E is not simple to project onto. Censor *et al* introduced the *subgradient extragradient method* in [6, 7] to overcome this drawback, where he replaced the second projection with a projection onto a constructible half - space which has an explicit formula to compute. The scheme is in the following form:

$$\begin{cases} z^n = P_E\left(x^n - \lambda \mathcal{A} x^n\right), \\ T^n = \{x \in H : \langle x^n - \lambda \mathcal{A} x^n - z^n, x - z^n \rangle\} \\ x^{n+1} = P_{T^n}\left(x^n - \lambda \mathcal{A} z^n\right) \qquad n \geq 1. \end{cases} \tag{1.5}$$

The subgradient extragradient method have been studied, modified and improved by a lot of researches to come up with varient methods. Most of these modifications used a fixed or variable step size which depend on the factorials of the underlying operator such as strongly or strongly inverse modules and lipschitz constant, therefore those algorithms require the prior knowledge of such factorials to be implemented. In a situation where such constants are hard to compute or does not exist, such algorithms may be difficult or impossible to implement. Recently, some authors present some algorithms with variable steps size and independent of the strongly pseudomonotone and the lipschitzs constant in [18, 38, 42, 46].

Recently, inertial schemes have received increasing interests (see for instance [2, 13, 14, 29, 30, 42, 45, 49]). Similar to the most of inertial type algorithms, the sequences generated by these algorithms are established to be weakly convergent to the solution of the problems. However, in the paper [24] for fixed point problems and recently in [39] and [42] introduced an inertial type algorithms with strong convergence.

In this paper, motivated and inspired by the above works, we proposed an accelerated subgradient extragradient algorithm by incorporating the inertial extrapolation step. The aim of this modification is to obtain an algorithm with faster and strong convergence properties which performs better under mild conditions imposed on the parameters. Furthermore, we present several numerical examples to illustrate the perfomance and the effect of the inertial step when compared to the existing algorithms in the literature.

This paper is organized as follows: In Section , we give some definitions and lemmas which we will use in our convergence analysis. In Section we present the convergence analysis of our proposed algorithm and lastly, in Section , we illustrate the inertial effect and the computational performance of our algorithms by giving some examples.

2. Preliminaries

This section, recalls some known facts and necessary tools that we need for the convergence analysis of our method.

Throughout this article $\mathcal{H}$ is a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, E is a nonempty, closed and convex subset of $\mathcal{H}$. The notation $x^n \rightharpoonup x$ (resp $x^n \longrightarrow x$) is used to indicate that, respectively, the sequence $\{x^n\}$ converges weakly (strongly) to x. The following are known to hold in a Hilbert space:

$$\|x \pm \mathfrak{y}\|^2 = \|x\|^2 + \|\mathfrak{y}\|^2 \pm 2\langle x, \mathfrak{y} \rangle \tag{2.1}$$

and

$$\|\alpha x + (1-\alpha)\mathfrak{y}\|^2 = \alpha\|x\|^2 + (1-\alpha)\|\mathfrak{y}\|^2 - \alpha(1-\alpha)\|x-\mathfrak{y}\|^2 \tag{2.2}$$

for all $x, \mathfrak{y} \in \mathcal{H}$ and $\alpha \in \mathbb{R}$ [5].

Definition 2.1. Let $\mathcal{A} : \mathcal{H} \longrightarrow \mathcal{H}$ be a mapping defined on a real Hilbert space $\mathcal{H}$. $\mathcal{A}$ is said to be:

(1) *Strongly monotone* if there exists $\gamma > 0$ such that

$$\langle \mathcal{A}(x) - \mathcal{A}(\mathfrak{y})\rangle \geq \gamma\|x-\mathfrak{y}\|^2 \quad \forall\, x, \mathfrak{y} \in E.$$

(2) *Monotone* if

$$\langle \mathcal{A}(x) - \mathcal{A}(\mathfrak{y}), x - \mathfrak{y}\rangle \geq 0, \quad \forall\, x, \mathfrak{y} \in E.$$

(3) *Strongly pseudomonotone* if if there exists $\gamma > 0$ such that

$$\langle \mathcal{A}(x), \mathfrak{y} - x\rangle \geq 0, \quad \Longrightarrow \langle \mathcal{A}(\mathfrak{y}), x - \mathfrak{y}\rangle \leq -\gamma\|x-\mathfrak{y}\|^2 \quad \forall\, x, \mathfrak{y} \in E.$$

(4) L - *Lipschitz* continuous on $\mathcal{H}$ if there exists a constant $L > 0$ such that

$$\|\mathcal{A}x - \mathcal{A}\mathfrak{y}\| \leq L\,\|x-\mathfrak{y}\|\,, \quad \forall\, x, \mathfrak{y} \in \mathcal{H}.$$

Lemma 2.2. [5] *Let E be a closed convex subset of $\mathcal{H}$ and P_E be the metric projection from $\mathcal{H}$ onto E (i.e., for $x \in \mathcal{H}, \langle x - P_E x\rangle = \inf\{\langle x - \mathfrak{y}\rangle : \mathfrak{y} \in E\}$. Then, for any $x \in \mathcal{H}$, $\mathfrak{y} = P_E x$ if and only if there holds the relation:*

$$\langle x - \mathfrak{y}, y - \mathfrak{y}\rangle \leq 0, \quad \forall\, y \in E$$

Lemma 2.3. [1] *Let $\{\alpha^n\}$ and $\{\beta^n\}$ be a sequences of nonnegative real numbers, if $\sum_{n=1}^{\infty} \alpha^n = \infty$ and $\sum_{n=1}^{\infty} \alpha^n \beta^n \leq \infty$ then $\lim_{n\to\infty} \beta^n = 0$.*

Lemma 2.4. [31] *Let $\{\varphi^n\}$, $\{\delta^n\}$ and $\{\alpha^n\}$ be the sequences in $[0, +\infty)$ such that, for each $n \geq 1$,*

$$\varphi^{n+1} \leq \varphi^n + \alpha^n(\varphi^n - \varphi^{n-1}) + \delta^n, \quad \sum \delta^n < +\infty$$

and there exists a real number α with $0 \leq \alpha^n \leq \alpha \leq 1$ for all $n \geq 1$. Then the following conclusions hold:

(i) $\sum[\varphi^n - \varphi^{n-1}]_+ < +\infty$, *where* $[t]_+ = \max\{t, 0\}$*;*

(ii) there exists $\varphi^* \in [0, +\infty)$ *such that* $\lim \varphi^n = \varphi^*$.

3. An algorithm for strongly psedumonotone variational inequality problems

Assumption 3.1. The following conditions are assumed for the convergence of our method:

(A1) The feasible set E is a nonempty closed and convex subset of the real Hilbert space $\mathcal{H}$.

(A3) The solution set Ω of the problem *VIP* (1.1) is nonempty.

(A2) $\mathcal{A} : \mathcal{H} \longrightarrow \mathcal{H}$ is a strongly psedumonotone and L - *Lipschitz* on $\mathcal{H}$.

(A4) The real sequence $\{\alpha^n\}$ is a non-decreasing and $\{\lambda^n\} \subset (0, 1]$ is a sequence of positive real numbers with $\lim_{n\to\infty} \lambda^n = 0$ and $\sum_{n=1}^{\infty} \lambda^n = \infty$.

Algorithm 1 Accelerated Subgradient Extragradient Algorithm

Initialization: Choose $x^0, x^1 \in H$ and $t^0 = 1$, $\lambda^n \in \left(0, \sqrt{2} - 1\right)$,

Iterative Steps: Assume that $x^{n-1}, x^n \in H$, and t^n are known, calculate t^{n+1}, w^n and y^n as follows:

Step 1. Compute

$$t^{n+1} = \frac{-0.1 + \sqrt{1 + 4(t^n)^2}}{2}$$

and

$$w^n = x^n - \alpha^n (x^n - x^{n-1}),$$

where

$$\alpha^n = \frac{t^n - 1}{t^{n+1}}.$$

Step 2. Compute

$$y^n = Pc\left(w^n - \lambda^n A w^n\right).$$

Step 3. Construct

$$T^n = \left\{x \in H : \langle w^n - \lambda^n A w^n - y^n, x - y^n \rangle \leq 0\right\}.$$

Compute

$$x^{n+1} = P_{T^n}\left(w^n - \lambda^n A y^n\right).$$

If $y^n =: x^{n+1} = y^n$ then stop and y^n is a solution of problem (VIP), otherwise set $n = n + 1$ and go back to **Step 1.**

Lemma 3.2. *The sequence $\{t^n\}$ generated by Algorithm 1 is monotonically increasing and bounded from below.*

$$\frac{n+1}{2} \leq t^n \quad \forall n \leq 0$$

Lemma 3.3. *The following holds for each $p \in VI(A, C)$ and $n \geq 0$,*

$$\|x^{n+1} - p\|^2 \leq \|w^n - p\|^2 - (1 - L\lambda^n)\|w^n - y^n\|^2 - (1 - L\lambda^n)\|x^{n+1} - y^n\|^2 - \gamma\|y^n - p\|^2. \tag{3.1}$$

Proof. Let $t^n = w^n - \lambda^n A y^n$, then

$$\begin{aligned}
\|x^{n+1} - p\|^2 &= \|P_{T^n}(t^n) - p\|^2 \\
&= \langle P_{T^n}(t^n) - t^n + t^n - p, P_{T^n}(t^n) - t^n + t^n - p \rangle \\
&= \|t^n - p\|^2 + \|P_{T^n}(t^n) - t^n\|^2 + 2\left\langle P_{T^n}(t^n) - t^n, t^n - p \right\rangle
\end{aligned}$$

Notice that

$$
\begin{aligned}
2\|P_{T^n}(t_n) - t^n\|^2 &+ 2\left\langle P_{T^n}(t^n) - t^n, t^n - p\right\rangle, \\
&= 2\left\langle P_{T^n}(t^n) - t^n, P_{T^n}(t^n) - t^n\right\rangle + 2\left\langle P_{T^n}(t^n) - t^n, t^n - p\right\rangle, \\
&= 2\left\langle P_{T^n}(t^n) - t^n, P_{T^n}(t^n) - p\right\rangle, \\
&= 2\left\langle x^{n+1} - (w^n - \lambda^n Ay^n), x^{n+1} - (w^n - \lambda^n Ay^n)\right\rangle \le 0.
\end{aligned} \tag{3.2}
$$

Therefore,

$$
\|P_{T^n}(t_n) - t^n\|^2 + 2\left\langle P_{T^n}(t^n) - t^n, t^n - p\right\rangle \le -\|P_{T^n}(t_n) - t^n\|^2, \tag{3.3}
$$

Hence,

$$
\begin{aligned}
\|x^{n+1} - p\|^2 \le \|t^n - p\|^2 &- \|P_{T^n}(t_n) - t^n\|^2, \\
&= \|w^n - \lambda^n Ay^n - p\|^2 - \|x^{n+1} - (w^n - \lambda^n Ay^n)\|^2 \\
&= \|w^n - p\|^2 - \|x^{n+1} - w^n\|^2 + 2\lambda^n \left\langle Ay^n, p - x^{n+1}\right\rangle
\end{aligned}
$$

Since p is a solution of problem (VIP), $\langle Ap, x - p\rangle \ge 0$ for all $x \in C$. It follows from the strong monotonicity of A that $\langle Ax, x - p\rangle \le \|x - p\|^2$. Thus, taking $x := y^n$ we have for all $y^n \in C$

$$
\langle Ay^n, y^n - p\rangle \le \gamma\|y^n - p\|^2
$$

, this implies that

$$
\begin{aligned}
\left\langle Ay^n, y^n - x^{n+1}\right\rangle &= \langle Ay^n, p - y^n\rangle + \left\langle Ay^n, y^n - x^{n+1}\right\rangle \\
&\le \left\langle Ay^n, y^n - x^{n+1}\right\rangle - \|y^n - p\|^2
\end{aligned} \tag{3.4}
$$

Combining the relation 3 with the relation 3, we obtain

$$
\begin{aligned}
&\|x^{n+1} - p\|^2 \\
&= \|w^n - p\|^2 - \|x^{n+1} - w^n\|^2 + 2\lambda^n \left\langle Ay^n, y^n - x^{n+1}\right\rangle - 2\lambda^n\gamma\|y^n - p\|^2, \\
&= \|w^n - p\|^2 - \|x^{n+1} - y^n\|^2 - \|y^n - w^n\|^2 + 2\left\langle x^{n+1} - y^n, y^n - w^n\right\rangle, \\
&\quad + 2\lambda^n \left\langle Ay^n, y^n - x^{n+1}\right\rangle - 2\lambda^n\gamma\|y^n - p\|^2, \\
&= \|w^n - p\|^2 - \|x^{n+1} - y^n\|^2 - \|y^n - w^n\|^2 + 2\left\langle w^n - \lambda^n Ay^n - y^n, x^{n+1} - y^n\right\rangle, \\
&\quad - \gamma\|y^n - p\|^2.
\end{aligned}
$$

From the definition of y^n and the fact that $x^{n+1} \in T^n$, we have

$$
\begin{aligned}
2\left\langle w^n - \lambda^n Ay^n - y^n, x^{n+1} - y^n\right\rangle &= 2\left\langle w^n - \lambda^n Aw^n - y^n, x^{n+1} - y^n\right\rangle, \\
&\quad + 2\lambda^n \left\langle Aw^n - Ay^n - y^n, x^{n+1} - y^n\right\rangle, \\
&\le 2\lambda^n \left\langle Aw^n - Ay^n - y^n, x^{n+1} - y^n\right\rangle, \\
&\le 2L\lambda^n\|w^n - y^n\|\|x^{n+1} - y^n\|, \\
&\le L\lambda^n\|w^n - y^n\|^2 + L\lambda^n\|x^{n+1} - y^n\|^2.
\end{aligned} \tag{3.5}
$$

Now, from () and (), we have

$$\|x^{n+1}-p\|^2 \le \|w^n-p\|^2-(1-L\lambda^n)\|w^n-y^n\|^2-(1-L\lambda^n)\|x^{n+1}-y^n\|^2-\gamma\|y^n-p\|^2.$$

Hence, the proof. ■

Theorem 3.4. *The sequence x^n, y^n generated by Algorithm 1 converges strongly to some solution of problem (VIP).*

Proof. From Lemma 3.3, we have

$$\begin{aligned}
\|x^{n+1}-p\|^2 &\le \|w^n-p\|^2-(1-L\lambda^n)\|w^n-y^n\|^2-(1-L\lambda^n)\|x^{n+1}-y^n\|^2 \\
&\quad -\gamma\|y^n-p\|^2. \\
&\le \|w^n-p\|^2-(1-L\lambda^n)\|w^n-y^n\|^2-(1-L\lambda^n)\|x^{n+1}-y^n\|^2 \\
&\le \|w^n-p\|^2-(1-L\lambda^n)\left[\|w^n-y^n\|^2+\|x^{n+1}-y^n\|^2\right] \\
&\le \|w^n-p\|^2-\frac{(1-L\lambda^n)}{2}\left[\|w^n-y^n\|^2+\|x^{n+1}-y^n\|^2\right] \\
&\le \|w^n-p\|^2-\frac{(1-L\lambda^n)}{2}\|x^{n+1}-w^n\|^2
\end{aligned}$$

It follows from the Assumption that $\lim_{n\to\infty}\lambda^n=0$. Therefore, there exist $N\in\mathbb{N}$ such that $\lambda^n \le \frac{1}{2L}$ for all $n\ge N$. It now follows from (**??**) that there exist $N\in\mathbb{N}$ such that $\frac{1-L\lambda^n}{2}\ge\frac{1}{4}$ for all $n\ge N$. Thus,

$$\|x^{n+1}-p\|^2 \le \|w^n-p\|^2-\frac{1}{4}\|x^{n+1}-w^n\|^2. \tag{3.6}$$

By the definition of w^n, we have

$$\begin{aligned}
\|w^n-p\|^2 &= \|x^n-\alpha^n(x^n-x^{n-1})-p\|^2, \\
&= \|(1+\alpha^n)(x^n-p)-\alpha^n(x^{n-1}-p)\|^2, \\
&= (1+\alpha^n)\|x^n-p\|^2-\alpha^n\|x^{n-1}-p\|^2+\alpha^n(1+\alpha^n)\|x^n-x^{n-1}\|^2.
\end{aligned} \tag{3.7}$$

On the other hand, we have

$$\begin{aligned}
&\|x^{n+1}-w^n\|^2 \\
&= \|x^{n+1}-x^n\|^2+(\alpha^n)^2\|x^n-x^{n-1}\|^2-2\alpha^n\left\langle x^{n+1}-x^n, x^n-x^{n-1}\right\rangle, \\
&\ge \|x^{n+1}-x^n\|^2+(\alpha^n)^2\|x^n-x^{n-1}\|^2-2\alpha^n\|x^{n+1}-x^n\|\|x^n-x^{n-1}\|, \\
&\ge (1-\alpha^n)\|x^{n+1}-x^n\|^2+((\alpha^n)^2-\alpha^n)\|x^n-x^{n-1}\|^2.
\end{aligned} \tag{3.8}$$

Combining (3.6), (3.7) and (3.8), we have

$$
\begin{aligned}
\|x^{n+1}-p\|^2 \leq\ & (1+\alpha^n)\|x^n-p\|^2 - \alpha^n\|x^{n-1}-p\|^2 + \alpha^n(1+\alpha^n)\|x^n-x^{n-1}\|^2 \\
& - \frac{1}{4}(1-\alpha^n)\|x^{n+1}-x^n\|^2 - \frac{1}{4}((\alpha^n)^2-\alpha^n)\|x^n-x^{n-1}\|^2, \\
=\ & (1+\alpha^n)\|x^n-p\|^2 - \alpha^n\|x^{n-1}-p\|^2 - \frac{1}{4}(1-\alpha^n)\|x^{n+1}-x^n\|^2 \\
& + \left[\alpha^n(1+\alpha^n) - \frac{1}{4}((\alpha^n)^2-\alpha^n)\right]\|x^n-x^{n-1}\|^2, \\
=\ & (1+\alpha^n)\|x^n-p\|^2 - \alpha^n\|x^{n-1}-p\|^2 - \frac{1}{4}(1-\alpha^n)\|x^{n+1}-x^n\|^2 \\
& + \left[\frac{3}{4}(\alpha^n)^2 + \frac{5}{4}\alpha^n\right]\|x^n-x^{n-1}\|^2, \\
=\ & (1+\alpha^n)\|x^n-p\|^2 - \alpha^n\|x^{n-1}-p\|^2 - \beta^n\|x^{n+1}-x^n\|^2 \\
& + \gamma^n\|x^n-x^{n-1}\|^2,
\end{aligned}
\tag{3.9}
$$

where $\beta^n = \frac{1}{4}(1-\alpha^n) \geq 0$ and $\gamma^n = \frac{3}{4}(\alpha^n)^2 + \frac{5}{4}\alpha^n \geq 0$.

Put $\Gamma^n = \|x^n-p\|^2 - \alpha^n\|x^{n-1}-p\|^2 + \gamma^n\|x^n-x^{n-1}\|^2$, it follows from (3.9) and the fact that the sequence α^n is a non-decreasing sequence that,

$$
\begin{aligned}
\Gamma^{n+1} - \Gamma^n =\ & \|x^{n+1}-p\|^2 - (1+\alpha^{n+1})\|x^{n-1}-p\|^2 + \alpha^n\|x^{n-1}-p\|^2, \\
& + \gamma^{n+1}\|x^{n+1}-x^n\|^2 - \gamma^n\|x^n-x^{n-1}\|^2, \\
=\ & \|x^{n+1}-p\|^2 - (1+\alpha^n)\|x^{n-1}-p\|^2 + \alpha^n\|x^{n-1}-p\|^2, \\
& + \gamma^{n+1}\|x^{n+1}-x^n\|^2 - \gamma^n\|x^n-x^{n-1}\|^2, \\
\leq\ & -\beta^n\|x^{n+1}-x^n\|^2 + \gamma^{n+1}\|x^{n+1}-x^n\|^2, \\
=\ & -(\beta^n-\gamma^{n+1})\|x^{n+1}-x^n\|^2.
\end{aligned}
\tag{3.10}
$$

It can be seen that for $n \geq N$, we have

$$
\begin{aligned}
\beta^n - \gamma^{n+1} =\ & \frac{1}{4}(1-\alpha^n) - \frac{3}{4}(\alpha^{n+1})^2 - \frac{5}{4}\alpha^{n+1}, \\
\geq\ & \frac{1}{4}(1-\alpha^{n+1}) - \frac{3}{4}(\alpha^{n+1})^2 - \frac{5}{4}\alpha^{n+1}, \\
\geq\ & \frac{1}{4}(1-\alpha) - \frac{3}{4}\alpha^2 - \frac{5}{4}\alpha, \\
\geq\ & \frac{1}{4} - \frac{6}{4}\alpha - \frac{3}{4}\alpha^2.
\end{aligned}
\tag{3.11}
$$

It follows from (3.10) and (3.11) that,

$$
\Gamma^{n+1} - \Gamma^n \leq -\tau\|x^{n+1}-x^n\|, \tag{3.12}
$$

where $\tau = \frac{1}{4} - \frac{6}{4}\alpha - \frac{3}{4}\alpha^2 \geq 0$. This implies that

$$
\Gamma^{n+1} - \Gamma^n \leq 0. \tag{3.13}
$$

It now follows that the sequence $\{\Gamma^n\}$ is a nonincreasing sequence. On the other hand, we have

$$\begin{aligned}\Gamma^n &= \|x^n - p\|^2 - \alpha^n\|x^{n-1} - p\|^2 + \gamma^n\|x^n - x^{n-1}\|, \\ &\geq \|x^n - p\|^2 - \alpha^n\|x^{n-1} - p\|^2.\end{aligned} \tag{3.14}$$

Therefore, for all $n \geq N$, we obtain

$$\begin{aligned}\|x^n - p\|^2 &\leq \alpha^n\|x^{n-1} - p\|^2 + \Gamma^n, \\ &\leq \alpha\|x^{n-1} - p\|^2 + \Gamma^N, \\ &\leq \ldots \leq \alpha^n\|x^N - p\|^2 + \Gamma^N(\alpha^{n-1} + \ldots + 1), \\ &\leq \alpha^{n-N}\|x^N - p\|^2 + \frac{\Gamma^N}{1-\alpha}.\end{aligned} \tag{3.15}$$

Also we have,

$$\begin{aligned}\Gamma^{n+1} &= \|x^{n+1} - p\|^2 - \alpha^{n+1}\|x^n - p\|^2 + \gamma^{n+1}\|x^{n+1} - x^n\|, \\ &\geq -\alpha^{n+1}\|x^n - p\|^2.\end{aligned} \tag{3.16}$$

It follows from (3.15) and (3.16), we have

$$\begin{aligned}-\Gamma^{n+1} &\leq \alpha^{n+1}\|x^n - p\|^2 \leq \alpha^n\|x^n - p\|^2 \leq \alpha^{n-N+1}\|x^N - p\|^2 + \frac{\alpha\Gamma^N}{1-\alpha}, \\ &\leq \|x^N - p\|^2 + \frac{\alpha\Gamma^N}{1-\alpha}.\end{aligned} \tag{3.17}$$

It now follows from (3.12) and (3.17) that

$$\tau\sum_{n=N}^{k}\|x^{n+1} - x^n\|^2 \leq \Gamma^N - \Gamma^{k+1} \leq \|x^N - p\|^2 + \frac{\Gamma^N}{1-\alpha}. \tag{3.18}$$

Letting $k \to \infty$, we have

$$\sum_{n=N}^{k}\|x^{n+1} - x^n\|^2 \leq +\infty, \tag{3.19}$$

therefore, we have

$$\|x^{n+1} - x^n\|^2 \to 0. \tag{3.20}$$

From (3.8), we have

$$\|x^{n+1} - w^n\|^2 = \|x^{n+1} - x^n\|^2 + (\alpha^n)^2\|x^n - x^{n-1}\|^2 - 2\alpha^n\left\langle x^{n+1} - x^n, x^n - x^{n-1}\right\rangle, \tag{3.21}$$

Therefore,

$$\|x^{n+1} - w^n\|^2 \to 0. \tag{3.22}$$

It follows from (3.9), (3.19) and Lemma (cite) we have

$$\lim_{n\to\infty}\|x^n - p\|^2 \to \infty. \tag{3.23}$$

By (3.7) we have

$$\lim_{n\to\infty} \|w^n - p\|^2 \to \infty. \tag{3.24}$$

From (3.20) and (3.22), we have

$$0 \le \|x^n - w^n\| = \|x^n - x^{n+1}\| + \|x^{n+1} - w^n\|. \tag{3.25}$$

It follows from (3.1), (3.23) and (3.24) thtat

$$(1 - \lambda^n L)\|y^n - w^n\|^2 \le \|w^n - p\|^2 - \|x^{n+1} - p\|^2 \to 0. \tag{3.26}$$

By (3.1) and (3.25) we get

$$\lim_{n\to\infty} \|y^n - w^n\| = 0. \tag{3.27}$$

From (3.25) and (3.27) we have

$$\lim_{n\to\infty} \|x^n - y^n\| \le \lim_{n\to\infty} \|x^n - w^n\| + \lim_{n\to\infty} \|w^n - y^n\| = 0. \tag{3.28}$$

∎

We can now show that the sequence $\{x^n\}$ converges strongly to p, It follows from (3.1) that

$$\begin{aligned}
2\lambda^n\gamma\|y^n - p\|^2 &\le -\|x^{n+1} - p\|^2 + \|w^n - p\|^2, \\
&\le -\|x^{n+1} - p\|^2 + (1+\alpha^n)\|x^n - p\|^2 - \alpha^n\|x^{n-1} - p\|^2 \\
&\quad + \alpha^n(1+\alpha^n)\|x^n - x^{n-1}\|^2, \\
&\le (\|x^n - p\|^2 - \|x^{n+1} - p\|^2) + 2\alpha\|x^n - x^{n-1}\|^2 \\
&\quad + (\alpha^n\|x^n - p\|^2 - \alpha^{n-1}\|x^{n-1} - p\|^2).
\end{aligned} \tag{3.29}$$

This implies

$$\begin{aligned}
\sum_{n=1}^{k} 2\lambda^n\gamma\|y^n - p\|^2 &\le \|x^1 - p\|^2 - \|x^{k+1} - p\|^2 + \alpha^k\|x^k - p\|^2 - \alpha^0\|x^0 - p\|^2 \\
&\quad + \sum_{n=1}^{k} 2\alpha\|x^n - x^{n-1}\|^2 \\
&\le \|x^1 - p\|^2 + \alpha^k\|x^k - p\|^2 + \sum_{n=1}^{k} 2\alpha\|x^n - x^{n-1}\|^2, \\
&\le M,
\end{aligned}$$

for some $M > 0$, therefore,

$$\sum_{n=1}^{k} 2\lambda^n\gamma\|y^n - p\|^2 \le +\infty.$$

It follows from the assumption that $\sum_{n=1}^{\infty} \lambda^n = \infty$ and Lemma (cite) that

$$\lim_{n\to\infty} \inf \|y^n - p\| = 0. \tag{3.30}$$

It follows from (3.29) that there exist a subsequence $\{y^{n_k}\}$ of $\{y^n\}$ such that

$$\lim_{n\to\infty} \|y^{n_k} - p\| = 0.$$

Table 1. The numerical results for example 4.1

		EGM		ISEGM		MISEGM	
λ_n	m	`Iter.`	`CPU(s)`	`Iter.`	`CPU(s)`	`Iter.`	`CPU(s)`
$\frac{1}{n+1}$	50	3046	67.6559	2501	23.2553	2294	21.5560
$\frac{1}{(n+1)\log(n+3)}$	50	7367	143.1685	3900	38.7113	3014	29.3706
$\frac{\log(n+3)}{(n+1)}$	50	2119	21.9511	1655	18.0232	1122	11.9152

Since $\lim_{n\to\infty}\|x^n - y^n\| = 0$, we get that $\lim_{n\to\infty}\|x^{n_k} - p\| = 0$. On the other hand we have $\lim_{n\to\infty}\|y^n - p\| = 0 \in \mathbb{R}$. Therefore, $x^n \to 0$ as $n \to \infty$. Hence, the proof.

4. Numerical Illustrations

Some numerical results will be presented in this section to examine the convergence of the algorithm 1 compared to the existing algorithms. The MATLAB program was run on a PC (with Intel(R) Core(TM)i5-6200U CPU @ 2.30GHz 2.40GHz, RAM 8.00 GB) in MATLAB version 9.5 (R2018b).

- Hieu algorithm 1 [16] (shortly, `EGM`) and $D_n = \|x_n - y_n\|^2$.
- Hieu algorithm 1 [42] (shortly, `ISEGM`), $\alpha_n = \frac{1}{10}$ and $D_n = \|w_n - y_n\|^2$.
- Our proposed algorithm 1 (shortly, `MISEGM`) and $D_n = \|w_n - y_n\|^2$.

Example 4.1. Consider the linear operator $F(x) := Mx+q$ where which is taken from [**?**] and has been considered by many authors for numerical experiments, see, for example, [47], where

$$M = BB^T + S + D$$

and B is an $m \times m$ matrix, S is an $m \times m$ skew-symmetric matrix, and D is an $m \times m$ diagonal matrix, whose diagonal entries are nonnegative. The feasible set $C \subset \mathbb{R}^m$ is closed and convex and defined as

$$C = \{x \in \mathbb{R}^m : -5 \le x_i \le 5\}.$$

For experiments, the starting point $x_{-1} = x_0 = (1, 1, \cdots, 1)^T \in \mathbb{R}^m$ and the matrices B, S, q are randomly generated in the interval $(-2, 2)$ and D is randomly generated in the interval $(0, 2)$. Figures 1-6 and table 1 illustrate the comparison of our proposed algorithm.

Example 4.2. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$F(x_1, x_2) = (x_1 + x_2 + \sin x_1; -x_1 + x_2 + \sin x_2) \quad \forall x \in \mathbb{R}^2,$$

and let $C = \{x \in \mathbb{R}^2 : -5 \le x_i \le 5\}$. It is not hard to check that F is strongly pseudomonotone and Lipschitz continuous. During this experiment 4.2, we take $x_{-1} = x_0 = (1, 1)^T$. Figures 7-12 and table 2 illustrate the comparison of our proposed algorithm.

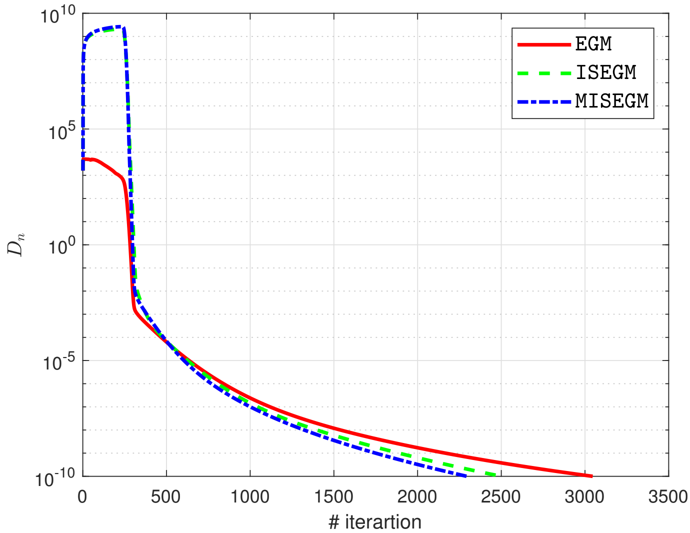

FIGURE 1. Example 4.1 for $\lambda_n = \frac{1}{n+1}$.

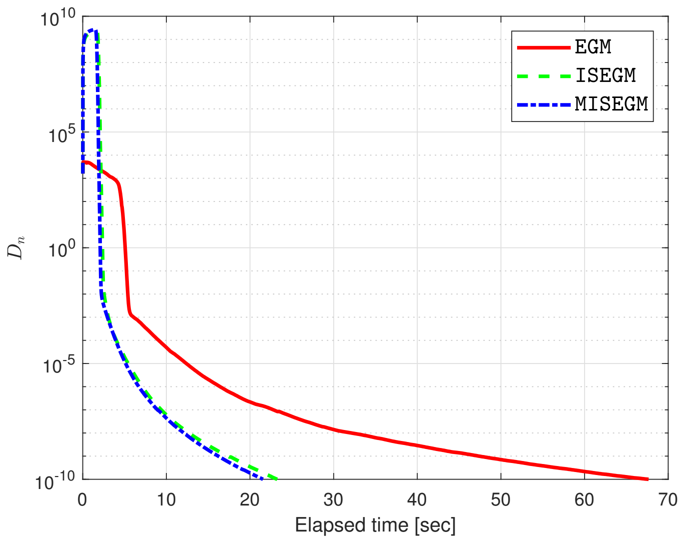

FIGURE 2. Example 4.1 for $\lambda_n = \frac{1}{n+1}$.

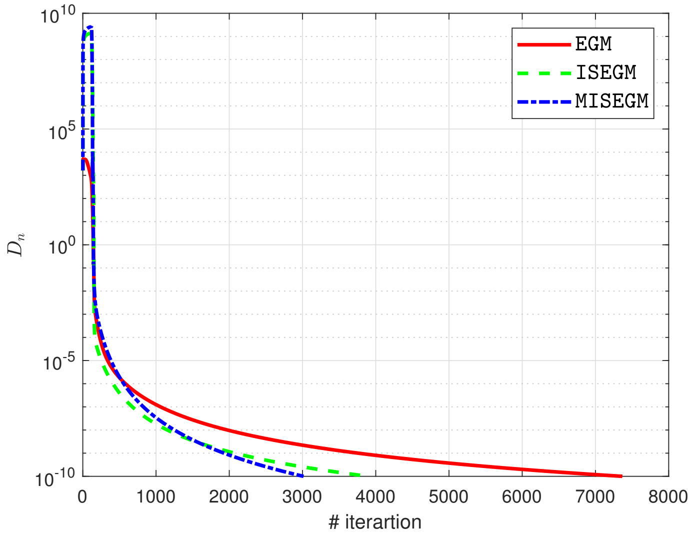

FIGURE 3. Example 4.1 for $\lambda_n = \frac{1}{(n+1)\log(n+3)}$.

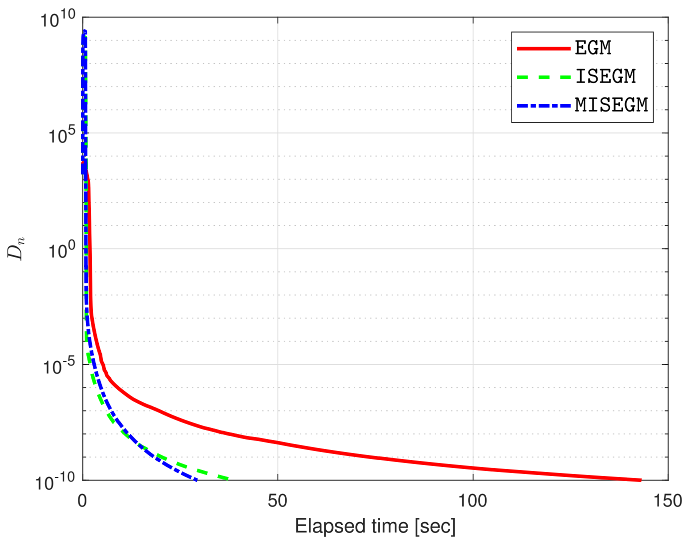

FIGURE 4. Example 4.1 for $\lambda_n = \frac{1}{(n+1)\log(n+3)}$.

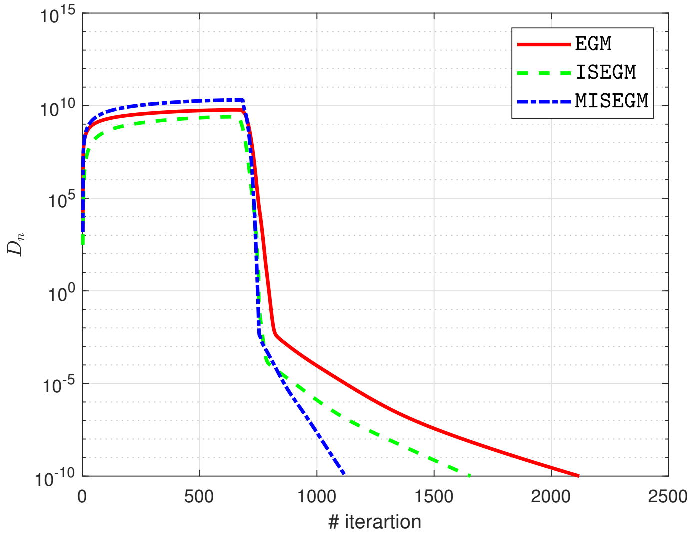

FIGURE 5. Example 4.1 for $\lambda_n = \frac{\log(n+3)}{(n+1)}$.

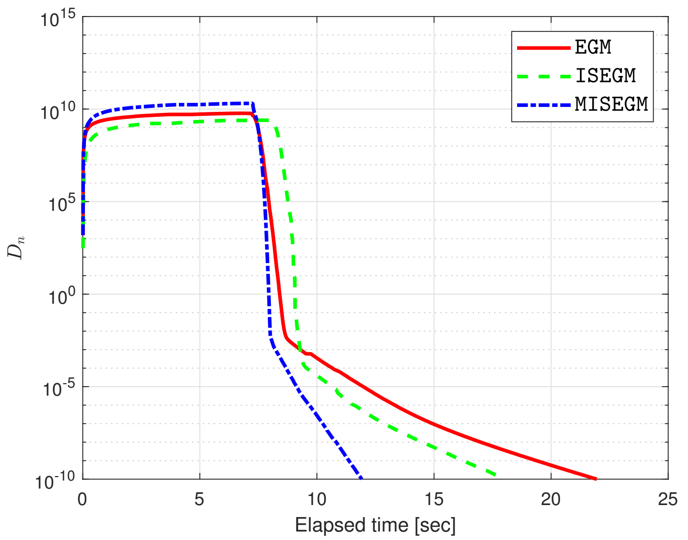

FIGURE 6. Example 4.1 for $\lambda_n = \frac{\log(n+3)}{(n+1)}$.

TABLE 2. The numerical results for example 4.2

		EGM		ISEGM		MISEGM	
λ_n	TOL	Iter.	CPU(s)	Iter.	CPU(s)	Iter.	CPU(s)
$\frac{1}{n+1}$	10^{-15}	1155	21.4736	598	4.7345	431	3.1472
$\frac{1}{(n+1)\log(n+3)}$	10^{-15}	8383	164.8228	6095	59.8131	4800	44.5205
$\frac{\log(n+3)}{(n+1)}$	10^{-20}	680	12.0765	417	3.0595	319	2.1016

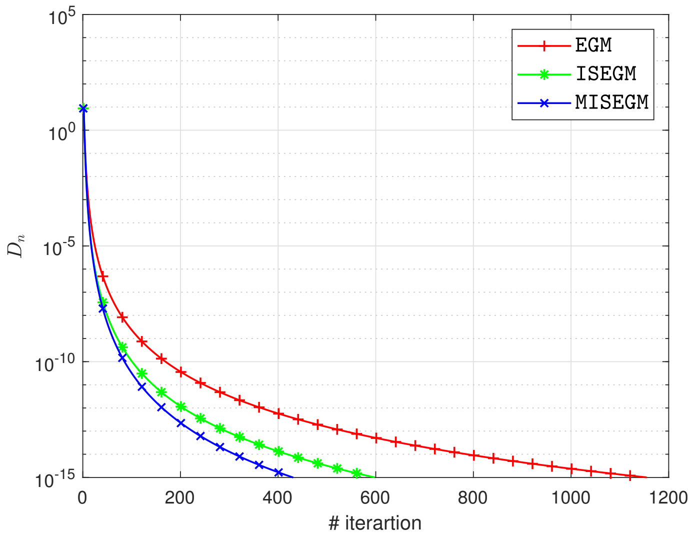

FIGURE 7. Example 4.2 for $\lambda_n = \frac{1}{n+1}$.

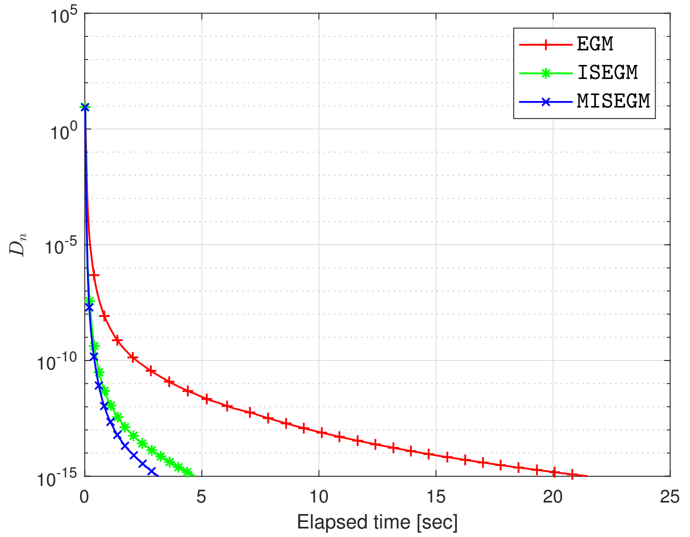

FIGURE 8. Example 4.2 for $\lambda_n = \frac{1}{n+1}$.

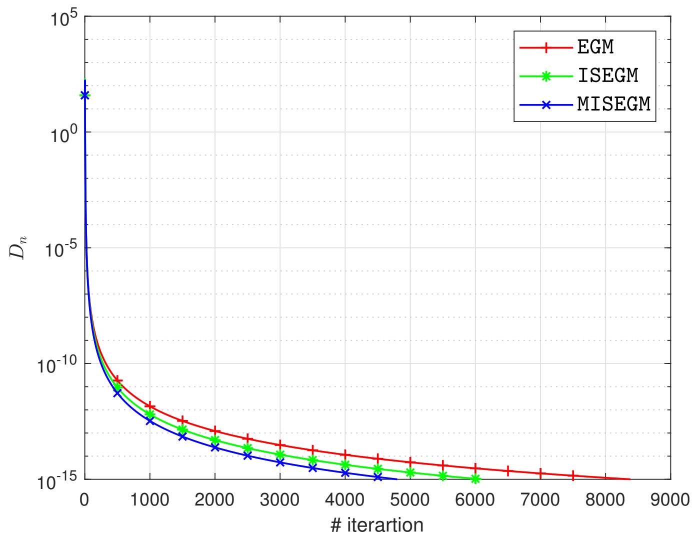

FIGURE 9. Example 4.2 for $\lambda_n = \frac{1}{(n+1)\log(n+3)}$.

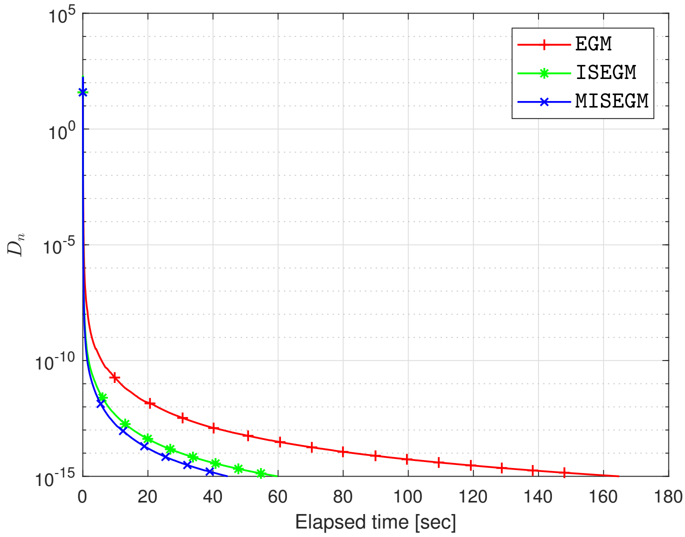

FIGURE 10. Example 4.2 for $\lambda_n = \frac{1}{(n+1)\log(n+3)}$.

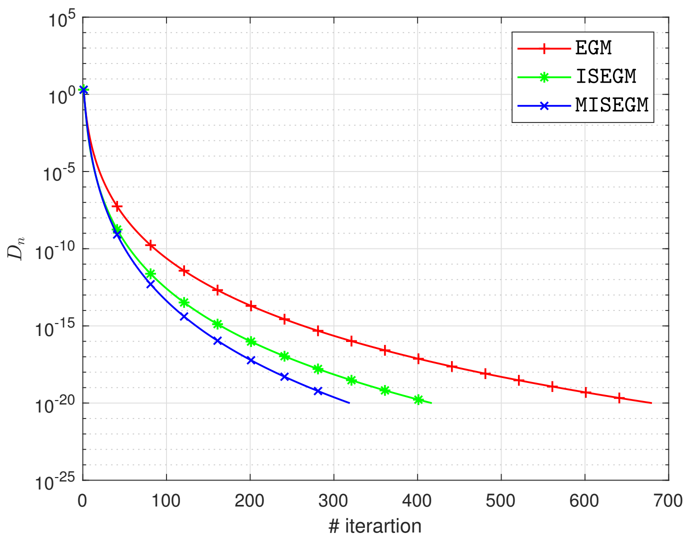

FIGURE 11. Example 4.2 for $\lambda_n = \frac{\log(n+3)}{n+1)}$.

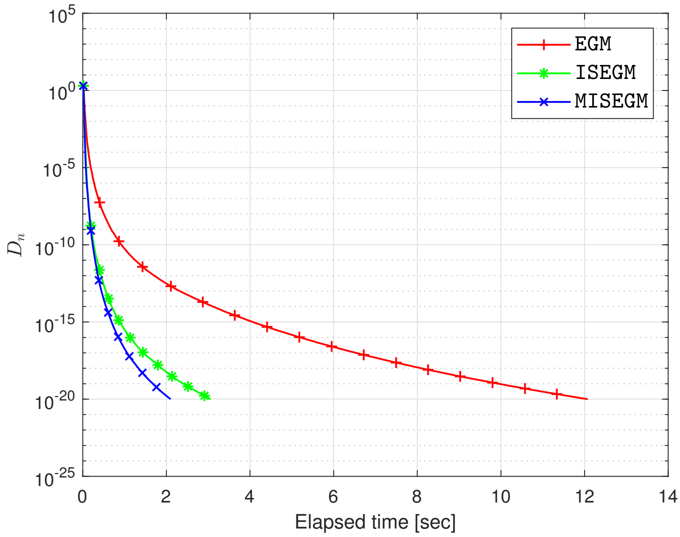

FIGURE 12. Example 4.2 for $\lambda_n = \frac{\log(n+3)}{n+1}$.

5. Conclusion

In this article, we presented a strong convergence inertial projection algorithm by incorporating the inertial term with the subgradient extragradient method for variational inequality problems. We have shown that the sequence generated by our proposed algorithm converges strongly under mild assumptions imposed on the underlying operator. The proposed algorithm is one of the few inertial algorithms whose iterates converge strongly to the solution of the given problem. We also presented some numerical examples to show the computational performance of our proposed algorithm. Moreover we compared the proposed algorithm with other algorithms in the literature, our proposed algorithms performed better in both number of iteration and computational time compared to these algorithms. As a future research, we will consider modifications of the proposed algorithm by introducing line search procedure such as the Armijo line search. We will give more computational experiments and consider probems in infinite Hilbert dimensional space.

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