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HYBRID EXTRAGRADIENT SCHEME FOR SPLIT VARIATIONAL INCLUSION IN HILBERT SPACES

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Abstract In this paper, we propose a hybrid extragradient method to approximate a common solution set of split variational inclusion problem, a general system of variational inequalities and fixed point set of a k-strictly pseudocontractive mapping in real Hilbert space. Our hybrid method is based on the well-known extragradient method, viscosity approximation method, and Mann-type iteration method. Strong convergence theorem is established under some suitable conditions in a real Hilbert space. Results presented in this paper may be viewed as a refinement and important generalizations of the previously known results announced by many other authors.

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1. Introduction

Let H_1 and H_2 be real Hilbert spaces with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let C and Q be nonempty closed convex subsets of H_1 and H_2, respectively. Let $\{x_n\}$ be a sequence in H_1, then $x_n \to x$ (respectively, $x_n \rightharpoonup x$) will denote strong (respectively, weak) convergence of the sequence $\{x_n\}$. A mapping $S : C \to C$ is called nonexpansive if $\|Sx - Sy\| \leq \|x - y\|, \forall x, y \in C$.

The fixed point problem (FPP) for the mapping S is to find $x \in C$ such that

$$Sx = x. \tag{1.1}$$

We denote $Fix(S) := \{x \in C : Sx = x\}$, the set of solutions of FPP.

Assume throughout the paper that S is a nonexpansive mapping such that $Fix(S) \neq \emptyset$. Recall that a self-mapping $f : C \to C$ is a contraction on C if there exists a constant $\alpha \in (0, 1)$ and $x, y \in C$ such that $\|f(x) - f(y)\| \leq \alpha \|x - y\|$.

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Given a nonlinear mapping $A : C \to H_1$. Then the variational inequality problem (VIP) is to find $u \in C$ such that

$$\langle Au, v - u\rangle \geq 0, \ \forall v \in C. \tag{1.2}$$

The solution of VIP (1.2) is denoted by $VI(A, C)$. It is well known that if A is strongly monotone and Lipschitz continuous mapping on C then VIP (1.2) has a unique solution. There are several different approaches towards solving this problem in finite dimensional and infinite dimensional spaces see [1–3] and the research in this direction is intensively continued. Then VIP satisfies the following Lemma;

Lemma 1.1. *For a given $z \in H_1, u \in C$ satisfies the inequality*

$$\langle u - z, v - u\rangle \geq 0, \ \forall v \in C, \ \textit{iff } u = P_C z, \tag{1.3}$$

where P_C is the projection of H_1 onto a closed convex set C.

For finding an element of $Fix(S) \cap VI(A, C)$ when C is closed and convex, S is nonexpansive and A is α-inverse strongly monotone, Takashi and Toyoda [4] introduced the following Mann-type itereative algorithm:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S P_C(x_n - \lambda_n A x_n), \ \forall n \geq 0, \tag{1.4}$$

where S is nonexpansive P_C is the metric projection of H onto $C, x_0 = x \in C, \{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\lambda_n\}$ is a sequence in $(0, 2\alpha)$. They showed that, if $Fix(S) \cap VI(A, C) \neq \emptyset$, then the sequence $\{x_n\}$ converges weakly to some $z \in Fix(S) \cap VI(A, C)$. Nadezhkina and Takahashi [5] and Zeng and Yao [6] propose extragardient methods motivated by Korpelevič [7] for finding a common element of the fixed point set of a nonexpansive mapping and the solution set of a variational inequality problem.

Let $D_1, D_2 : C \to H$ be two mappings. Now we consider the following problem of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{aligned} &\langle \mu_1 D_1 y^* + x^* - y^*, x - x^*\rangle \geq 0, \ \forall x \in C \\ &\langle \mu_2 D_2 x^* + y^* - x^*, x - y^*\rangle \geq 0, \ \forall x \in C, \end{aligned} \tag{1.5}$$

which is called a general system of variational inequalities where $\mu_1 > 0$ and $\mu_2 > 0$ are two constants. The set of solutions of problem (1.5) is denoted by $GSVI(D_1, D_2, C)$. In particular, if $D_1 = D_2 = A$, then problem (1.5) reduces to the problem of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{aligned} &\langle \mu_1 A y^* + x^* - y^*, x - x^*\rangle \geq 0, \ \forall x \in C \\ &\langle \mu_2 A x^* + y^* - x^*, x - y^*\rangle \geq 0, \ \forall x \in C, \end{aligned} \tag{1.6}$$

which was defined by Verma [8] and is called the new system of variational inequalities. Further, if $x^* = y^*$ additionally, the problem (1.6) reduces to the classical variational inequality problem (1.2).

Ceng et al. [9] studied the problem (1.5) by transforming it into a fixed-point problem. Precisely and for easy reference, we state their results in following lemma and theorem.

Lemma CWY [9] For given $\bar{x}, \bar{y} \in C, (\bar{x}, \bar{y})$ is a solution of (1.5) if and only if $\bar{x}$ is a fixed point of the mapping $G : C \to C$ defined by

$$G(x) = P_C[P_C(x - \mu_1 D_2 x) - \mu_1 D_1 P_C(x - \mu_2 D_2 x)], \ \forall x \in C, \tag{1.7}$$

where $\bar{y} = P_C(\bar{x} - \mu_2 D_2 \bar{x})$. In particular, if the mapping $D_i : C \to H$ is μ_i-inverse strongly monotone for $i = 1, 2$, then the mapping G is nonexpansive provided $\mu_i \in (0, 2\mu_i)$ for $i = 1, 2$.

Throughout this paper, the fixed-point set of the mapping G is denoted by $\mathcal{G}$. Utilizing Lemma CWY, they introduced and studied a relaxed extragradient method for solving problem (1.5).

Theorem CWY [9] Let C be a nonempty closed convex subset of a real Hilbert space H. Let the mapping $D_i : C \to H$ be η_i-inverse strongly monotone for $i = 1, 2$. Let $S : C \to H$ be a nonexpansive mapping with $Fix(S) \cap \mathcal{G} \neq \emptyset$. Suppose $x_1 = u \in C$ and $\{x_n\}$ is generated by

$$\begin{aligned} & y_n = P_C(x_n - \mu_2 D_2 x_n), \\ & x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n S P_C(y_n - \mu_1 D_1 y_n), \end{aligned} \tag{1.8}$$

where $\mu_i \in (0, 2\eta_i)$ for $i = 1, 2$, and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are three sequences in $[0, 1]$ such that

(i) $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \geq 1$;

(ii) $\lim\limits_{n\to\infty} \alpha_n = 0, \ \sum\limits_{n=1}^{\infty} \alpha_n = \infty$;

(iii) $0 < \liminf\limits_{n\to\infty} \beta_n \leq \limsup\limits_{n\to\infty} \beta_n < 1$.

Then $\{x_n\}$ converges strongly to $\bar{x} = P_{Fix(S)\cap\mathcal{G}} u$ and $(\bar{x}, \bar{y})$ is a solution of problem (1.5), where $\bar{y} = P_C(\bar{x} - \mu_2 D_2 \bar{x})$.

It is clear that the above result unifies and extends some corresponding result in the literature.

Based on the relaxed extragradient method and viscosity approximation method, Yao et al. [10] proposed and analyzed an iterative algorithm for finding a common element of strictly pseudocontractive mapping in a real Hilbert space H.

Theorem YLK [10] Let C be a nonempty closed convex subset of a real Hilbert space H. Let the mapping $D_i : C \to H$ be η_i-inverse strongly monotone for $i = 1, 2$. Let $S : C \to H$ be a k-strictly pseudocontractive mapping with $Fix(S) \cap \mathcal{G} \neq \emptyset$. Let $Q : C \to C$ be a ρ-contraction with $\rho \in [0, \frac{1}{2})$. For given $x_0 \in C$ arbitrarily, let $\{x_n\}, \{y_n\}$, and $\{z_n\}$ be generated iterative by

$$\begin{aligned} & z_n = P_C(x_n - \mu_2 D_2 x_n), \\ & y_n = \alpha_n Q x_n + (1 - \alpha_n) P_C(z_n - \mu_1 D_1 z_n), \\ & x_{n+1} = \beta_n x_n + \gamma_n P_C(z_n - \mu_1 D_1 z_n) + \delta_n S y_n, \ \forall n \geq 0, . \end{aligned} \tag{1.9}$$

where $\mu_i \in (0, 2\eta_i)$ for $i = 1, 2$, and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\}$ are four sequences in $[0, 1]$ such that

(i) $\beta_n + \gamma_n + \delta_n = 1$ and $(\gamma_n + \delta_n)k \leq \gamma_n < (1 - 2\rho)\delta_n$ for all $n \geq 0$;

(ii) $\lim\limits_{n\to\infty} \alpha_n = 0, \ \sum\limits_{n=1}^{\infty} \alpha_n = \infty$;

(iii) $0 < \liminf\limits_{n\to\infty} \beta_n \leq \limsup\limits_{n\to\infty} \beta_n < 1$ and $\liminf\limits_{n\to\infty} \delta_n > 0$;

(iv) $\lim\limits_{n\to\infty} (\gamma_{n+1}/(1 - \beta_{n+1}) - \gamma_n/(1 - \beta_n)) = 0$.

Then the sequence $\{x_n\}$ generated by (1.9) converges strongly to $\bar{x} = P_{Fix(S)\cap\mathcal{G}} Q \bar{x}$ and $(\bar{x}, \bar{y})$ is a solution of the general system (1.5) of variational inequalities, where $\bar{y} = P_C(\bar{x} - \mu_2 D_2 \bar{x})$.

Recall also a multi-valued mapping $M : H_1 \to 2^{H_1}$ is called monotone if, for all $x, y \in H_1, u \in Mx$ and $v \in My$ such that

$$\langle x - y, u - v\rangle \geq 0.$$

A monotone mapping M is maximal if the Graph(M) is not properly contained in the graph of any other monotone mapping. It is well known that a monotone mapping M is maximal if and only if for $(x, u) \in H_1 \times H_1, \langle x-y, u-v\rangle \geq 0$ for every $(y, v) \in$ Graph(M) implies that $u \in Mx$.

Let $M : H_1 \to 2^{H_1}$ be a multi-valued maximal monotone mapping. Then the resolvent mapping $J_\lambda^M : H_1 \to H_1$ associated with M is defined by

$$J_\lambda^M(x) := (I + \lambda M)^{-1}(x),\ \forall x \in H_1,$$

for some $\lambda > 0$, where I stands for the identity operator on H_1. Note that for all $\lambda > 0$ the resolvent operator J_λ^M is single-valued, nonexpansive, and firmly nonexpansive.

In 2011, Moudafi [11] introduced the following split monotone variational inclusion problem: Find $x^* \in H_1$ such that

$$\begin{cases} 0 \in f_1(x^*) + B_1(x^*), \\ y^* = Ax^* \in H_2 :\ 0 \in f_2(y^*) + B_2(y^*), \end{cases} \tag{1.10}$$

where $B_1 : H_1 \to 2^{H_1}, B_2 : H_2 \to 2^{H_2}$ are multi-valued maximal monotone mappings and A is bounded linear operator, $f_1 : H_1 \to H_1$ and $f_2 : H_2 \to H_2$ are two given operators.

The split monotone variational inclusion problem (1.10) includes as special cases: the split common fixed point problem, the split variational inequality problem, the split zero problem, and the split feasibility problem, which have already been studied and used in practice as a model in intensity-modulated radiation therapy treatment planning, see [12] This formalism is also at the core of the modeling of many inverse problems arising for phase retrieval and other real-world problems; for instance, in sensor networks in computerized tomography and data compression.

If $f_1 \equiv 0$ and $f_2 \equiv 0$, the problem (1.10) reduces to the following split variational inclusion problem: Find $x^* \in H_1$ such that

$$\begin{cases} 0 \in B_1(x^*), \\ y^* = Ax^* \in H_2 :\ 0 \in B_2(y^*), \end{cases} \tag{1.11}$$

which constitutes a pair of variational inclusion problems connected with a bounded linear operator A in two different Hilbert spaces H_1 and H_2. The solution set of problem (1.11) is denoted by $\bar{\Gamma} = \{x^* \in H_1 : 0 \in B_1(x^*), y^* = Ax^* \in H_2 : 0 \in B_2(y^*)\}$.

Very recently, Byrne et al. [13] studied the weak and strong convergence of the following iterative method for problem (1.11): For given $x_0 \in H_1$ and $\lambda > 0$, compute iterative sequence $\{x_n\}$ generated by the following scheme:

$$x_{n+1} = J_\lambda^{B_1}(x_n + \epsilon A^*(J_\lambda^{B_2} - I)Ax_n). \tag{1.12}$$

In 2013, Kazmi and Rivi [14] modified scheme (1.11) to the case of a split variational inclusion and the fixed point problem of a nonexpansive mapping. To be more precise, they proved the following strong convergence theorem.

Theorem KR [14] Let H_1 and H_2 be two real Hilbert spaces and $A : H_1 \to H_2$ be a bounded linear operator. Let $f : H_1 \to H_1$ be a contraction mapping with constant $\rho \in (0, 1)$ and $T : H_1 \to H_1$ be a nonexpansive mapping such that $\Omega = Fix(T) \cap \bar{\Gamma} \neq \emptyset$.

For a given $x_0 \in H_1$ arbitrarily, let the iterative sequences $\{u_n\}$ and $\{x_n\}$ be generated by

$$\begin{cases} u_n = J_\lambda^{B_1}(x_n + \epsilon A^*(J_\lambda^{B_2} - I)Ax_n), \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tu_n, \end{cases} \tag{1.13}$$

where $\lambda > 0$ and $\epsilon \in (0, \frac{1}{L}), L$ is the spectral radius of the operator A^*A, and A^* is the adjoint of $A, \{\alpha_n\}$ is a sequence in $(0, 1)$ such that $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$. Then the sequence $\{u_n\}$ and $\{x_n\}$ both convergence strongly to $z \in \Omega$, where $z = P_\Omega f(z)$.

Inspired and motivation by research going on in this area, a modified general iterative method for a split variational inclusion and k-strictly pseudo-contractive mapping, which is defined in the following way:

$$\begin{cases} z_n = J_\lambda^{B_1}(x_n + \xi A^*(J_\lambda^{B_2} - I)Ax_n), \\ y_n = \alpha_n K x_n + (1 - \alpha_n)P_C[P_C(z_n - \mu_2 D_2 z_n) - \mu_1 D_1 P_C(z_n - \mu_2 D_2 z_n)] \\ x_{n+1} = \beta_n x_n + \gamma_n y_n + \delta_n S y_n, \ \forall n \geq 0, \end{cases} \tag{1.14}$$

where $\mu_i \in (0, 2\eta_i)$ for $i = 1, 2, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset [0, 1], \lambda > 0$ and $\xi \in (0, \frac{1}{L}), L$ is the spectral radius of the operator A^*A, A^* is the adjoint of bounded linear operation A, $D_i : C \to H_1$ are η_i-inverse strongly monotone for $i = 1, 2$, $S : C \to C$ is a k-strictly pseudocontractive mapping and $K : C \to C$ be ρ-contraction with $\rho \in [0, \frac{1}{2})$.

Furthermore, we prove that the sequences generated by the iterative scheme converge strongly to a common solution set of split variational inclusion problem, a general system of variational inequalities and fixed point set of a strictly pseudocontractive mapping in real Hilbert space.

2. Preliminaries

In this section, we collect some notations and lemmas. Let C be a nonempty closed convex subset of a real Hilbert space H_1. A mapping $D : C \to H_1$ is called *monotone* if

$$\langle Dx - Dy, x - y\rangle \geq 0, \ \forall x, y \in C. \tag{2.1}$$

A mapping $D : C \to H_1$ is called *Lipschitz continuous* if there exists a real number $L > 0$ such that

$$\|Dx - Dy\| \leq L\|x - y\|, \ \forall x, y \in C. \tag{2.2}$$

Recall that a mapping $D : C \to H_1$ is called *α-inverse strongly monotone* if there exists a real number $\alpha > 0$ such that

$$\langle Dx - Dy, x - y\rangle \geq \alpha\|Dx - Dy\|^2, \ \forall x, y \in C. \tag{2.3}$$

It is clear that every inverse strongly monotone mapping is a monotone and Lipschitz continuous mapping. Also, recall that a mapping $S : C \to C$ is said to be *k-strictly pseudocontractive* if there exists a constant $0 \leq k < 1$ such that

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + k\|(I - S)x - (I - S)y\|^2, \ \forall x, y \in C. \tag{2.4}$$

For such a case, we also say that S is a k-strict pseudo-contraction [15]. It is clear that, in a real Hilbert space H_1, inequality (2.4) is equivalent to the following:

$$\langle Sx - Sy, x - y\rangle \leq \|x - y\|^2 - \frac{1-k}{2}\|(I - S)x - (I - S)y\|^2, \ \forall x, y \in C. \tag{2.5}$$

This immediately implies that if S is a k-strictly pseudocontractive mapping, then $I - S$ is $\frac{1-k}{2}$-inverse strongly monotone.

Let H_1 be a real Hilbert space. Then

$$\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y\rangle, \tag{2.6}$$

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \tag{2.7}$$

and

$$\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2, \tag{2.8}$$

for all $x, y \in H_1$ and $y \in [0, 1]$.

We recall some concepts and results which are needed in sequel. A mapping P_C is said to be *metric projection* of H_1 onto C if for every point $x \in H_1$, there exists a unique nearest point in C denoted by $P_C x$ such that

$$\|x - P_C x\| \leq \|x - y\|, \ \forall y \in C. \tag{2.9}$$

It is well known that P_C is a nonexpansive mapping and is characterized by the following property:

$$\|P_C x - P_C y\|^2 \leq \langle x - y, P_C x - P_C y\rangle, \ \forall x, y \in H_1. \tag{2.10}$$

Moreover, $P_C x$ is characterized by the following properties:

$$\langle x - P_C x, y - P_C x\rangle \leq 0, \tag{2.11}$$

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2, \ \forall x \in H_1, y \in C, \tag{2.12}$$

and

$$\|(x - y) - (P_C x - P_C y)\|^2 \geq \|x - y\|^2 - \|P_C x - P_C y\|^2, \ \forall x, y \in H_1. \tag{2.13}$$

It is known that every nonexpansive operator $S : H_1 \to H_1$ satisfies, for all $(x, y) \in H_1 \times H_1$, the inequality

$$\langle (x - S(x)) - (y - S(y)), S(y) - S(x)\rangle \leq \frac{1}{2}\|(S(x) - x) - (S(y) - y)\|^2, \tag{2.14}$$

and therefore, we get, for all $(x, y) \in H_1 \times Fix(S)$,

$$\langle x - S(x), y - S(x)\rangle \leq \frac{1}{2}\|S(x) - x\|^2. \tag{2.15}$$

Lemma 2.1. *[16] Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1 - \beta_n)z_n + \beta_n x_n$ for all integers $n \geq 0$ and*

$$\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Then, $\lim_{n\to\infty} \|z_n - x_n\| = 0$.

Lemma 2.2. [17] *Let $\{a_n\}$ be a sequence of nonnegative numbers satisfying the condition*

$$a_{n+1} \leq (1-\delta_n)a_n + \delta_n\sigma_n, \ \forall n \geq 1,$$

where $\{\delta_n\}, \{\sigma_n\}$ are sequences of real numbers such that

(i) $\{\delta_n\} \subset [0,1]$ and $\sum_{n=1}^{\infty}\delta_n = \infty$, or equivalently, $\prod_{n=1}^{\infty}(1-\delta_n) = \lim_{n\to\infty}\prod_{k=1}^{\infty}(1-\delta_k) = 0$;

(ii) $\limsup_{n\to\infty}\sigma_n \leq 0$ or $\sum_{n=1}^{\infty}\delta_n\sigma_n$ is convergent.

Then, $\lim_{n\to\infty} a_n = 0$.

Lemma 2.3. [18] *Assume that T is nonexpansive self mapping of a closed convex subset C of a Hilbert space H_1. If T has a fixed point, then $I-T$ is demiclosed, i.e., whenever $\{x_n\}$ converges strongly to some y, it follows that $(I-T)x = y$. Here I is the identity mapping on H_1.*

Lemma 2.4. [19] *Let C be a nonempty closed convex subset of a real Hilbert space H and $S : C \to C$ be a self-mapping of C.*

(i) If S is a k-strict pseudocontractive mapping, then S satisfies the Lipschitz condition

$$\|Sx - Sy\| \leq \frac{1+k}{1-k}\|x-y\|, \ \forall x, y \in C. \tag{2.16}$$

(ii) If S is a k-strict pseudocontractive mapping, then the mapping $I-S$ is demiclosed at 0, that is, if $\{x_n\}$ is a sequence in C such that $x_n \to \tilde{x}$ weakly and $(I-S)x_n \to 0$ strongly, then $(I-S)\tilde{x} = 0$.

(iii) If S is k-(quasi)strict pseudo-contraction, then the fixed-point set $Fix(S)$ of S is closed and convex so thst the projection $P_{Fix(S)}$ is well defined.

Lemma 2.5. [10] *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $S : C \to C$ be a k strictly pseudocontractive mapping. Let γ and δ be two nonnegative real numbers. Assume $(\gamma+\delta)k \leq \gamma$. Then*

$$\|\gamma(x-y) + \delta(Sx - Sy)\| \leq (\gamma+\delta)\|x-y\|, \ \forall x, y \in C. \tag{2.17}$$

3. Main Result

Theorem 3.1. *Let H_1 and H_2 be two real Hilbert spaces and let $C \subseteq H_1$ and $Q \subseteq H_2$ be nonempty closed convex subsets. Let $A : H_1 \to H_2$ be a bounded linear operator and $D_i : C \to H$ be η_i-inverse strongly monotone for $i = 1, 2$. Let $S : C \to C$ be a k-strictly pseudocontractive mapping such that $Fix(S) \cap \bar{\Gamma} \cap \mathcal{G} \neq \emptyset$. Let $K : C \to C$ be ρ-contraction with $\rho \in [0, \frac{1}{2})$. For given $x_0 \in C$ arbitrarily, let the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ be generated iteratively by*

$$\begin{cases} z_n = J_\lambda^{B_1}(x_n + \xi A^*(J_\lambda^{B_2} - I)Ax_n), \\ y_n = \alpha_n Kx_n + (1-\alpha_n)P_C[P_C(z_n - \mu_2 D_2 z_n) - \mu_1 D_1 P_C(z_n - \mu_2 D_2 z_n)] \\ x_{n+1} = \beta_n x_n + \gamma_n y_n + \delta_n S y_n, \ \forall n \geq 0, \end{cases} \tag{3.1}$$

where $\mu_i \in (0, 2\eta_i)$ *for* $i = 1, 2, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subset [0, 1], \lambda > 0$ *and* $\xi \in (0, \frac{1}{L}), L$ *is the spectral radius of the operator* A^*A, A^* *is the adjoint of* A*. Assume that the following conditions are satisfied:*

(i) $\beta_n + \gamma_n + \delta_n = 1$ *and* $(\gamma_n + \delta_n)k \leq \gamma_n$ *for all* $n \geq 0$*;*

(ii) $\lim_{n\to\infty} \alpha_n = 0, \ \sum_{n=1}^{\infty} \alpha_n = \infty$*;*

(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$*; and* $\liminf_{n\to\infty} \delta_n > 0$*;*

(iv) $\lim_{n\to\infty} (\gamma_{n+1}/(1 - \beta_{n+1}) - \gamma_n/(1 - \beta_n)) = 0$.

Then, the sequence $\{x_n\}$ *generated by (3.1) converges strongly to* $\bar{x} \in P_{Fix(S)\cap\bar{\Gamma}\cap\mathcal{G}} K\bar{x}$ *if and only if* $\lim_{n\to\infty} \|y_n - z_n\| = 0$*. Furthermore* $(\bar{x}, \bar{y})$ *is a solution of the general system (1.5) of variational inequalities, where* $\bar{y} = P_C(\bar{x} - \mu_2 D_2 \bar{x})$*.*

Proof. We devide the proof into 6 steps.

Step 1. First we will prove that $\{x_n\}$ is bounded.

Indeed, take $x^* \in Fix(S) \cap \bar{\Gamma} \cap \mathcal{G}$ arbitrarily. Then $Sx^* = x^*$, $x^* \in \bar{\Gamma}$, and $x^* = P_C[P_C(x^* - \mu_2 D_2 x^*) - \mu_1 D_1 P_C(x^* - \mu_2 D_2 x^*)]$.

From $x^* \in \bar{\Gamma}$, we have $x^* = J_\lambda^{B_1} x^*, Ax^* = J_\lambda^{B_2}(Ax^*)$. We estimate

$$\begin{aligned}
& \|z_n - x^*\|^2 \\
= \ & \|J_\lambda^{B_1}(x_n + \xi A^*(J_\lambda^{B_2} - I)Ax_n) - x^*\|^2 \\
= \ & \|J_\lambda^{B_1}(x_n + \xi A^*(J_\lambda^{B_2} - I)Ax_n) - J_\lambda^{B_1} x^*\|^2 \\
\leq \ & \|x_n + \xi A^*(J_\lambda^{B_2} - I)Ax_n - x^*\|^2 \\
\leq \ & \|x_n - x^*\|^2 + \xi^2 \|A^*(J_\lambda^{B_2} - I)Ax_n\|^2 \\
& + 2\xi\langle x_n - x^*, A^*(J_\lambda^{B_2} - I)Ax_n\rangle. \qquad (3.2)
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\|z_n - x^*\|^2 \ \leq \ & \|x_n - x^*\|^2 + \xi^2\langle (J_\lambda^{B_2} - I)Ax_n, AA^*(J_\lambda^{B_2} - I)Ax_n\rangle \\
& + 2\xi\langle x_n - x^*, A^*(J_\lambda^{B_2} - I)Ax_n\rangle. \qquad (3.3)
\end{aligned}$$

Now, we have

$$\begin{aligned}
\xi^2\langle (J_\lambda^{B_2} - I)Ax_n, AA^*(J_\lambda^{B_2} - I)Ax_n\rangle \ \leq \ & L\xi^2\langle (J_\lambda^{B_2} - I)Ax_n, (J_\lambda^{B_2} - I)Ax_n\rangle \\
= \ & L\xi^2\|(J_\lambda^{B_2} - I)Ax_n\|^2. \qquad (3.4)
\end{aligned}$$

Setting $\Lambda := 2\xi\langle x_n - x^*, A^*(J_\lambda^{B_2} - I)Ax_n\rangle$ and using (2.15), we have

$$\begin{aligned}
\Lambda \ = \ & 2\xi\langle x_n - x^*, A^*(J_\lambda^{B_2} - I)Ax_n\rangle \\
= \ & 2\xi\langle A(x_n - x^*), (J_\lambda^{B_2} - I)Ax_n\rangle \\
= \ & 2\xi\langle A(x_n - x^*) + (J_\lambda^{B_2} - I)Ax_n - (J_\lambda^{B_2} - I)Ax_n, (J_\lambda^{B_2} - I)Ax_n\rangle \\
= \ & 2\xi\left\{\langle J_\lambda^{B_2}Ax_n - Ax^*, (J_\lambda^{B_2} - I)Ax_n\rangle - \|(J_\lambda^{B_2} - I)Ax_n\|^2\right\} \\
\leq \ & 2\xi\left\{\frac{1}{2}\|(J_\lambda^{B_2} - I)Ax_n\|^2 - \|(J_\lambda^{B_2} - I)Ax_n\|^2\right\} \\
\leq \ & -\xi\|(J_\lambda^{B_2} - I)Ax_n\|^2. \qquad (3.5)
\end{aligned}$$

Using (3.3), (3.4) and (3.5), we obtain

$$\|z_n - x^*\|^2 \leq \|x_n - x^*\|^2 + \xi(L\xi - 1)\|(J_\lambda^{B_2} - I)Ax_n\|^2. \tag{3.6}$$

Since $\xi \in (0, \frac{1}{L})$, we obtain

$$\|z_n - x^*\|^2 \leq \|x_n - x^*\|^2. \tag{3.7}$$

For simplicity, we write $y^* = P_C(x^* - \mu_2 D_2 x^*)$ and $u_n = P_C(z_n - \mu_2 D_2 z_n)$ for all $n \geq 0$. Since $D_i : C \to H_1$ be η_i-inverse strongly monotone for $i = 1, 2$ and $0 < \mu_i < 2\eta_i$ for $i = 1, 2$, we know that for all $n \geq 0$,

$$\begin{aligned}
& \|P_C[P_C(z_n - \mu_2 D_2 z_n) - \mu_1 D_1 P_C(z_n - \mu_2 D_2 z_n)] - x^*\|^2 \\
= & \|P_C[P_C(z_n - \mu_2 D_2 z_n) - \mu_1 D_1 P_C(z_n - \mu_2 D_2 z_n)] \\
& -P_C[P_C(x^* - \mu_2 D_2 x^*) - \mu_1 D_1 P_C(x^* - \mu_2 D_2 x^*)]\|^2 \\
\leq & \|[P_C(z_n - \mu_2 D_2 z_n) - \mu_1 D_1 P_C(z_n - \mu_2 D_2 z_n)] \\
& -[P_C(x^* - \mu_2 D_2 x^*) - \mu_1 D_1 P_C(x^* - \mu_2 D_2 x^*)]\|^2 \\
= & \|[P_C(z_n - \mu_2 D_2 z_n) - P_C(x^* - \mu_2 D_2 x^*)] \\
& -\mu_1[D_1 P_C(z_n - \mu_2 D_2 z_n) - D_1 P_C(x^* - \mu_2 D_2 x^*)]\|^2 \\
\leq & \|P_C(z_n - \mu_2 D_2 z_n) - P_C(x^* - \mu_2 D_2 x^*)\|^2 \\
& -\mu_1(2\eta_1 - \mu_1)\|D_1 P_C(z_n - \mu_2 D_2 z_n) - D_1 P_C(x^* - \mu_2 D_2 x^*)\|^2 \\
\leq & \|(z_n - \mu_2 D_2 z_n) - (x^* - \mu_2 D_2 x^*)\|^2 - \mu_1(2\eta_1 - \mu_1)\|D_1 u_n - D_1 y^*\|^2 \\
= & \|(z_n - x^*) - \mu_2(D_2 z_n - D_2 x^*)\|^2 - \mu_1(2\eta_1 - \mu_1)\|D_1 u_n - D_1 y^*\|^2 \\
\leq & \|z_n - x^*\|^2 - \mu_2(2\eta_2 - \mu_2)\|D_2 z_n - D_2 x^*\|^2 - \mu_1(2\eta_1 - \mu_1)\|D_1 u_n - D_1 y^*\|^2 \\
\leq & \|z_n - x^*\|^2 \\
\leq & \|x_n - x^*\|^2.
\end{aligned} \tag{3.8}$$

Hence, we get

$$\begin{aligned}
& \|y_n - x^*\| \\
= & \|\alpha_n(Kx_n - x^*) + (1 - \alpha_n)(P_C[P_C(z_n - \mu_2 D_2 z_n) \\
& -\mu_1 D_1 P_C(z_n - \mu_2 D_2 z_n)] - x^*)\| \\
\leq & \alpha_n\|Kx_n - x^*\| + (1 - \alpha_n)\|P_C[P_C(z_n - \mu_2 D_2 z_n) \\
& -\mu_1 D_1 P_C(z_n - \mu_2 D_2 z_n)] - x^*)\| \\
\leq & \alpha_n(\rho\|x_n - x^*\| + \|Kx^* - x^*\|) + (1 - \alpha_n)\|x_n - x^*\| \\
= & (1 - (1 - \rho)\alpha_n)\|x_n - x^*\| + (1 - \rho)\alpha_n \frac{\|Kx^* - x^*\|}{1 - \rho} \\
\leq & \max\left\{\|x_n - x^*\|, \frac{\|Kx^* - x^*\|}{1 - \rho}\right\}.
\end{aligned} \tag{3.9}$$

Since $(\gamma_n + \delta_n)k \leq \gamma_n$ for all $n \geq 0$, utilizing Lemma 2.5, we obtain from (3.9)

$$\begin{aligned} \|x_{n+1}-x^*\| &= \|\beta_n(x_n-x^*)+\gamma_n(y_n-x^*)+\delta_n(Sy_n-x^*)\| \\ &\leq \beta_n\|x_n-x^*\|+\|\gamma_n(y_n-x^*)+\delta_n(Sy_n-x^*)\| \\ &\leq \beta_n\|x_n-x^*\|+(\gamma_n+\delta_n)\|y_n-x^*\| \\ &\leq \beta_n\|x_n-x^*\|+(\gamma_n+\delta_n)\max\left\{\|x_n-x^*\|, \frac{\|Kx^*-x^*\|}{1-\rho}\right\} \\ &\leq \max\left\{\|x_n-x^*\|, \frac{\|Kx^*-x^*\|}{1-\rho}\right\}. \end{aligned} \tag{3.10}$$

By induction, we obtain that for all $n \geq 0$,

$$\|x_n-x^*\| \leq \max\left\{\|x_0-x^*\|, \frac{\|Kx^*-x^*\|}{1-\rho}\right\}. \tag{3.11}$$

Hence, $\{x_n\}$ is bounded. Consequently, we deduce immediately that $\{z_n\}, \{y_n\}, \{Sy_n\}$ and $\{u_n\}$ are bounded, where $u_n = P_C(z_n - \mu_2 D_2 z_n)$ for all $n \geq 0$.

Now, put

$$t_n := P_C[P_C(z_n-\mu_2D_2z_n)-\mu_1D_1P_C(z_n-\mu_2D_2z_n)],\ \forall n \geq 0. \tag{3.12}$$

Then it is easy to see that $\{t_n\}$ is bounded because P_C, D_1, and D_2 are Lipschitz continuous and $\{z_n\}$ is bounded.

Step 2. We will prove that $\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0$.

Indeed, define $x_{n+1} = \beta_n x_n + (1-\beta_n)w_n$ for all $n \geq 0$, so we get $w_n = \frac{x_{n+1}-\beta_n x_n}{1-\beta_n}$. It follows that

$$\begin{aligned} w_{n+1}-w_n &= \frac{\gamma_{n+1}(y_{n+1}-y_n)+\delta_{n+1}(Sy_{n+1}-Sy_n)}{1-\beta_{n+1}} + \left(\frac{\gamma_{n+1}}{1-\beta_{n+1}}-\frac{\gamma_n}{1-\beta_n}\right)y_n \\ &\quad + \left(\frac{\delta_{n+1}}{1-\beta_{n+1}}-\frac{\delta_n}{1-\beta_n}\right)Sy_n. \end{aligned} \tag{3.13}$$

Since $(\gamma_n+\delta_n)k \leq \gamma_n$ for all $n \geq 0$, utilizing Lemma 2.5 we have

$$\|\gamma_{n+1}(y_{n+1}-y_n)+\delta_{n+1}(Sy_{n+1}-Sy_n)\| \leq (\gamma_{n+1}+\delta_{n+1})\|y_{n+1}-y_n\|. \tag{3.14}$$

Next, we estimate $\|y_{n+1}-y_n\|$. Observe that

$$\begin{aligned} &\|z_{n+1}-z_n\| \\ =\ &\|J_\lambda^{B_1}(x_{n+1}+\gamma A^*(J_\lambda^{B_2}-I)Ax_{n+1}-J_\lambda^{B_1}(x_n+\gamma A^*(J_\lambda^{B_2}-I)Ax_n\| \\ \leq\ &\|J_\lambda^{B_1}(I+\gamma A^*(J_\lambda^{B_2}-I)A)x_{n+1}-J_\lambda^{B_1}(I+\gamma A^*(J_\lambda^{B_2}-I)A)x_n\| \\ \leq\ &\|x_{n+1}-x_n\|. \end{aligned} \tag{3.15}$$

And

$$\begin{aligned}
& \|t_{n+1}-t_n\|^2 \\
= & \|P_C[P_C(z_{n+1}-\mu_2 D_2 z_{n+1})-\mu_1 D_1 P_C(z_{n+1}-\mu_2 D_2 z_{n+1})] \\
& -P_C[P_C(z_n-\mu_2 D_2 z_n)-\mu_1 D_1 P_C(z_n-\mu_2 D_2 z_n)]\|^2 \\
\leq & \|[P_C(z_{n+1}-\mu_2 D_2 z_{n+1})-\mu_1 D_1 P_C(z_{n+1}-\mu_2 D_2 z_{n+1})] \\
& -[P_C(z_n-\mu_2 D_2 z_n)-\mu_1 D_1 P_C(z_n-\mu_2 D_2 z_n)]\|^2 \\
= & \|[P_C(z_{n+1}-\mu_2 D_2 z_{n+1})-P_C(z_n-\mu_2 D_2 z_n)] \\
& -\mu_1[D_1 P_C(z_{n+1}-\mu_2 D_2 z_{n+1})-D_1 P_C(z_n-\mu_2 D_2 z_n)]\|^2 \\
\leq & \|P_C(z_{n+1}-\mu_2 D_2 z_{n+1})-P_C(z_n-\mu_2 D_2 z_n)\|^2 \\
& -\mu_1(2\eta_1-\mu_1)\|D_1 P_C(z_{n+1}-\mu_2 D_2 z_{n+1})-D_1 P_C(z_n-\mu_2 D_2 z_n)\|^2 \\
\leq & \|P_C(z_{n+1}-\mu_2 D_2 z_{n+1})-P_C(z_n-\mu_2 D_2 z_n)\|^2 \\
\leq & \|(z_{n+1}-\mu_2 D_2 z_{n+1})-(z_n-\mu_2 D_2 z_n)\|^2 \\
= & \|(z_{n+1}-z_n)-\mu_2(D_2 z_{n+1}-D_2 z_n)\|^2 \\
\leq & \|z_{n+1}-z_n\|^2-\mu_2(2\eta_2-\mu_2)\|D_2 z_{n+1}-D_2 z_n\|^2 \\
\leq & \|z_{n+1}-z_n\|^2.
\end{aligned} \tag{3.16}$$

Comblining (3.15) with (3.16), we get

$$\|t_{n+1}-t_n\| \leq \|x_{n+1}-x_n\| \tag{3.17}$$

This together with (3.17) implies that

$$\begin{aligned}
& \|y_{n+1}-y_n\| \\
= & \|(t_{n+1}-t_n)+\alpha_{n+1}(Kx_{n+1}-t_{n+1})-\alpha_n(Kx_n-t_n)\| \\
\leq & \|t_{n+1}-t_n\|+\alpha_{n+1}\|Kx_{n+1}-t_{n+1}\|+\alpha_n\|Kx_n-t_n\| \\
\leq & \|x_{n+1}-x_n\|+\alpha_{n+1}\|Kx_{n+1}-t_{n+1}\|+\alpha_n\|Kx_n-t_n\|.
\end{aligned} \tag{3.18}$$

From $w_n = \frac{\gamma_n y_n + \delta_n S y_n}{1-\beta_n}$, and it follows from (3.13) , (3.14) and (3.18) that

$$\begin{aligned}
& \|w_{n+1} - w_n\| \\
= & \left\| \frac{\gamma_{n+1} y_{n+1} + \delta_{n+1} S y_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n y_n + \delta_n S y_n}{1-\beta_n} \right\| \\
= & \left\| \frac{\gamma_{n+1} y_{n+1} + \delta_{n+1} S y_{n+1}}{1-\beta_{n+1}} + \frac{\gamma_{n+1} y_n}{1-\beta_{n+1}} - \frac{\gamma_{n+1} y_n}{1-\beta_{n+1}} \right. \\
& \left. + \frac{\delta_{n+1} S y_n}{1-\beta_{n+1}} - \frac{\delta_{n+1} S y_n}{1-\beta_{n+1}} - \frac{\gamma_n y_n + \delta_n S y_n}{1-\beta_n} \right\| \\
= & \left\| \left(\frac{\gamma_{n+1} y_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_{n+1} y_n}{1-\beta_{n+1}} \right) + \left(\frac{\delta_{n+1} S y_{n+1}}{1-\beta_{n+1}} - \frac{\delta_{n+1} S y_n}{1-\beta_{n+1}} \right) \right. \\
& \left. + \frac{\gamma_{n+1} y_n}{1-\beta_{n+1}} - \frac{\gamma_n y_n}{1-\beta_n} + \frac{\delta_{n+1} S y_n}{1-\beta_{n+1}} - \frac{\delta_n S y_n}{1-\beta_n} \right\| \\
\leq & \left\| \frac{\gamma_{n+1}(y_{n+1} - y_n) + \delta_{n+1}(S y_{n+1} - S y_n)}{1-\beta_{n+1}} \right\| \\
& + \left| \frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n} \right| \|y_n\| + \left| \frac{\delta_{n+1}}{1-\beta_{n+1}} - \frac{\delta_n}{1-\beta_n} \right| \|S y_n\| \\
\leq & \frac{\gamma_{n+1} + \delta_{n+1}}{1-\beta_{n+1}} \|y_{n+1} - y_n\| + \left| \frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n} \right| \|y_n\| \\
& + \left| \frac{\delta_{n+1}}{1-\beta_{n+1}} - \frac{\delta_n}{1-\beta_n} \right| \|S y_n\| \\
\leq & \frac{\gamma_{n+1} + \delta_{n+1}}{1-\beta_{n+1}} \|y_{n+1} - y_n\| + \left| \frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n} \right| (\|y_n\| + \|S y_n\|) \\
= & \|y_{n+1} - y_n\| + \left| \frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n} \right| (\|y_n\| + \|S y_n\|) \\
\leq & \|x_{n+1} - x_n\| + \alpha_{n+1} \|K x_{n+1} - t_{n+1}\| + \alpha_n \|K x_n - t_n\| \\
& + \left| \frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n} \right| (\|y_n\| + \|S y_n\|).
\end{aligned} \tag{3.19}$$

Since $\{x_n\}, \{y_n\}$ and $\{t_n\}$ are bounded, it follows from conditions (ii) and (iv) that

$$\begin{aligned}
& \limsup_{n\to\infty} (\|w_{n+1} - w_n\| - \|x_{n+1} - x_n\|) \\
\leq & \limsup_{n\to\infty} \left\{ \alpha_{n+1} \|K x_{n+1} - t_{n+1}\| + \alpha_n \|K x_n - t_n\| \right. \\
& \left. + \left| \frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n} \right| (\|y_n\| + \|S y_n\|) \right\} \\
= & \ 0.
\end{aligned} \tag{3.20}$$

By Lemma 2.1 , we get

$$\lim_{n\to\infty} \|w_n - x_n\| = 0. \tag{3.21}$$

From $x_{n+1} = (1-\beta_n) w_n + \beta_n x_n$ we get $\|x_{n+1} - x_n\| = (1-\beta_n)\|w_n - x_n\|$ so

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} (1-\beta_n)\|w_n - x_n\| = 0. \tag{3.22}$$

Step 3. We will prove that $\lim_{n\to\infty}\|D_2z_n - D_2x^*\| = 0$, $\lim_{n\to\infty}\|D_1u_n - D_1y^*\| = 0$ where $y^* = P_C(x^* - \mu_2 D_2 x^*)$.

Indeed, utilizing Lemma 2.5 and the convexity of $\|\cdot\|^2$, we get from (3.1) and (3.8)

$$\begin{aligned}
& \|x_{n+1} - x^*\|^2 \\
= \; & \|\beta_n(x_n - x^*) + \gamma_n(y_n - x^*) + \delta_n(Sy_n - x^*)\|^2 \\
\leq \; & \beta_n\|x_n - x^*\|^2 + (\gamma_n + \delta_n)\left\|\frac{1}{\gamma_n + \delta_n}[\gamma_n(y_n - x^*) + \delta_n(Sy_n - x^*)]\right\|^2 \\
\leq \; & \beta_n\|x_n - x^*\|^2 + (\gamma_n + \delta_n)\|y_n - x^*\|^2 \\
\leq \; & \beta_n\|x_n - x^*\|^2 + (\gamma_n + \delta_n)[\alpha_n\|Kx_n - x^*\|^2 + (1 - \alpha_n)\|t_n - x^*\|^2] \\
\leq \; & \beta_n\|x_n - x^*\|^2 + \alpha_n\|Kx_n - x^*\|^2 + (\gamma_n + \delta_n)\|t_n - x^*\|^2 \\
\leq \; & \beta_n\|x_n - x^*\|^2 + \alpha_n\|Kx_n - x^*\|^2 + (\gamma_n + \delta_n) \\
& \times[\|z_n - x^*\|^2 - \mu_2(2\eta_2 - \mu_2)\|D_2z_n - D_2x^*\|^2 \\
& -\mu_1(2\eta_1 - \mu_1)\|D_1u_n - D_1y^*\|^2] \\
\leq \; & \beta_n\|x_n - x^*\|^2 + \alpha_n\|Kx_n - x^*\|^2 + (\gamma_n + \delta_n) \\
& \times[\|x_n - x^*\|^2 - \mu_2(2\eta_2 - \mu_2)\|D_2z_n - D_2x^*\|^2 \\
& -\mu_1(2\eta_1 - \mu_1)\|D_1u_n - D_1y^*\|^2] \\
= \; & \|x_n - x^*\|^2 + \alpha_n\|Kx_n - x^*\|^2 \\
& -(\gamma_n + \delta_n)[\mu_2(2\eta_2 - \mu_2)\|D_2z_n - D_2x^*\|^2 \\
& +\mu_1(2\eta_1 - \mu_1)\|D_1u_n - D_1y^*\|^2].
\end{aligned} \tag{3.23}$$

Therefore,

$$\begin{aligned}
& (\gamma_n + \delta_n)[\mu_2(2\eta_2 - \mu_2)\|D_2z_n - D_2x^*\|^2 + \mu_1(2\eta_1 - \mu_1)\|D_1u_n - D_1y^*\|^2] \\
\leq \; & \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \alpha_n\|Kx_n - x^*\|^2 \\
\leq \; & (\|x_n - x^*\| + \|x_{n+1} - x^*\|)\|x_n - x_{n+1}\| + \alpha_n\|Kx_n - x^*\|^2.
\end{aligned} \tag{3.24}$$

Since $\alpha_n \to 0$, $\|x_n - x_{n+1}\| \to 0$ and $\lim_{n\to\infty}(\gamma_n + \delta_n) > 0$, we have

$$\lim_{n\to\infty}\|D_1u_n - D_1y^*\| = 0, \quad \lim_{n\to\infty}\|D_2z_n - D_2x^*\| = 0. \tag{3.25}$$

Step 4. We will prove that $\lim_{n\to\infty}\|Sy_n - y_n\| = 0$.

From (3.1), we obtain

$$\begin{aligned}
&\|u_n - y^*\|^2 \\
= \;& \|P_C(z_n - \mu_2 D_2 z_n) - P_C(x^* - \mu_2 D_2 x^*)\|^2 \\
\leq \;& \langle (z_n - \mu_2 D_2 z_n) - (x^* - \mu_2 D_2 x^*), u_n - y^* \rangle \\
= \;& \frac{1}{2}[\|z_n - x^* - \mu_2(D_2 z_n - D_2 x^*)\|^2 + \|u_n - y^*\|^2 \\
& -\|(z_n - x^*) - \mu_2(D_2 z_n - D_2 x^*) - (u_n - y^*)\|^2] \\
\leq \;& \frac{1}{2}[\|z_n - x^*\|^2 + \|u_n - y^*\|^2 \\
& -\|(z_n - u_n) - \mu_2(D_2 z_n - D_2 x^*) - (x^* - y^*)\|^2] \\
= \;& \frac{1}{2}[\|z_n - x^*\|^2 + \|u_n - y^*\|^2 - \|z_n - u_n - (x^* - y^*)\|^2 \\
& +2\mu_2 \langle z_n - u_n - (x^* - y^*), D_2 z_n - D_2 x^* \rangle \\
& -\mu_2^2 \|D_2 z_n - D_2 x^*\|^2],
\end{aligned} \tag{3.26}$$

that is

$$\begin{aligned}
\|u_n - y^*\|^2 \leq \;& \|z_n - x^*\|^2 - \|z_n - u_n - (x^* - y^*)\|^2 \\
& +2\mu_2 \|z_n - u_n - (x^* - y^*)\| \|D_2 z_n - D_2 x^*\|.
\end{aligned} \tag{3.27}$$

Substituting (3.6) in (3.27), we have

$$\begin{aligned}
&\|u_n - y^*\|^2 \\
\leq \;& \|x_n - x^*\|^2 + \xi(L\xi - 1)\|(J_\lambda^{B_2} - I)Ax_n\|^2 - \|z_n - u_n - (x^* - y^*)\|^2 \\
& +2\mu_2 \|z_n - u_n - (x^* - y^*)\| \|D_2 z_n - D_2 x^*\|.
\end{aligned} \tag{3.28}$$

Further, similarly to above argument, we derive

$$\begin{aligned}
&\|t_n - x^*\|^2 \\
= \;& \|P_C(u_n - \mu_1 D_1 u_n) - P_C(y^* - \mu_1 D_1 y^*)\|^2 \\
\leq \;& \langle (u_n - \mu_1 D_1 u_n) - (y^* - \mu_1 D_1 y^*), t_n - x^* \rangle \\
= \;& \frac{1}{2}[\|u_n - y^* - \mu_1(D_1 u_n - D_1 y^*)\|^2 + \|t_n - x^*\|^2 \\
& -\|(u_n - y^*) - \mu_1(D_1 u_n - D_1 y^*) - (t_n - x^*)\|^2] \\
\leq \;& \frac{1}{2}[\|u_n - y^*\|^2 + \|t_n - x^*\|^2 - \|(u_n - t_n) \\
& -\mu_1(D_1 u_n - D_1 y^*) + (x^* - y^*)\|^2] \\
= \;& \frac{1}{2}[\|u_n - y^*\|^2 + \|t_n - x^*\|^2 - \|u_n - t_n + (x^* - y^*)\|^2 \\
& +2\mu_1 \langle u_n - t_n + (x^* - y^*), D_1 u_n - D_1 y^* \rangle \\
& -\mu_1^2 \|D_1 u_n - D_1 y^*\|^2]
\end{aligned} \tag{3.29}$$

that is,

$$\begin{aligned}
\|t_n - x^*\|^2 \leq \;& \|u_n - y^*\|^2 - \|u_n - t_n + (x^* - y^*)\|^2 \\
& +2\mu_1 \|u_n - t_n + (x^* - y^*)\| \|D_1 u_n - D_1 y^*\|.
\end{aligned} \tag{3.30}$$

Substituting (3.28) in (3.30), we have

$$\begin{aligned} &\|t_n - x^*\|^2 \\ \le\ & \|x_n - x^*\|^2 + \xi(L\xi - 1)\|(J_\lambda^{B_2} - I)Ax_n\|^2 \\ & -\|z_n - u_n - (x^* - y^*)\|^2 + 2\mu_2\|z_n - u_n - (x^* - y^*)\| \\ & \times\|D_2z_n - D_2x^*\| - \|u_n - t_n + (x^* - y^*)\|^2 \\ & +2\mu_1\|u_n - t_n + (x^* - y^*)\|\|D_1u_n - D_1y^*\|. \end{aligned} \tag{3.31}$$

Thus from (3.1) and (3.31), it follows that

$$\begin{aligned} &\|x_{n+1} - x^*\|^2 \\ \le\ & \beta_n\|x_n - x^*\|^2 + (\gamma_n + \delta_n)\|y_n - x^*\|^2 \\ \le\ & \beta_n\|x_n - x^*\|^2 + (1 - \beta_n)[\alpha_n\|Kx_n - x^*\|^2 + (1 - \alpha_n)\|t_n - x^*\|^2] \\ \le\ & \beta_n\|x_n - x^*\|^2 + \alpha_n\|Kx_n - x^*\|^2 + (1 - \beta_n)\|t_n - x^*\|^2 \\ \le\ & \beta_n\|x_n - x^*\|^2 + \alpha_n\|Kx_n - x^*\|^2 \\ & +(1 - \beta_n)[\|x_n - x^*\|^2 + \xi(L\xi - 1)\|(J_\lambda^{B_2} - I)Ax_n\|^2 \\ & -\|z_n - u_n - (x^* - y^*)\|^2 + 2\mu_2\|z_n - u_n - (x^* - y^*)\|\|D_2z_n - D_2x^*\| \\ & -\|u_n - t_n + (x^* - y^*)\|^2 + 2\mu_1\|u_n - t_n + (x^* - y^*)\|\|D_1u_n - D_1y^*\|] \\ =\ & \|x_n - x^*\|^2 + \alpha_n\|Kx_n - x^*\|^2 \\ & +(1 - \beta_n)[\xi(L\xi - 1)\|(J_\lambda^{B_2} - I)Ax_n\|^2 - \|z_n - u_n - (x^* - y^*)\|^2 \\ & +2\mu_2\|z_n - u_n - (x^* - y^*)\|\|D_2z_n - D_2x^*\| \\ & -\|u_n - t_n + (x^* - y^*)\|^2 + 2\mu_1\|u_n - t_n + (x^* - y^*)\|\|D_1u_n - D_1y^*\|] \end{aligned} \tag{3.32}$$

which hence implies that

$$\begin{aligned} & (1 - \beta_n)[\xi(1 - L\xi)\|(J_\lambda^{B_2} - I)Ax_n\|^2 \\ & +\|z_n - u_n - (x^* - y^*)\|^2 + \|u_n - t_n + (x^* - y^*)\|^2] \\ \le\ & \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \alpha_n\|Kx_n - x^*\|^2 \\ & +(1 - \beta_n)[2\mu_2\|z_n - u_n - (x^* - y^*)\|\|D_2z_n - D_2x^*\|] \\ & +2\mu_1\|u_n - t_n + (x^* - y^*)\|\|D_1u_n - D_1y^*\|. \end{aligned} \tag{3.33}$$

Since $\xi(1 - L\xi) > 0, \alpha_n \to 0, \limsup_{n\to\infty}\beta_n < 1, \|D_2z_n - D_2x^*\| \to 0, \|D_1u_n - D_1y^*\| \to 0, \|x_{n+1} - x_n\| \to 0$, it follows from the boundedness of $\{x_n\}, \{z_n\}, \{u_n\}$ and $\{t_n\}$ that

$$\begin{aligned} & (1 - \beta_n)[\xi(1 - L\xi)\|(J_\lambda^{B_2} - I)Ax_n\|^2] \\ & +(1 - \beta_n)[\|z_n - u_n - (x^* - y^*)\|^2 + \|u_n - t_n + (x^* - y^*)\|^2] \\ \le\ & (\|x_n - x^*\| + \|x_{n+1} - x^*\|)\|x_n - x_{n+1}\| + \alpha_n\|Kx_n - x^*\|^2 \\ & +(1 - \beta_n)[2\mu_2\|z_n - u_n - (x^* - y^*)\|\|D_2z_n - D_2x^*\|] \\ & +2\mu_1\|u_n - t_n + (x^* - y^*)\|\|D_1u_n - D_1y^*\|. \end{aligned} \tag{3.34}$$

We get

$$\lim_{n\to\infty}\|(J_\lambda^{B_1} - I)Ax_n\| = 0, \tag{3.35}$$

$$\lim_{n\to\infty}\|u_n - t_n + (x^* - y^*)\| = 0, \tag{3.36}$$

$$\lim_{n\to\infty} \|z_n - u_n - (x^* - y^*)\| = 0. \tag{3.37}$$

Furthermore, using (3.1) and $\gamma \in (0, \frac{1}{L})$, we observe that

$$\begin{aligned}
& \|z_n - x^*\|^2 \\
= \; & \|J_\lambda^{B_1}(x_n + \xi A^*(J_\lambda^{B_2} - I)Ax_n) - J_\lambda^{B_1}x^*\|^2 \\
\leq \; & \langle z_n - x^*, x_n + \gamma A^*(J_\lambda^{B_2} - I)Ax_n - x^*\rangle \\
= \; & \frac{1}{2}\{\|z_n - x^*\|^2 + \|x_n + \xi A^*(J_\lambda^{B_2} - I)Ax_n - x^*\|^2 \\
& -\|(z_n - x^*) - [x_n + \xi A^*(J_\lambda^{B_2} - I)Ax_n - x^*]\|^2\} \\
= \; & \frac{1}{2}\{\|z_n - x^*\|^2 + \|x_n - x^*\|^2 + \xi(L\gamma - 1)\|(J_\lambda^{B_2} - I)Ax_n\|^2 \\
& -\|z_n - x_n - \xi A^*(J_\lambda^{B_2} - I)Ax_n\|^2\} \\
\leq \; & \frac{1}{2}\{\|z_n - x^*\|^2 + \|x_n - x^*\|^2 - [\|z_n - x_n\|^2 + \xi^2\|A^*(J_\lambda^{B_2} - I)Ax_n\|^2 \\
& -2\xi\langle z_n - x_n, A^*(J_\lambda^{B_2} - I)Ax_n\rangle]\} \\
\leq \; & \frac{1}{2}\{\|z_n - x^*\|^2 + \|x_n - x^*\|^2 - \|z_n - x_n\|^2 \\
& +2\xi\|A(z_n - x_n)\|\|(J_\lambda^{B_2} - I)Ax_n\|\}.
\end{aligned}$$

Hence, we obtain

$$\|z_n - x^*\|^2 \leq \|x_n - x^*\|^2 - \|z_n - x_n\|^2 + 2\xi\|A(z_n - x_n)\|\|(J_\lambda^{B_2} - I)Ax_n\|. \tag{3.38}$$

Substituting (3.38) in (3.27), we get

$$\begin{aligned}
& \|u_n - y^*\|^2 \\
\leq \; & \|x_n - x^*\|^2 - \|z_n - x_n\|^2 + 2\xi\|A(z_n - x_n)\|\|(J_\lambda^{B_2} - I)Ax_n\| \\
& -\|z_n - u_n - (x^* - y^*)\|^2 + 2\mu_2\|z_n - u_n - (x^* - y^*)\| \\
& \times\|D_2z_n - D_2x^*\|.
\end{aligned} \tag{3.39}$$

Substituting (3.39) in (3.30), we get

$$\begin{aligned}
& \|t_n - x^*\|^2 \\
\leq \; & \|x_n - x^*\|^2 - \|z_n - x_n\|^2 + 2\xi\|A(z_n - x_n)\|\|(J_\lambda^{B_2} - I)Ax_n\| \\
& -\|z_n - u_n - (x^* - y^*)\|^2 + 2\mu_2\|z_n - u_n - (x^* - y^*)\|\|D_2z_n - D_2x^*\| \\
& -\|u_n - t_n + (x^* - y^*)\|^2 + 2\mu_1\|u_n - t_n + (x^* - y^*)\| \\
& \times\|D_1u_n - D_1y^*\|.
\end{aligned} \tag{3.40}$$

From (3.23) and (3.40), we get

$$
\begin{aligned}
& \|x_{n+1}-x^*\|^2 \\
\leq\ & \beta_n\|x_n-x^*\|^2+(\gamma_n+\delta_n)\|y_n-x^*\|^2 \\
=\ & \beta_n\|x_n-x^*\|^2+(1-\beta_n)\|y_n-x^*\|^2 \\
\leq\ & \beta_n\|x_n-x^*\|^2+(1-\beta_n)[\alpha_n\|Kx_n-x^*\|^2+(1-\alpha_n)\|t_n-x^*\|^2] \\
\leq\ & \beta_n\|x_n-x^*\|^2+\alpha_n\|Kx_n-x^*\|^2+(1-\beta_n)\|t_n-x^*\|^2 \\
\leq\ & \beta_n\|x_n-x^*\|^2+\alpha_n\|Kx_n-x^*\|^2+(1-\beta_n)\Big[\|x_n-x^*\|^2-\|z_n-x_n\|^2 \\
& +2\xi\|A(z_n-x_n)\|\|(J_\lambda^{B_2}-I)Ax_n\|-\|z_n-u_n-(x^*-y^*)\|^2 \\
& +2\mu_2\|z_n-u_n-(x^*-y^*)\|\|D_2z_n-D_2x^*\|-\|u_n-t_n+(x^*-y^*)\|^2 \\
& +2\mu_1\|u_n-t_n+(x^*-y^*)\|\|D_1u_n-D_1y^*\|\Big] \\
=\ & \|x_n-x^*\|^2+\alpha_n\|Kx_n-x^*\|^2+(1-\beta_n)\Big[2\xi\|A(z_n-x_n)\|\|(J_\lambda^{B_2}-I)Ax_n\| \\
& +2\mu_2\|z_n-u_n-(x^*-y^*)\|\|D_2z_n-D_2x^*\| \\
& +2\mu_1\|u_n-t_n+(x^*-y^*)\|\|D_1u_n-D_1y^*\|\Big] \\
& -(1-\beta_n)\Big[\|x_n-z_n\|^2+\|z_n-u_n-(x^*-y^*)\|^2 \\
& +\|u_n-t_n+(x^*-y^*)\|^2\Big],
\end{aligned}
\tag{3.41}
$$

which hence implies that

$$
\begin{aligned}
& (1-\beta_n)\Big[\|x_n-z_n\|^2+\|z_n-u_n-(x^*-y^*)\|^2+\|u_n-t_n+(x^*-y^*)\|^2\Big] \\
\leq\ & \|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2+\alpha_n\|Kx_n-x^*\|^2 \\
& +(1-\beta_n)\Big[2\xi\|A(z_n-x_n)\|\|(J_\lambda^{B_2}-I)Ax_n\| \\
& +2\mu_2\|z_n-u_n-(x^*-y^*)\|\|D_2z_n-D_2x^*\| \\
& +2\mu_1\|u_n-t_n+(x^*-y^*)\|\|D_1u_n-D_1y^*\|\Big] \\
\leq\ & (\|x_n-x^*\|+\|x_{n+1}-x^*\|)\|x_n-x_{n+1}\|+\alpha_n\|Kx_n-x^*\|^2 \\
& +(1-\beta_n)\Big[2\xi\|A(z_n-x_n)\|\|(J_\lambda^{B_2}-I)Ax_n\| \\
& +2\mu_2\|z_n-u_n-(x^*-y^*)\|\|D_2z_n-D_2x^*\| \\
& +2\mu_1\|u_n-t_n+(x^*-y^*)\|\|D_1u_n-D_1y^*\|\Big].
\end{aligned}
\tag{3.42}
$$

Since $\alpha_n \to 0, \|x_n-x_{n+1}\| \to 0, \|(J_\lambda^{B_2}-I)Ax_n\| \to 0, \|D_2z_n-D_2x^*\| \to 0, \|D_1u_n-D_1y^*\| \to 0, \|u_n-t_n+(x^*-y^*)\| \to 0$ and $\|z_n-u_n-(x^*-y^*)\| \to 0$ implies that

$$\lim_{n\to\infty}\|x_n-z_n\|=0. \tag{3.43}$$

Also since $y_n = \alpha_n Kx_n + (1-\alpha_n)t_n$ and $\alpha_n \to 0$ and $\|y_n - z_n\| \to 0$ thus

$$\|y_n - t_n\| \leq \alpha_n \|Kx_n - t_n\|,$$

implies that

$$\lim_{n\to\infty} \|y_n - t_n\| = 0. \tag{3.44}$$

And

$$(1-\alpha_n)\|t_n - z_n\| = \|y_n - z_n - \alpha_n(Kx_n - z_n)\| \leq \|y_n - z_n\| + \alpha_n\|Kx_n - z_n\|,$$

since $\alpha_n \to 0$ and $\|y_n - z_n\| \to 0$ implies that

$$\lim_{n\to\infty} \|t_n - z_n\| = 0. \tag{3.45}$$

Observe that

$$\|t_n - x_n\| \leq \|t_n - z_n\| + \|z_n - x_n\|,$$

since (3.43) and (3.45), we get

$$\lim_{n\to\infty} \|t_n - x_n\| = 0. \tag{3.46}$$

And

$$\|y_n - x_n\| \leq \|y_n - t_n\| + \|t_n - x_n\|,$$

since (3.44) and (3.46), we get

$$\lim_{n\to\infty} \|y_n - x_n\| = 0. \tag{3.47}$$

Note that from $x_{n+1} = \beta_n x_n + \gamma_n y_n + \delta_n S y_n$,

$$\|\delta_n(Sy_n - x_n)\| \leq \|x_{n+1} - x_n\| + \gamma_n\|y_n - x_n\|.$$

Since (3.22) and (3.47), it follows that

$$\lim_{n\to\infty} \|Sy_n - x_n\| = 0. \tag{3.48}$$

Note that

$$\|Sy_n - y_n\| \leq \|Sy_n - x_n\| + \|x_n - y_n\|,$$

from (3.47) and (3.48), we get

$$\lim_{n\to\infty} \|Sy_n - y_n\| = 0. \tag{3.49}$$

Step 5. We will prove that $\limsup_{n\to\infty}\langle K\bar{x} - \bar{x}, x_n - \bar{x}\rangle \leq 0$, where $\bar{x} = P_{Fix(S)\cap\bar{\Gamma}\cap\mathcal{G}} K\bar{x}$ Indeed, since $\{x_n\}$ is bounded, there exists a bounded $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle K\bar{x} - \bar{x}, x_n - \bar{x}\rangle = \lim_{i\to\infty}\langle K\bar{x} - \bar{x}, x_{n_i} - \bar{x}\rangle. \tag{3.50}$$

Also, since H is reflexive and $\{y_n\}$ is bounded, without loss generality we may assume that $y_{n_i} \to p$ weakly for some $p \in C$. First, it is clear from Lemma 2.4 that $p \in Fix(S)$.

Now let us show that $p \in \mathcal{G}$. We note that

$$\begin{aligned} & \|y_n - G(y_n)\| \\ \leq \ & \alpha_n\|Kx_n - G(y_n)\| + (1-\alpha_n)\|P_C[P_C(z_n - \mu_2 D_2 z_n) \\ & -\mu_1 D_1 P_C(z_n - \mu_2 D_2 z_n)] - G(y_n)\| \\ = \ & \alpha_n\|Kx_n - G(y_n)\| + (1-\alpha_n)\|G(z_n) - G(y_n)\| \\ \leq \ & \alpha_n\|Kx_n - G(y_n)\| + (1-\alpha_n)\|x_n - y_n\| \\ \longrightarrow \ & 0. \end{aligned} \tag{3.51}$$

According to Lemma 2.4 we obtain $p \in \mathcal{G}$. Further, let us show that $p \in \bar{\Gamma}$. On the other hand $z_{n_k} = J_\lambda^{B_1}(x_{n_k} + \xi A^*(J_\lambda^{B_2} - I)Ax_{n_k})$ can be rewritten as

$$\frac{(x_{n_k} - z_{n_k}) + \xi A^*(J_\lambda^{B_2} - I)Ax_{n_k}}{\lambda} \in B_1 z_{n_k}. \tag{3.52}$$

By passing to limit $k \to \infty$ in (3.52) and by taking into account (3.35) and (3.43) and the fact that the graph of a maximal monotone operator is weakly-strongly closed, we obtain $0 \in B_1(p)$, i.e., $p \in \text{SOLVIP}(B_1)$. Furthermore, since $\{x_n\}$ and $\{z_n\}$ have the same asymptotical behavior, $\{Ax_{n_k}\}$ weakly converges to Ap. Again, by (3.35) and the fact that the resolvent $J_\lambda^{B_2}$ is nonexpansive and Lemma 2.3, we obtain that $Ap \in B_2(Ap)$, i.e., $Ap \in \text{SOLVIP}(B_2)$. Thus, $p \in Fix(S) \cap \bar{\Gamma} \cap \mathcal{G}$.

Hence it follows from (2.11) and (3.50) that

$$\begin{aligned} \limsup_{n\to\infty}\langle K\bar{x} - \bar{x}, x_n - \bar{x}\rangle &= \lim_{i\to\infty}\langle K\bar{x} - \bar{x}, x_{n_i} - \bar{x}\rangle \\ &= \langle K\bar{x} - \bar{x}, p - \bar{x}\rangle \\ &\leq 0. \end{aligned} \tag{3.53}$$

Step 6. We will prove that $\lim_{n\to\infty} x_n = \bar{x}$.

Indeed, since $G : C \to C$ is nonexpansive, we have

$$\|t_n - \bar{x}\| = \|G(z_n) - G(\bar{x})\| \leq \|x_n - \bar{x}\|. \tag{3.54}$$

Note that

$$\begin{aligned} & \langle Kx_n - \bar{x}, y_n - \bar{x}\rangle \\ = \ & \langle Kx_n - \bar{x}, x_n - \bar{x}\rangle + \langle Kx_n - \bar{x}, y_n - x_n\rangle \\ = \ & \langle Kx_n - K\bar{x}, x_n - \bar{x}\rangle + \langle K\bar{x} - \bar{x}, x_n - \bar{x}\rangle + \langle Kx_n - \bar{x}, y_n - x_n\rangle \\ \leq \ & \rho\|x_n - \bar{x}\|^2 + \langle K\bar{x} - \bar{x}, x_n - \bar{x}\rangle + \|Kx_n - \bar{x}\|\|y_n - x_n\|. \end{aligned} \tag{3.55}$$

Utilizing (2.7) and Lemma 2.5, we obtain from (3.54), (3.55) and the convexity of $\|\cdot\|^2$

$$\begin{aligned}
& \|x_{n+1}-\bar{x}\|^2 \\
= \; & \|\beta_n(x_n-\bar{x})+\gamma_n(y_n-\bar{x})+\delta_n(Sy_n-\bar{x})\|^2 \\
\leq \; & \beta_n\|x_n-\bar{x}\|^2+(\gamma_n+\delta_n)\left\|\frac{1}{\gamma_n+\delta_n}[\gamma_n(y_n-\bar{x})+\delta_n(Sy_n-\bar{x})]\right\|^2 \\
\leq \; & \beta_n\|x_n-\bar{x}\|^2+(\gamma_n+\delta_n)\|y_n-\bar{x}\|^2 \\
\leq \; & \beta_n\|x_n-\bar{x}\|^2+(\gamma_n+\delta_n)[(1-\alpha_n)^2\|t_n-\bar{x}\|^2+2\alpha_n\langle Kx_n-\bar{x},y_n-\bar{x}\rangle] \\
\leq \; & \beta_n\|x_n-\bar{x}\|^2+(\gamma_n+\delta_n)[(1-\alpha_n)\|x_n-\bar{x}\|^2+2\alpha_n\langle Kx_n-\bar{x},y_n-\bar{x}\rangle] \\
= \; & (1-(\gamma_n+\delta_n)\alpha_n)\|x_n-\bar{x}\|^2+(\gamma_n+\delta_n)2\alpha_n\langle Kx_n-\bar{x},y_n-\bar{x}\rangle \\
\leq \; & (1-(\gamma_n+\delta_n)\alpha_n)\|x_n-\bar{x}\|^2 \\
& +(\gamma_n+\delta_n)2\alpha_n[\rho\|x_n-\bar{x}\|^2+\langle K\bar{x}-\bar{x},x_n-\bar{x}\rangle+\|Kx_n-\bar{x}\|\|y_n-\bar{x}\|] \\
\leq \; & [1-(1-2\rho)(\gamma_n+\delta_n)\alpha_n]\|x_n-\bar{x}\|^2 \\
& +(\gamma_n+\delta_n)2\alpha_n[\langle K\bar{x}-\bar{x},x_n-\bar{x}\rangle+\|Kx_n-\bar{x}\|\|y_n-\bar{x}\|] \\
= \; & [1-(1-2\rho)(\gamma_n+\delta_n)\alpha_n]\|x_n-\bar{x}\|^2 \\
& +(1-2\rho)(\gamma_n+\delta_n)\alpha_n\frac{2[\langle K\bar{x}-\bar{x},x_n-\bar{x}\rangle+\|Kx_n-\bar{x}\|\|y_n-\bar{x}\|]}{1-2\rho}. \qquad (3.56)
\end{aligned}$$

Note that $\liminf_{n\to\infty}(1-2\rho)(\gamma_n+\delta_n)>0$. It follows that $\sum_{n=0}^{\infty}(1-2\rho)(\gamma_n+\delta_n)\alpha_n=\infty$. It is clear that

$$\limsup_{n\to\infty}\frac{2[\langle K\bar{x}-\bar{x},x_n-\bar{x}\rangle+\|Kx_n-\bar{x}\|\|y_n-\bar{x}\|]}{1-2\rho}\leq 0 \qquad (3.57)$$

because $\limsup_{n\to\infty}\langle K\bar{x}-\bar{x},x_n-\bar{x}\rangle\leq 0$ and $\lim_{n\to\infty}\|x_n-y_n\|=0$. Therefore, all conditions of Lemma 2.2 are satisfied. Consequently, we immediately deduce that $x_n\to\bar{x}$. This completes the proof.

■

4. Consequently results

* Let H_1 and H_2 be two real Hilbert spaces and let $C\subseteq H_1$ and $Q\subseteq H_2$ be nonempty closed convex subsets. Let $A:H_1\to H_2$ be a bounded linear operator and $D_i:C\to H$ be η_i-inverse strongly monotone for $i=1,2$. Let $S:C\to C$ be a k-strictly pseudocontractive mapping such that $Fix(S)\cap\bar{\Gamma}\cap\mathcal{G}\neq\emptyset$. Let $K:C\to C$ be ρ-contraction with $\rho\in[0,\frac{1}{2})$. For fixed $u\in C$ and given $x_0\in C$ arbitrarily, let the sequences $\{x_n\},\{y_n\}$ and $\{z_n\}$ be generated iteratively by

$$\begin{cases} z_n=J_\lambda^{B_1}(x_n+\xi A^*(J_\lambda^{B_2}-I)Ax_n), \\ y_n=\alpha_nu+(1-\alpha_n)P_C[P_C(z_n-\mu_2D_2z_n)-\mu_1D_1P_C(z_n-\mu_2D_2z_n)] \\ x_{n+1}=\beta_nx_n+\gamma_ny_n+\delta_nSy_n,\ \forall n\geq 0, \end{cases} \qquad (4.1)$$

where $\mu_i\in(0,2\eta_i)$ for $i=1,2,\{\alpha_n\},\{\beta_n\},\{\gamma_n\}\subset[0,1],\lambda>0$ and $\xi\in(0,\frac{1}{L}),L$ is the spectral radius of the operator A^*A, A^* is the adjoint of A. Assume that the following conditions are satisfied:

(i) $\beta_n+\gamma_n+\delta_n=1$ and $(\gamma_n+\delta_n)k\leq\gamma_n$ for all $n\geq 0$;

(ii) $\lim_{n\to\infty} \alpha_n = 0,\ \sum_{n=1}^{\infty} \alpha_n = \infty$;

(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$; and $\liminf_{n\to\infty} \delta_n > 0$;

(iv) $\lim_{n\to\infty} (\gamma_{n+1}/(1-\beta_{n+1}) - \gamma_n/(1-\beta_n)) = 0$.

Then, the sequence $\{x_n\}$ generated by (4.1) converges strongly to $\bar{x} \in P_{Fix(S)\cap\bar{\Gamma}\cap\mathcal{G}}u$ if and only if $\lim_{n\to\infty} \|y_n - z_n\| = 0$. Furthermore $(\bar{x}, \bar{y})$ is a solution of the general system (1.5) of variational inequalities, where $\bar{y} = P_C(\bar{x} - \mu_2 D_2 \bar{x})$.

* Let H_1 and H_2 be two real Hilbert spaces and let $C \subseteq H_1$ and $Q \subseteq H_2$ be nonempty closed convex subsets. Let $A : H_1 \to H_2$ be a bounded linear operator and $D_i : C \to H$ be η_i-inverse strongly monotone for $i = 1, 2$. Let $S : C \to C$ be a nonexpansive mapping such that $Fix(S) \cap \bar{\Gamma} \cap \mathcal{G} \neq \emptyset$. Let $K : C \to C$ be ρ-contraction with $\rho \in [0, \frac{1}{2})$. For given $x_0 \in C$ arbitrarily, let the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ be generated iteratively by

$$\begin{cases} z_n = J_\lambda^{B_1}(x_n + \xi A^*(J_\lambda^{B_2} - I)Ax_n), \\ y_n = \alpha_n K x_n + (1-\alpha_n)P_C[P_C(z_n - \mu_2 D_2 z_n) - \mu_1 D_1 P_C(z_n - \mu_2 D_2 z_n)] \\ x_{n+1} = \beta_n x_n + \gamma_n y_n + \delta_n S y_n, \ \forall n \geq 0, \end{cases} \tag{4.2}$$

where $\mu_i \in (0, 2\eta_i)$ for $i = 1, 2, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset [0, 1], \lambda > 0$ and $\xi \in (0, \frac{1}{L}), L$ is the spectral radius of the operator A^*A, A^* is the adjoint of A. Assume that the following conditions are satisfied:

(i) $\beta_n + \gamma_n + \delta_n = 1$ and $(\gamma_n + \delta_n)k \leq \gamma_n$ for all $n \geq 0$;

(ii) $\lim_{n\to\infty} \alpha_n = 0,\ \sum_{n=1}^{\infty} \alpha_n = \infty$;

(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$; and $\liminf_{n\to\infty} \delta_n > 0$;

(iv) $\lim_{n\to\infty} (\gamma_{n+1}/(1-\beta_{n+1}) - \gamma_n/(1-\beta_n)) = 0$.

Then, the sequence $\{x_n\}$ generated by (4.2) converges strongly to $\bar{x} \in P_{Fix(S)\cap\bar{\Gamma}\cap\mathcal{G}}K\bar{x}$ if and only if $\lim_{n\to\infty} \|y_n - z_n\| = 0$. Furthermore $(\bar{x}, \bar{y})$ is a solution of the general system (1.5) of variational inequalities, where $\bar{y} = P_C(\bar{x} - \mu_2 D_2 \bar{x})$.

Corollary 4.1. *Let H_1 and H_2 be two real Hilbert spaces and let $C \subseteq H_1$ and $Q \subseteq H_2$ be nonempty closed convex subsets. Let $A : H_1 \to H_2$ be a bounded linear operator and $D_i : C \to H$ be η_i-inverse strongly monotone for $i = 1, 2$. Let $S : C \to C$ be a nonexpansive mapping such that $Fix(S) \cap \bar{\Gamma} \cap \mathcal{G} \neq \emptyset$. Let $K : C \to C$ be ρ-contraction with $\rho \in [0, \frac{1}{2})$. For fixed $u \in C$ and given $x_0 \in C$ arbitrarily, let the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ be generated iteratively by*

$$\begin{cases} z_n = J_\lambda^{B_1}(x_n + \xi A^*(J_\lambda^{B_2} - I)Ax_n), \\ y_n = \alpha_n u + (1-\alpha_n)P_C[P_C(z_n - \mu_2 D_2 z_n) - \mu_1 D_1 P_C(z_n - \mu_2 D_2 z_n)] \\ x_{n+1} = \beta_n x_n + \gamma_n y_n + \delta_n S y_n, \ \forall n \geq 0, \end{cases} \tag{4.3}$$

where $\mu_i \in (0, 2\eta_i)$ for $i = 1, 2, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset [0, 1], \lambda > 0$ and $\xi \in (0, \frac{1}{L}), L$ is the spectral radius of the operator A^*A, A^* is the adjoint of A. Assume that the following conditions are satisfied:

(i) $\beta_n + \gamma_n + \delta_n = 1$ and $(\gamma_n + \delta_n)k \leq \gamma_n$ for all $n \geq 0$;

(ii) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(iii) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$; *and* $\liminf_{n\to\infty} \delta_n > 0$;

(iv) $\lim_{n\to\infty} (\gamma_{n+1}/(1-\beta_{n+1}) - \gamma_n/(1-\beta_n)) = 0$.

Then, the sequence $\{x_n\}$ *generated by (4.3) converges strongly to* $\bar{x} \in P_{Fix(S)\cap\bar{\Gamma}\cap\mathcal{G}}u$ *if and only if* $\lim_{n\to\infty} \|y_n - z_n\| = 0$. *Furthermore* $(\bar{x}, \bar{y})$ *is a solution of the general system (1.5) of variational inequalities, where* $\bar{y} = P_C(\bar{x} - \mu_2 D_2 \bar{x})$.

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References

[1] L.C. Ceng, J.C. Yao, An extragradient like approximation method for variational inequality problems and fixed point problems, Appl. Math. Comput. 190(2007) 205–215.

[2] R. Wangkeeree, R. Wangkeeree, A general iterative methods for variational inequality problems and mixed equilibrium problems and fixed point problems of strictly pseudocontractive mappings in Hilbert spaces, Fixed Point Theory Appl. 32(2007), Article ID 519065.

[3] S.S. Zeng, N.C. Wong, J.C. Yao, Convergence of hybrid steepest-descent methods for generalized variational inequalities, Acta Math. Sin. English Ser. 22(1)(2006) 1–12.

[4] W. Takahashi, M. Toyoda, Weak convergence theorems for nonexpansive mappings and monotone mappings, Journal of Optimization Theory and Applications 118(2)(2003) 417–428.

[5] N. Nadezhkina, W. Takahashi, Weak convergence theorem by an extragradient method for nonexpansive mappings and monotone mappings, Journal of Optimization Theory and Applications 128(1)(2006) 191–201.

[6] L.C. Zeng, J.C. Yao, Strong convergence theorem by an extragradient method for fixed point problems and variational inequality problems, Taiwanese Journal of Mathematics 10(5)(2006) 1293–1303.

[7] G.M. Korpelevič, An extragradient method for finding saddle points and for other problems, E'konomikai Matematicheskie Metody 12(4)(1976) 747–756.

[8] R.U. Verma, On a new system of nonlinear variational inequalities and associated iterative algorithms, Mathematical Sciences Research Hot-Line 3(8)(1999) 65–68.

[9] L.C. Ceng, C. Wang, J.C. Yao, Strong convergence theorems by a relaxed extragradient method for a general system of variational inequalities, Mathematical Methods of Operations Research 67(3)(2008) 375–390.

[10] Y. Yao, Y.C. Liou, S.M. Kang, Approach to common elements of variational inequality problems and fixed point problems via a relaxed extragradient method, Computers & Mathematics with Applications 59(11)(2010) 3472–3480.

[11] A. Moudafi, Split monotone variational inclusions. J. Optim. Theory Appl. 150(2011) 275–283.

[12] Y. Censor, T. Elfving, A multiprojection algorithm using Bregman projections in a product space, Numerical Algorithms 8(1994) 221–239.

[13] C. Byrne, Y. Censor, A. Gibali and S. Reich, The split common null point problem. J. Nonlinear Convex Anal. 13(4)(2012) 759–775.

[14] K.R. Kazmi, S.H. Rizvi, An iterative method for split variational inclusion problem and fixed point problem for a nonexpansive mapping. Optim. Lett. 2013; https://doi.org/10.1007/s11590-013-0629-2.

[15] G.L. Acedo, H.K. Xu, Iterative methods for strict pseudo-contractions in Hilbert spaces, Nonlinear Analysis: Theory, Methods & Applications 67(7)(2007) 2258–2271.

[16] T. Suzuki, Strong Convergence of Krasnoselskii and Mann's Type Sequences for One-Parameter Nonexpansive Semigroups Without Bochner Integrals, Journal of Mathematical Analysis and Applications 305(2005) 227–239.

[17] H.K. Xu, T.H. Kim, Convergence of hybrid steepest-descent methods for variational inequalities, J. Optim. Theory Appl. 119(2003) 659–678.

[18] K. Geobel, W.A. Kirk, Topics in Metric Fixed Point Theory, Cambridge Studies in Advanced Mathematics, cambridge University Press, 1990.

[19] G. Marino, H.K. Xu, Weak and strong convergence theorems for strict pseudo-contractions in Hilbert spaces, Journal of Mathematical Analysis and Applications 329(1)(2007) 336–346.