Mathematical modeling of piezoelectric thin plates and slender beams through functional analysis

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Abstract : A simplified but accurate modeling of linearly piezoelectric thin plates or slender beam are derived by a rigorous study of the asymptotic behavior of a three-dimensional body when some of its dimensions, considered as parameters, tend to zero. The study is carried out by some tools of applied functional analysis like singular perturbations in variational equations.

1 Introduction

A modeling of thin linearly piezoelectric plates was proposed in [1], [2]. This modeling, as previously done in [3] for thin linearly elastic plates, is derived by a rigorous study of the asymptotic behavior of a three-dimensional body when its thickness, considered as a parameter, tends to zero. The obtained model is simpler that the genuine three-dimensional one because it involves two-dimensional problems set over the mean surface of the plate. But this model is also accurate because of some convergence results of the fields when the thickness goes to zero. The derivation of these models is as follows. First, the linear boundary value problem associated with the equilibrium of the plate of thickness 2ε is transformed in a variational equation whose existence and uniqueness of a solution s^ε is obtained by the Lions-Stampacchia theorem. This problem set on a variable set $\Omega^\varepsilon = \omega \times (-\varepsilon, \varepsilon)$ is transformed by a suitable scaling $S(\varepsilon)$ (a change of coordinates and of unknowns) in an equivalent problem set on a fixed domain $\Omega = \omega \times (-1, 1)$ with $s(\varepsilon) = S(\varepsilon)s^\varepsilon$ as unique solution. Next, a technique of singular perturbations in variational equations permits the determination of the limit $\bar{s}$ of $s(\varepsilon)$ when ε goes to zero. Finally, the proposal of model is the problem, set on Ω^ε , solved by the inverse scaling $\bar{s}^\varepsilon$ of $\bar{s}$. Actually, depending on the boundary conditions, two types of models are obtained: they correspond to the cases of sensors or of actuators.

Here, we try to extend this modeling to the case of slender linearly piezoelectric rods. As it is known for slender linearly elastic rods, the models and their derivations are more complex, thus we confine to the case of a rod working as a

sensor and made of a particular class of monoclinic piezoelectric material.

2   Setting the problem

For all $\zeta = (\zeta_1, \zeta_2, \zeta_3)$ in R^3 we denote (ζ_1, ζ_2) by $\hat{\zeta}$ and $\mathfrak{H}$ stands for the space $S^3 \times R^3$ where S^3 denotes 3×3 the space of all symmetric matrices. The greek indices run in $\{1, 2\}$ while the latin ones run in $\{1, 2, 3\}$. The reference configuration of a linearly piezoelectric slender rod is the closure in R^3 of the set $\Omega^\varepsilon := \varepsilon\omega \times (0, L)$, where L is a real positive number , ε a small one and ω is a bounded domain of R^2 with a Lipschitz boundary $\partial\omega$ and such that:
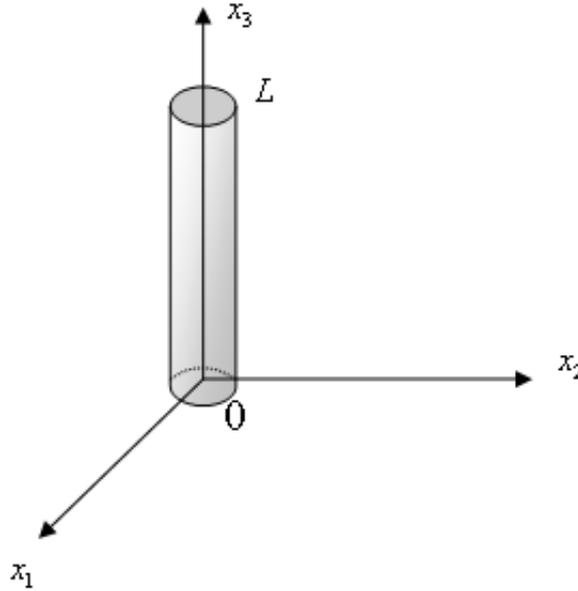

Figure 1: A piezoelectric slender beam

$$\int_\omega x_\alpha d\hat{x} \;=\; \int_\omega x_1 x_2 d\hat{x} \;= 0 \tag{2.1}$$

The lateral part of the boundary of the plate $\varepsilon\partial\omega \times (0, L)$ is denoted by Γ^ε_{lat} while its basis are $\Gamma^\varepsilon_0 = \varepsilon\partial\omega \times \{0\}$ and $\Gamma^\varepsilon_L = \varepsilon\partial\omega \times \{L\}$. Let $(\Gamma^\varepsilon_{mD}, \Gamma^\varepsilon_{mN})$ and $(\Gamma^\varepsilon_{eD}, \Gamma^\varepsilon_{eN})$ two suitable partitions of $\partial\Omega^\varepsilon$ with both Γ^ε_{mD} and Γ^ε_{eD} of strictly positive surface measure. The beam is, one hand, clamped along Γ^ε_{mD} and at an electrical potential j^ε_0 on Γ^ε_{eD} , on the other hand, subjected to body forces f^ε in

Ω^ε . Furthermore, the rod is subjected to surface forces g^ε and electrical loading d^ε on Γ^ε_{mN} and Γ^ε_{eN} respectively. We denoted the outward unit normal to $\partial\Omega^\varepsilon$ by n^ε and assume that $\Gamma^\varepsilon_{mD} = \Gamma^\varepsilon_0$. Then the equations determining the state $s^\varepsilon = (u^\varepsilon, \phi^\varepsilon)$ at equilibrium read as:

$$\begin{cases} div\sigma^\varepsilon \,+\, f^\varepsilon \,=\, 0 \text{ in } \Omega^\varepsilon, \sigma^\varepsilon \cdot n^\varepsilon = g^\varepsilon \text{ on } \Gamma^\varepsilon_{\text{m,N}}, u^\varepsilon = 0 \text{ on } \Gamma^\varepsilon_{\text{m,D}} \\ divD^\varepsilon \,=\, 0 \text{ in } \Omega^\varepsilon, D^\varepsilon \cdot n^\varepsilon = d^\varepsilon \text{ on } \Gamma^\varepsilon_{\text{e,N}}, \varphi^\varepsilon = \varphi^\varepsilon_0 \text{ on } \Gamma^\varepsilon_{\text{e,D}} \\ (\sigma^\varepsilon, D^\varepsilon) = M^\varepsilon\,(x)\,(e(u^\varepsilon), \nabla\varphi^\varepsilon) \text{ in } \Omega^\varepsilon \end{cases} \qquad (2.2)$$

where $u^\varepsilon, \varphi^\varepsilon, e\,(u^\varepsilon)\,, \sigma^\varepsilon$ and D^ε respectively denote the displacement and electrical fields, the linearized strain tensor, the stress tensor and the electrical displacement. As previously M^ε , is an element of $\mathcal{L}\,(\mathcal{H}, \mathcal{H})$ such that:

$$\begin{cases} \sigma^\varepsilon \,=\, a^\varepsilon e(u^\varepsilon) \,-\, b^\varepsilon \nabla\varphi^\varepsilon \,, \\ D^\varepsilon \,=\, (\,b^\varepsilon)^T e(u^\varepsilon) \,+\, c^\varepsilon \nabla\varphi^\varepsilon \end{cases} \qquad (2.3)$$

with $(b^\varepsilon)^T$ the transpose of b^ε , a^ε , and c^ε symmetric and positive. To give a variational formulation of (2.3) we make the following regularity hypothesis on the exterior loading:

$$(\text{H}_1)\begin{cases} (f^\varepsilon, g^\varepsilon, d^\varepsilon) \in L^2\,(\Omega^\varepsilon)^3 \times L^2\,(\Gamma^\varepsilon_{mN})^3 \times L^2\,(\Gamma^\varepsilon_{eN}) \\ \varphi^\varepsilon_0 \text{ has an } \text{H}^1\,(\Omega^\varepsilon) \text{ extension into } \Omega^\varepsilon, \text{ still denoted by } \varphi^\varepsilon_0 \end{cases}$$

and define on the space of electromechanical states

$$V^\varepsilon := \left\{ r = (v, \psi) \in H^1_{\Gamma^\varepsilon_{mD}}\,(\Omega^\varepsilon)^3 \times H^1_{\Gamma^\varepsilon_{eD}}\,(\Omega^\varepsilon) \right\} \qquad (2.4)$$

a bilinear form m^ε :

$$m^\varepsilon\,(r, q) = m^\varepsilon\,((v, \psi)\,, (w, \varphi)) := \int\limits_{\Omega^\varepsilon} M^\varepsilon\,(e\,(v)\,, \nabla\psi) \cdot (e\,(w)\,, \nabla\varphi)\,dx^\varepsilon \qquad (2.5)$$

and a linear form L^ε

$$L^\varepsilon = L^\varepsilon\,((v, \psi)) := \int\limits_{\Omega^\varepsilon} f^\varepsilon \cdot v dx^\varepsilon + \int\limits_{\Gamma^\varepsilon_{m,N}} g^\varepsilon \cdot v ds^\varepsilon + \int\limits_{\Gamma^\varepsilon_{e,N}} d^\varepsilon \psi ds^\varepsilon \qquad (2.6)$$

$$\wp\,(\Omega^\varepsilon):\begin{cases} \text{Find } s^\varepsilon = (u^\varepsilon, \phi^\varepsilon) \in (0, \phi^\varepsilon_0) + V^\varepsilon \text{ such that} \\ m^\varepsilon\,(s^\varepsilon, r) \,=\, L^\varepsilon\,(r)\,, \forall \text{r} \in V^\varepsilon. \end{cases}$$

Then, the physical problem, set on the real beam $a^\varepsilon, b^\varepsilon, c^\varepsilon$ takes the form: Thus, with the additional and realistic assumptions of boundedness of and of uniform ellipticity of a^ε and c^ε :

$$(\text{H}_2)\begin{cases} M^\varepsilon \in L^\infty\,(\Omega^\varepsilon, \mathcal{L}\,(\mathcal{H})) \\ \exists \kappa^\varepsilon > 0; M^\varepsilon\,(x^\varepsilon)\,h \cdot h \geqslant \kappa^\varepsilon\,|h|^2_{\mathcal{H}}\,, \forall h \in \mathcal{H}, \text{ a.e. } x \in \Omega^\varepsilon \end{cases}$$

the theorem of Lions-Stampacchia implies the

Theorem 2.1. *Under assumptions* $(H_1) - (H_2)$ *the problem* $\wp(\Omega^\varepsilon)$ *has a unique solution.*

To derive a simplified and accurate model, the very question is to study the behavior of s^ε when ε, considered as a parameter, tends to 0.

3   The scaling operation

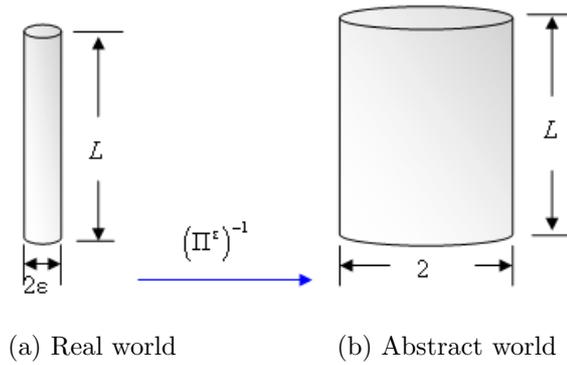Classically we come down to a fixed domain $\Omega = \omega \times (0, L)$ through the mapping Π^ε :

(a) Real world   (b) Abstract world

Figure 2: (a) Real state, physical state, defined on Ω^ε
(b) Abstract state or scaled state defined on Ω

$$x = (\hat{x}, x_3) \in \bar{\Omega} \mapsto \Pi^\varepsilon x = (\varepsilon\hat{x}, x_3) \in \bar{\Omega}^\varepsilon \tag{3.1}$$

Also, we drop the index ε for the images by $(\Pi^\varepsilon)^{-1}$ of the previous geometric sets. We make the following assumptions concerning i) the electromechanical coefficients:

(H_3) $\begin{cases} \exists M \in L^\infty(0, L; \mathcal{L}(\mathfrak{H})) \text{ such that } M^\varepsilon(\Pi^\varepsilon x) = M(x) = M(x_3), \forall x \in \Omega \\ \text{and } \exists \kappa > 0 : M(x_3) h \cdot h \geqslant \kappa |h|^2_{\mathfrak{H}} \ \forall h \in \mathfrak{H} \text{ a.e. } x \in \Omega. \end{cases}$

Moreover the material is monoclinic: $a_3 = a_{333} = b_3 = b_{33} = 0$.
ii) the electromechanical loading:

$$(H_4) \begin{cases} \text{there exist } (f, g, d) \in L^2(\Omega)^3 \times L^2(\Gamma_{mN})^3 \times L^2(\Gamma_{eN}) \text{ such that} \\ f_\alpha^\varepsilon(\pi^\varepsilon x) = \varepsilon^2 f_\alpha(x), f_3^\varepsilon(\pi^\varepsilon x) = \varepsilon f_3(x), \forall x \in \Omega, \\ g_\alpha^\varepsilon(\pi^\varepsilon x) = \varepsilon^2 g_\alpha(x), g_3^\varepsilon(\pi^\varepsilon x) = \varepsilon g_3(x), \forall x \in \Gamma_L, \\ g_\alpha^\varepsilon(\pi^\varepsilon x) = \varepsilon^3 g_\alpha(x), g_3^\varepsilon(\pi^\varepsilon x) = \varepsilon^2 g_3(x), \forall x \in \Gamma_{mN} \cap \Gamma_{lat}, \\ d^\varepsilon(\pi^\varepsilon x) = \varepsilon d(x), \forall x \in \Gamma_{eN} \cap (\Gamma_0 \cup \Gamma_L), \\ d^\varepsilon(\pi^\varepsilon x) = \varepsilon^2 d(x), \forall x \in \Gamma_{eN} \cap \Gamma_{lat}, \\ \varphi_0^\varepsilon(\pi^\varepsilon x) = \varepsilon \varphi_0(x), \forall x \in \Gamma_{eD} \end{cases}$$

iii) the boundedness of the exterior electrical loading:

$$(H_5) \begin{cases} \text{the extension of } \varphi_0 \text{ into } \Omega \text{ does not depend on } \hat{x} \text{ and } \Gamma_{eD} \subset \Gamma_0 \cup \Gamma_L \\ \text{with meas } (\Gamma_{eD} \cap \Gamma_0) > 0 \end{cases}$$

(this implies that φ_0 may take two different constant values on $\Gamma_{eD} \cap \Gamma_0$ and $\Gamma_{eD} \cap \Gamma_L$ whenever the last set is of positive surface measure).

Also, we associate a scaled electromechanical state $s(\varepsilon) := (u(\varepsilon), \varphi(\varepsilon))$ defined on Ω with the true physical electromechanical state $s^\varepsilon = (u^\varepsilon, \varphi^\varepsilon)$ defined on Ω^ε. This scaled electromechanical state, without any physical meaning , is defined by:

$$\hat{u}(x^\varepsilon) = \hat{u}(\varepsilon)(x), \ u_3^\varepsilon(x^\varepsilon) = \varepsilon u(\varepsilon)_3(x), \ \varphi^\varepsilon(x^\varepsilon) = \varepsilon \varphi(\varepsilon)(x) \qquad (3.2)$$

for all $x^\varepsilon = \Pi^\varepsilon x$ in Ω^ε. This allows us to introduce the scaling operator S^ε :

$$S^\varepsilon s^\varepsilon(x^\varepsilon) := s(\varepsilon)(x), \forall x^\varepsilon = \Pi^\varepsilon x \in \Omega^\varepsilon \qquad (3.3)$$

Assumptions $(H_3), (H_4)$ together with the scaling mapping from equation (3.2) are classical. Actually, they are justified by the convergence results they lead to. If we just consider the displacement these hypotheses are the ones made in [4] and supply a mathematical justification of the Bernoulli-Navier theory of slender linearly elastic beams.

4 Variational formulation of the scaled problem

Let V be the space of the scaled electromechanical states:

$$V := \left\{ r = (v, \psi) \in H^1_{\Gamma_{mD}}(\Omega)^3 \times H^1_{\Gamma_{eD}}(\Omega) \right\} \qquad (4.1)$$

of course , $r \in V^\varepsilon$ if and only if $\Pi^\varepsilon r \in V$. Now , for all $r = (v, \psi) \in V$, we define the scaled strain tensor $e(\varepsilon, v)$ and the scaled electrical potential gradient $\nabla(\varepsilon, \psi)$ by:

$$e_{\alpha\beta}(\varepsilon, v) := \varepsilon^{-2} e_{\alpha\beta}(v), \ e_{\alpha 3}(\varepsilon, v) := \varepsilon^{-1} e_{\alpha 3}(v), \ e_{33}(\varepsilon, v) := e_{33}(v)$$

$$\nabla(\varepsilon, \psi)_\alpha := \varepsilon^{-1}\partial_\alpha\psi, \ \nabla(\varepsilon, \psi)_3 := \partial_3\psi \tag{4.2}$$

To simplify the notations, we set

$$k(r) := (e(v), \nabla\psi), \ k(\varepsilon, r) := (e(\varepsilon, v), \nabla(\varepsilon, \psi)) \tag{4.3}$$

and, as in (2.5) and (2.6), we introduce a bilinear form $m(\varepsilon)$ and a linear form L on V :

$$m(\varepsilon)(r, q) := \int_\Omega M(x)k(\varepsilon, r).k(\varepsilon, q)\, dx,$$

$$L(r) := \int_\Omega f.v\, dx + \int_{\Gamma_{mN}} g.v\, ds + \int_{\Gamma_{eN}} d\psi\, ds \tag{4.4}$$

so that, under the assumptions $(H_1) - (H_4)$ the scaled electromechanical state $s(\varepsilon) = (u(\varepsilon), \phi(\varepsilon))$ is the unique solution of the mathematical problem

$$\wp(\varepsilon, \Omega) : \begin{cases} \text{Find } s(\varepsilon) \in (0, \phi_0) + V \text{ such that} \\ m(\varepsilon)(s(\varepsilon), r) = L(r) \ \forall r \in V. \end{cases}$$

5 Asymptotic behavior of the scaled electromechanical state

5.1 The method

We will proceed in a similar way as in the case of plates by first introducing a suitable orthogonal decomposition of $\mathfrak{H}$ in three subspaces $\mathfrak{H}^1$, $\mathfrak{H}^2$ and $\mathfrak{H}^3$:

$$\begin{cases} \mathfrak{H}^1 := \{h = (e, g) \in \mathfrak{H}; e_{i3} = g_3 = 0\}, \mathfrak{H}^2 := \{h = (e, g) \in \mathfrak{H}; e_{\alpha\beta} = e_{33} = g_i = 0\} \\ \mathfrak{H}^3 := \{h = (e, g) \in \mathfrak{H}; e_{\alpha\beta} = e_{\alpha3} = g_\alpha = 0\} \end{cases}$$

$$\tag{5.1}$$

We denote the projection of any element h of $\mathfrak{H}$ on $\mathfrak{H}^1$, $\mathfrak{H}^2$, $\mathfrak{H}^3$ by h^1, h^2, h^3 respectively and consider the operators $M^{kl} \in \mathcal{L}(\mathfrak{H}^l, \mathfrak{H}^k)$ generated by M. The Hypothesis (H_3) implies that M^{11} and M^{33} are positive invertible operators on $\mathfrak{H}^1$ and $\mathfrak{H}^3$, this same assumption also implies:

$$(Mh)^1 = M^{11}h^1 + M^{13}h^3$$
$$(Mh)^3 = M^{31}h^1 + M^{33}h^3$$

and consequently:

$$(Mh)^3 = \left(M^{33} - M^{31}\left(M^{11}\right)^{-1}M^{13}\right)h^3 + M^{31}\left(M^{11}\right)^{-1}(Mh)^1 \tag{5.2}$$

The operator (again a kind of Schur complement!) of $\mathcal{L}(\mathfrak{H}^3, \mathfrak{H}^3)$

$$\tilde{M} := M^{33} - M^{31}\left(M^{11}\right)^{-1}M^{13} \tag{5.3}$$

can be represented by a 2×2 matrice while h^3 can be represented by an element of R^2 , this will be done from now on. Note that neither M^{33} nor $\tilde{M}$ are necessarily symmetric but nevertheless

$$\kappa \left| h^3 \right|^2 \leqslant \tilde{M}(x) h^3 \cdot h^3 \ , \forall h^3 \in R^2 \text{ a.e. } x \in \Omega \tag{5.4}$$

this is implied by the coercivity of and by the fundamental relation:

$$h^2 = 0, \ (Mh)^1 = 0 \ \Rightarrow \ \tilde{M}h^3 = (Mh)^3$$

$$\tilde{M}h^3 \cdot h^3 = (Mh)^3 \cdot h^3 = Mh \cdot h \geqslant \kappa \left| h \right|^2 \geqslant \kappa \left| h^3 \right|^2 \tag{5.5}$$

As in the case of plates, the key point of the asymptotic study will be to show that if $\bar{k}$ is the limit (in a suitable topology) of $k(\varepsilon, s(\varepsilon))$, then $\int_\omega \left(M(x_3) \bar{k}(x) \right)^1 d\hat{x} = 0$.

This will enable us to show that $\tilde{M}$ supplies the limit constitutive equations.

5.2 The functional framework

We will show that the limit displacements live in the space V_{BN} of Bernoulli Navier displacements:

$$V_{BN} := \left\{ v \in H^1_{\Gamma mD}(\Omega)^3 \, ; e_{\alpha\beta}(v) = e_{\alpha 3}(v) = 0 \right\} \tag{5.6}$$

while the limit electrical potential belongs to

$$\Phi := \left\{ \psi \in H^1_{\Gamma_{eD}}(\Omega) \, ; \hat{\nabla}\psi = 0 \right\} = \left\{ \begin{array}{l} \psi; \psi(x) = \psi(x_3), \psi \in H^1(0, L), \\ \psi(0) = 0 \text{ and } \psi(L) = 0 \text{ if meas}(\Gamma_{eD} \cap \Gamma_L) > 0 \end{array} \right\} \tag{5.7}$$

We recall that for all v in V_{BN} , there exists a unique couple $\left(v^b, v^s\right)$ in $H^{1,0}(0, L)^2 \times H^{2,0}(0, L)^2$ such that:

$$\hat{v}(x) = v^b(x_3), v_3(x) = v^s(x_3) - x_\alpha \frac{dv^b_\alpha}{dx_3}(x_3) \tag{5.8}$$

with

$$H^{1,0}(0, L) = \left\{ v \in H^1(0, L) \, ; v(0) = 0 \right\}, H^{2,0}(0, L) = \left\{ v \in H^1(0, L) \, ; \frac{dv}{dx_3} \in H^{1,0}(0, L) \right\}$$

and note that for all v in V_{BN} :

$$e_{33}(v) = \frac{dv^s}{dx_3}(x_3) - x_\alpha \frac{d^2 v^b_\alpha}{dx_3^2}(x_3) \tag{5.9}$$

Finally, let

$$S := V_{BN} \times \Phi, \ X := H^1_{\Gamma_{mD}}(\Omega)^3 \times H^1_{\Gamma_{eD}}(\Omega) \tag{5.10}$$

the Korn and Poincaré inequality allows us to define on S and X the following hilbertian norms:

$$
\begin{aligned}
|(v, \psi)|_S^2 &= |e\,(v)|_{L^2(\Omega, S^3)}^2 + |\nabla \psi|_{L^2(\Omega)^3}^2 \\
|(v, \psi)|_X^2 &= |e\,(v)|_{L^2(\Omega, S^3)}^2 + |\psi|_{L^2(\Omega)}^2 + |\nabla \psi|_{L^2(\Omega)^3}^2
\end{aligned}
\tag{5.11}
$$

The set $(0, \phi_0) + S$ will appear to be the limit set of scaled electromechanical states.

5.3   The limit scaled problem

We define the following linear and bilinear forms:

$$
\begin{aligned}
\cdot\,(v^s, \psi) \in H_{(0,L)}^{1,0} \times \Phi \mapsto L^s\,(v^s, \psi) = &\int_0^L \left(\int_\omega f_3\,(\hat{x}, x_3)\,d\hat{x} + \int_{\partial\omega} g_3\,(\hat{x}, x_3)\,d\hat{l} \right) v^s\,(x_3)\,dx_3 \\
&+ \int_0^L \left(\int_{\partial\omega} d\,(\hat{x}, x_3)\,d\hat{l} \right) \psi\,(x_3)\,dx_3 \\
&+ \left(\int_\omega g_3\,(\hat{x}, L)\,d\hat{l} \right) v^s\,(L) + \theta_L \psi\,(L)
\end{aligned}
\tag{5.12}
$$

$$
\theta_L = \begin{cases} 0 \text{ if meas } (\Gamma_{eD} \cap \Gamma_L) > 0 \\[2mm] \displaystyle\int_\omega d\,(\hat{x}, L)\,d\hat{x} \text{ if meas}\,(\Gamma_{eD} \cap \Gamma_L) = 0 \end{cases}
$$

$$
\begin{aligned}
\cdot\,v_\alpha^b \in H^{2,0}\,(0, L) \mapsto L_\alpha^b\,(v_\alpha^b) = &\int_0^L \left(\int_\omega \hat{f}\,(\hat{x}, x_3)\,d\hat{x} + \int_{\partial\omega} \hat{g}\,(\hat{x}, x_3)\,d\hat{l} \right) v_\alpha^b\,(x_3)\,dx_3 \\
&- \left[\int_\omega x_\alpha f_3\,(\hat{x}, x_3)\,d\hat{x} + \int_\omega x_\alpha g_3\,(\hat{x}, x_3)\,d\hat{l} \right] \frac{dv_\alpha^b}{dx_3}\,(x_3)\,dx_3 \\
&+ \left(\int_\omega \hat{g}\,(\hat{x}, L)\,d\hat{x} \right) v_\alpha^b\,(L) - \left(\int_{\partial\omega} x_\alpha g_3\,(\hat{x}, x_3)\,d\hat{l} \right) \frac{dv_\alpha^b}{dx_3}\,(L)
\end{aligned}
\tag{5.13}
$$

$$
\cdot\,((u^s, \varphi), (v^s, \psi)) \in \left(H^{1,0}\,(0, L) \times \Phi \right)^2 \mapsto m^s\,((u^s, \varphi), (v^s, \psi))
$$

$$
|\omega| \int_0^L \tilde{M}\,(x_3) \left(\frac{du^s}{dx_3}, \frac{d\phi}{dx_3} \right) (x_3) \cdot \left(\frac{dv^s}{dx_3}, \frac{d\psi}{dx_3} \right) (x_3)\,dx_3
\tag{5.14}
$$

$$\cdot\left(u_\alpha^b, v_\alpha^b\right) \in H^{2,0}\left(0, L\right)^2 \mapsto m_\alpha^b\left(u_\alpha^b, v_\alpha^b\right) := I_\alpha^2\left(\omega\right) \int\limits_0^L \tilde{M}_{22}\left(x_3\right) \frac{d^2 u_\alpha^b}{dx_3^2}\left(x_2\right) \frac{d^2 v_\alpha^b}{dx_3^2}\left(x_3\right) dx_3$$

$$(5.15)$$

where the convention of summation on the repeated indices is not used and $I_\alpha^2(\omega) = \int_\omega x_\alpha^2\, d\hat{x}$. We have the following convergence result :

Theorem 5.1. *Under the assumptions and when ? tends to 0, the family of the unique solutions of strongly converges in X to the unique solution of :*

$$\bar{\wp}\left(\Omega\right) : \begin{cases} \textit{Find } s \in \left(0, \phi_0\right) + S \textit{ such that} \\ m^s\left(\left(u^s, \phi\right), \left(v^s, \psi\right)\right) = L^s\left(v^s, \psi\right) \; \forall\left(v^s, \psi\right) \in H^{1,0}\left(0, L\right) \times \Phi \\ m^b\left(u_\alpha^b, v_\alpha^b\right) = L_\alpha^b\left(v_\alpha^b\right) \; \forall v_\alpha^b \in H^{2,0}\left(0, L\right) \; \alpha = 1, 2 \end{cases}$$

Proof. 1st step: The family $\left(s\left(\varepsilon\right)\right)_{\varepsilon > 0}$ is bounded in X and, therefore, there exists a subsequence, not relabeled, such that

$$\left(s\left(\varepsilon\right), k\left(\varepsilon, s\left(\varepsilon\right)\right)\right) \rightharpoonup \left(\bar{s}, \bar{k}\right) \; \text{in } X \times L^2\left(\Omega, \mathfrak{H}\right) \; k^\alpha\left(s\left(\varepsilon\right)\right) \to 0 \; \text{in } L^2\left(\Omega, \mathfrak{H}^\alpha\right)$$

$$k\left(\bar{s}\right)^3 = \bar{k}^3. \tag{5.16}$$

where the $\rightharpoonup$ and $\to$ symbols stand for the weak and strong convergence respectively. This is an obvious consequence of the assumptions $\left(H_3\right), \left(H_5\right)$ the Poincaré and Korn inequalities and of the trace theory in Sobolev spaces.

2nd step: If $\left(M\bar{k}\right)^1 = \left(\bar{\sigma}_{\alpha\beta}, \bar{D}_\alpha\right)$, then

$$\int_\omega \bar{\sigma}_{\alpha\beta}(x)\, d\hat{x} = \int_\omega x_\gamma \bar{\sigma}_{\alpha\beta}(x)\, d\hat{x} = \int_\omega \bar{D}_\alpha(x)\, d\hat{x} = \int_\omega x_\gamma \bar{D}_\alpha(x)\, d\hat{x} = 0 \qquad (5.17)$$

Indeed let h arbitrary in $C_0^\infty\left(0, L\right)$ and $\alpha,\ \beta = \{\ 1,\ 2\}$ and consider the element $E^{\alpha\beta}$ of the space S^2 of 2×2 symmetric matrices such that

$$\left(E^{\alpha\beta}\right)_{\mu\nu} = \frac{1}{2}\left(\delta_{\mu\alpha}\delta_{\gamma\beta} + \delta_{\gamma\alpha}\delta_{\mu\beta}\right)$$

δ being the Kronecker symbol $\left(\delta_{\mu\gamma} = 0 \text{ if } \mu \neq \gamma, \delta_{\mu\gamma} = 1 \text{ if } \mu = \gamma\right)$. Let v defined by $\hat{v}\left(x\right) = \varepsilon^2\left(E^{\alpha\beta}\hat{x}\right)h\left(x_3\right), v_3\left(x\right) = 0$, we note that

$$e_{\alpha\beta}\left(\varepsilon, v\right)\left(x\right) = E^{\alpha\beta}h\left(x_3\right), e_{\alpha 3}\left(\varepsilon, v\right) = \frac{\varepsilon}{2}\left(E^{\alpha\beta}\hat{x}\right)\frac{dh}{dx_3}\left(x_3\right), e_{33}\left(\varepsilon, v\right) = 0$$

So that taking $r = \left(v, 0\right)$ in the formulation of $\wp\left(\varepsilon, \Omega\right)$ and letting ε go to zero yield

$$\int\limits_0^L \left(\int_\omega \bar{\sigma}_{\alpha\beta}\left(x\right) d\hat{x}\right) h\left(x_3\right) dx_3 = 0, \; \forall h \in C_0^\infty\left(0, L\right)$$

that is to say

$$r(x) = (0, \varepsilon x_\alpha h(x_3))$$

Next, we use $r(x) = (0, \varepsilon x_\alpha h(x_3))$ and $r(x) = (0, \varepsilon x_\gamma^2 h(x_3))$ to deduce

$$\int_\omega \bar{D}_\alpha(x)\, d\hat{x} = \int_\omega x_\gamma \bar{D}_\alpha(x)\, d\hat{x} = 0$$

We obtain the remaining of (6.6) by using scaled displacement fields like $\left(\varepsilon^2 x_\gamma^2 (E^{\alpha\alpha}\hat{x}) h(x_3), 0\right)$ associated with vanishing scaled electrical potential fields.

3rd step: the limit problem
Choosing $r = (v, \psi)$ arbitrary in S in $\wp(\varepsilon, \Omega)$ the formulation of $\wp(\varepsilon, \Omega)$ and going to the limit imply that $\bar{s} \in (0, \varphi_0) + S$ and

$$L(r) = \int_\Omega \left(M\bar{k}\right)^3 k(r)^3, \forall r \in S$$

But, (2.1), (H_3), (4.2), (4.3), (5.2), (5.3), (5.8), (5.9), (5.12)-(5.15) and fundamentally (5.17) yield

$$L(r) = L^s(v^s, \psi) + \sum_{i=1}^{2} L_\alpha^b(v_\alpha^b)$$

$$\int_\Omega \left(M\bar{k}\right)^3 k(r)^3 = m^s((\bar{u}^s, \bar{\varphi}), (v^s, \psi)) + \sum_{\alpha=1}^{2} m_a^b(\bar{u}_a^b, v_a^b)$$

which proves that $\bar{s}$ solves the problem $\bar{\wp}(\Omega)$. As previously noted $\tilde{M}$ can be considered as a 2×2 positively defined matrix, hence the Lions-Stampacchia theorem implies that $\bar{\wp}(\Omega)$ has a unique solution and, consequently, the whole family $(s(\varepsilon))_{\varepsilon > 0}$ converges to it!

4th step: strong convergence
It suffices to proceed as in the case of plates (see [1],[2]) by taking $h = k(\varepsilon, s(\varepsilon)) - \bar{k}$ in (H_3).   □

6   Back to the problem $\wp(\Omega^\varepsilon)$: a proposal of a simplified and accurate model

We come back to the reference configuration $\bar{\Omega}^\varepsilon$ of the real slender beam through the operators Π^ε and $(S^\varepsilon)^{-1}$. With the solution $\bar{s}$ of $\bar{\wp}(\Omega)$ is associated a physical electromechanical state $\bar{s}^\varepsilon$ defined on Ω^ε by :

$$\bar{s}^\varepsilon(\Pi^\varepsilon x) = (S^\varepsilon)^{-1}\bar{s}(x), \ \forall x \in \Omega$$

This electromechanical state is the solution of a problem posed over Ω^ε which is the transportation by Π^ε of the (limit scaled) problem $\bar{\wp}(\Omega)$. This transported problem, set on Ω^ε, is our proposal to model slender piezoelectric beams of cross-section $\varepsilon\omega$ and length L. This model is simplified because the kinematics of the competing fields are simplified: Bernoulli-Navier displacements with electrical potential depending only of x_3. It is also accurate due to the previous convergence result.

Remark 6.1. *The assumptions (2.1) and (H_3) yield a decoupling between the (stretch-electric) behavior and the two bending ones.*

Remark 6.2. *It is possible to consider other electrical boundary conditions to get a model in the spirit of the case $p = 2$ for the plates, but the model and its derivation is rather more complex (see [5]).*

7   Conclusion

We have presented various models of structures with one or two small dimensions (thin flat plates and slender rods) and made of piezoelectric materials. These models are derived from the classical three-dimensional ones by a rigorous mathematical analysis of their asymptotic behaviors when the small dimensions, considered as parameters, tend to zero. These models are simpler than the genuine ones because they involve two-dimensional or one-dimensional problems and, consequently, are easier to implement numerically. They also are accurate because of some convergence results: the thinner the plate or the more slender the rod the sharper the model! This study demonstrates the power of applied functional or variational analysis to deal with problems stemming from continuum physics or engineering. In the case of a plate and depending on the boundary conditions, two models are obtained. They correspond to the physical situations where the plate acts as a sensor or as an actuator. For a slender rod, we confined to the sensor case.

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