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CONVERGENCE OF INERTIAL MODIFIED KRASNOSELSKII-MANN ITERATION WITH APPLICATION TO IMAGE RECOVERY

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Abstract The Krasnoselskii-Mann iteration plays an important role in the approximation of fixed points of nonexpansive operators. In this paper, we present a convergence rate analysis for the inertial modified Krasnoselskii-Mann iteration built from nonexpansive operators and give some application to image recovery.

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1. Introduction

Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\|\cdot\|$. Suppose that C is a nonempty closed and convex of H. A mapping $T : C \to C$ is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|,$$

for all $x, y \in C$. The set of fixed points of a mapping $T : C \to C$ is defined by $F(T) := \{x \in C : Tx = x\}$.

A significant body of work on iteration methods for fixed points problems has accumulated in literature (for example, see [2–5]). Specifically, the Mann algorithm [6, 7]:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n, \tag{1.1}$$

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for some suitably chosen scalars $\alpha_n \in [0,1]$. The iterative sequence $\{x_n\}$ converges weakly to a fixed point of T provided that $\alpha_n \in [0,1]$ satisfies

$$\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty. \tag{1.2}$$

Combettes [8] therefore considers the convergence of the inexact KrasnoselskiiMann iteration

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n(Tx_n + \varepsilon_n), \tag{1.3}$$

with a given starting point $x_1 \in H$, where ε_n represents an error in the evaluation of Tx_n. He proves weak convergence of the sequence $\{x_n\}$ under the assumptions that $F(T)$ is nonempty, $\alpha_n \in (0,1)$ satisfies (1.2), and the additional error condition

$$\sum_{n=1}^{\infty} \alpha_n \varepsilon_n < \infty.$$

Apart from the error due to the inexact evaluation of T, implementations of the KrasnoselskiiMann iteration produce an additional error due to the finite precision arithmetic of the computer. To get a complete picture of the practical numerical behaviour of the Krasnoselskii-Mann iteration, we are therefore forced to analyse the convergence properties of a scheme like

$$x_{n+1} := (1-\alpha_n)x_n + \alpha_n(Tx_n + \varepsilon_n) + \widetilde{\varepsilon}_n, \tag{1.4}$$

where, again, ε_n represents the error in the evaluation of Tx_n, whereas $\widetilde{\varepsilon}_n$ denotes the error resulting from the finite precision arithmetic. To keep the notation simple, we can write this as

$$x_{n+1} := (1-\alpha_n)x_n + \alpha_n Tx_n + r_n, \tag{1.5}$$

for some vector r_n that we call the residual since it represents the difference between the exact Krasnoselskii-Mann iteration and its inexact counterpart.

Kanzow and Shehu [9] consider the more general inexact scheme

$$x_{n+1} := \alpha_n x_n + \beta_n Tx_n + r_n, \tag{1.6}$$

where $\alpha_n, \beta_n \in [0,1]$ are suitable numbers satisfying $\alpha_n + \beta_n \leq 1$, hence these two numbers do not necessarily sum up to one, and r_n is again called the residual vector. Despite the fact that this generalizes existing choices, it turns out in their subsequent analysis that, to some extent, the particular choice $\alpha_n + \beta_n < 1$ also bridges the gap between weak and strong convergence results.

In 2008, Mainge [10] introduced the following inertial Mann algorithm by unifying the Mann algorithm and the inertial extrapolation:

$$\begin{cases} w_n = x_n + \alpha_n(x_n - x_{n-1}) \\ x_{n+1} = w_n + \lambda_n(Tw_n - w_n), \end{cases} \tag{1.7}$$

for each $n \geq 1$. He showed that the iterative sequence $\{x_n\}$ converges weakly to a fixed point of T under the following conditions:

(A1) $\alpha_n \in [0, \alpha)$ for any $\alpha \in [0,1)$;
(A2) $\sum_{n=1}^{\infty} \alpha_n \|x_n - x_{n-1}\|^2 < \infty$;
(A3) $0 < \inf_{n\geq 1} \lambda_n \leq \sup_{n \geq 1} \lambda_n < 1$.

To satisfy the summability condition (A2) of the sequence $\{x_n\}$, one needs to calculate $\{\alpha_n\}$ at each step (see [16]).

Very recently, Bot and Csetnek [12] got rid of the condition (A2) and replaced (A1) and (A3) by the following conditions, respectively:

(B1) $\delta > \frac{\alpha^2(1+\alpha)+\alpha\sigma}{1-\alpha^2}$;

(B2) $0 < \lambda \leq \lambda_n \leq \frac{\delta-\alpha[\alpha(1+\alpha)+\alpha\delta+\sigma]}{\delta[1+\alpha(1+\alpha)+\alpha\delta+\sigma]}$.

Inspired by the above work, in this article, we propose the inertial modified Krasnoselskii-Mann iteration, analyze the convergence of the proposed algorithms and give some application to image recovery.

2. Preliminaries

Lemma 2.1. *Let X be a real inner product space. Then the following statements hold:*

(a) $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle, \quad \forall\, x, y \in X$;

(b) $\|x-y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x-y, y\rangle, \quad \forall\, x, y \in X$;

(c) $\|\alpha x+\beta y\|^2 = \alpha(\alpha+\beta)\|x\|^2+\beta(\alpha+\beta)\|y\|^2-\alpha\beta\|x-y\|^2, \quad \forall x, y \in X, \forall \alpha, \beta \in \mathbb{R}$.

Lemma 2.2 (see [1])**.** *Let $\{\zeta_n\}$ and $\{\gamma_n\}$ be nonnegative sequences satisfying $\sum_{n=1}^{\infty} \zeta_n < \infty$ and $\gamma_{n+1} \leq \gamma_n + \zeta_n, n = 1, 2, \ldots$. Then, $\{\gamma_n\}$ is a convergent sequence.*

Lemma 2.3 (see [17], Lemma 3)**.** *Let $\{\mu_n\}, \{\psi_n\}$ and $\{\alpha_n\}$ be sequences in $[0,\infty)$ such that $\mu_{n+1} \leq \mu_n + \alpha_n(\mu_n - \mu_{n+1}) + \psi_n$ for all $n \geq 1$, $\sum_{n=1}^{\infty} \psi_n < \infty$ and there exists a real number α with $0 \leq \alpha_n \leq \alpha < 1$ for all $n \geq 1$. Then the following hold:*

(a) $\sum_{n=1}^{\infty}[\mu_n - \mu_{n-1}]_+ < \infty$, *where* $[t]_+ = \max\{t, 0\}$*;*

(b) *there exists* $\mu^* \in [0,\infty)$ *such that* $\lim_{n\to\infty} \mu_n = \mu^*$.

Lemma 2.4. *(Opial) Let C be a nonempty set of H and $\{x_n\}$ be a sequence in H such that the following two conditions hold:*

(a) *for every $x \in C$, $\lim_{n\to\infty} \|x_n - x\|$ exists;*

(b) *every sequential weak cluster point of $\{x_n\}$ is in C.*

Then $\{x_n\}$ converges weakly to a point in C.

Lemma 2.5 (see [18], Corollary 4.18)**.** *Let C be a nonempty closed convex subset of a real Hilbert space $H, T : C \to H$ be a nonexpansive mapping. Let $\{x_n\}$ be a sequence in C and $x \in H$ such that $x_n \rightharpoonup x$ and $Tx_n - x_n \to 0$ as $n \to \infty$. Then $x \in F(T)$.*

3. Main Results

Theorem 3.1. *Suppose that $T : H \to H$ is a nonexpansive mapping such that its set of fixed points $F(T)$ is nonempty. Let the sequence $\{x_n\}$ in H be generated by choosing $x_0 = x_1 \in H$ and using the recursion*

$$\begin{cases} w_n = x_n + \alpha_n(x_n - x_{n-1}) \\ x_{n+1} = \beta_n w_n + \delta_n T w_n + r_n, \end{cases} \tag{3.1}$$

where r_n denotes the residual vector (which represents the difference between the exact Krasnoselskii-Mann iteration and its inexact counterpart), $\{\alpha_n\}$ is chosen such that given

$\alpha \in [0,1)$, we have $0 \le \alpha_n \le \bar{\alpha}_n$ with $\bar{\alpha}_n$ defined by

$$\bar{\alpha}_n := \begin{cases} \min\left\{\alpha, \frac{\epsilon_n}{\|x_n - x_{n-1}\|}\right\} & \text{if } x_n \neq x_{n-1}, \\ \alpha & \text{otherwise,} \end{cases} \tag{3.2}$$

$\{\beta_n\}, \{\delta_n\}$ are sequences in $[0,1]$ and $\{\epsilon_n\}$ is a positive sequence satisfying

(a) $\beta_n + \delta_n \le 1$, $\sum_{n=1}^{\infty} \delta_n \beta_n = \infty$;
(b) $\sum_{n=1}^{\infty}(1 - \beta_n - \delta_n) < \infty$;
(c) $\sum_{n=1}^{\infty} \|r_n\| < \infty$;
(d) $\sum_{n=1}^{\infty} \epsilon_n < \infty$.

Then the following statements are true:

(i) *$\{x_n\}$ is bounded;*
(ii) *the sequence $\{x_n\}$ converges weakly to a point of $F(T)$.*

Proof. (i) Note that $\alpha_n\|x_n - x_{n-1}\| \le \epsilon_n$ and this implies that $\sum_{n=1}^{\infty} \alpha_n\|x_n - x_{n-1}\| \le \sum_{n=1}^{\infty} \epsilon_n < \infty$.

Taking arbitrarily $x^* \in F(T)$. We obtain

$$\begin{aligned} \|x_{n+1} - x^*\| &= \|\beta_n w_n + \delta_n T w_n + r_n - x^*\| \\ &= \|\beta_n(w_n - x^*) + \delta_n(Tw_n - x^*) + r_n - (1 - \beta_n - \delta_n)x^*\| \\ &\le \|\beta_n(w_n - x^*) + \delta_n(Tw_n - x^*)\| + \|r_n\| + (1 - \beta_n - \delta_n)\|x^*\| \\ &\le \beta_n\|w_n - x^*\| + \delta_n\|Tw_n - x^*\| + \|r_n\| + (1 - \beta_n - \delta_n)\|x^*\| \\ &\le (\beta_n + \delta_n)\|w_n - x^*\| + \|r_n\| + (1 - \beta_n - \delta_n)\|x^*\| \\ &\le \|w_n - x^*\| + \|r_n\| + (1 - \beta_n - \delta_n)\|x^*\| \\ &= \|x_n - x^* + \alpha_n(x_n - x_{n-1})\| + \|r_n\| + (1 - \beta_n - \delta_n)\|x^*\| \\ &\le \|x_n - x^*\| + \alpha_n\|x_n - x_{n-1}\| + \|r_n\| + (1 - \beta_n - \delta_n)\|x^*\|. \end{aligned} \tag{3.3}$$

Take $\zeta_n := \alpha_n\|x_n - x_{n-1}\| + \|r_n\| + (1 - \beta_n - \delta_n)\|x^*\|$. Then $\sum_{n=1}^{\infty} \delta_n < \infty$ and (3.3) becomes

$$\|x_{n+1} - x^*\| \le \|x_n - x^*\| + \zeta_n.$$

By Lemma 2.2, we get that $\lim_{n\to\infty} \|x_n - x^*\|$ exists and consequently, $\{x_n\}$ is bounded.

(ii)Since T is nonexpansive, it follows from Lemma 2.1 that

$$\begin{aligned}
&\|x_{n+1}-x^*\|^2\\
&=\|\beta_n w_n+\delta_n T w_n + r_n - x^*\|^2\\
&=\|\beta_n(w_n-x^*)+\delta_n(Tw_n-x^*)+r_n-(1-\beta_n-\delta_n)x^*\|^2\\
&\le \|\beta_n(w_n-x^*)+\delta_n(Tw_n-x^*)\|^2+2\langle r_n-(1-\beta_n-\delta_n)x^*, x_{n+1}-x^*\rangle\\
&=\beta_n(\beta_n+\delta_n)\|w_n-x^*\|^2+\delta_n(\beta_n+\delta_n)\|Tw_n-x^*\|^2\\
&\quad-\delta_n\beta_n\|w_n-Tw_n\|^2+2\langle r_n-(1-\beta_n-\delta_n)x^*, x_{n+1}-x^*\rangle\\
&\le (\beta_n+\delta_n)^2\|w_n-x^*\|^2-\delta_n\beta_n\|w_n-Tw_n\|^2\\
&\quad+2\langle r_n-(1-\beta_n-\delta_n)x^*, x_{n+1}-x^*\rangle\\
&\le \|w_n-x^*\|^2-\delta_n\beta_n\|w_n-Tw_n\|^2+2\langle r_n-(1-\beta_n-\delta_n)x^*, x_{n+1}-x^*\rangle\\
&=\|w_n-x^*\|^2-\delta_n\beta_n\|w_n-Tw_n\|^2+2(1-\beta_n-\delta_n)\langle r_n-x^*, x_{n+1}-x^*\rangle\\
&\quad+2(\beta_n+\delta_n)\langle r_n, x_{n+1}-x^*\rangle\\
&\le \|w_n-x^*\|^2-\delta_n\beta_n\|w_n-Tw_n\|^2+2[(1-\beta_n-\delta_n)\|r_n-x^*\|\\
&\quad+(\beta_n+\delta_n)\|r_n\|]\|x_{n+1}-x^*\|\\
&\le \|w_n-x^*\|^2-\delta_n\beta_n\|w_n-Tw_n\|^2+2[(1-\beta_n-\delta_n)\|r_n-x^*\|\\
&\quad+\|r_n\|]\|x_{n+1}-x^*\|\\
&=\|w_n-x^*\|^2-\delta_n\beta_n\|w_n-Tw_n\|^2+\varphi_n,
\end{aligned} \tag{3.4}$$

where $\varphi_n := 2[(1-\beta_n-\delta_n)\|r_n-x^*\|+\|r_n\|]\|x_{n+1}-x^*\|$. Observe that $\sum_{n=1}^{\infty}\varphi_n<\infty$ by Conditions (b)and (c) above. By using Lemma 2.1 and equation(3.1), we get

$$\begin{aligned}
\|w_n-x^*\|^2&=\|x_n+\alpha_n(x_n-x_{n-1})-x^*\|^2\\
&=\|(1+\alpha_n)(x_n-x^*)-\alpha_n(x_{n-1}-x^*)\|^2\\
&=(1+\alpha_n)\|x_n-x^*\|^2-\alpha_n\|x_{n-1}-x^*\|^2+\alpha_n(1+\alpha_n)\|x_n-x_{n-1}\|^2.
\end{aligned} \tag{3.5}$$

Using equation (3.5) in (3.4), we get

$$\begin{aligned}
&\|x_{n+1}-x^*\|^2-(1+\alpha_n)\|x_n-x^*\|^2+\alpha_n\|x_{n-1}-x^*\|^2\\
&\le -\delta_n\beta_n\|w_n-Tw_n\|^2+\alpha_n(1+\alpha_n)\|x_n-x_{n-1}\|^2+\varphi_n.
\end{aligned} \tag{3.6}$$

Observe that

$$\begin{aligned}
\alpha_n(1+\alpha_n)\|x_n-x_{n-1}\|^2&=\alpha_n\|x_n-x_{n-1}\|\Big((1+\alpha_n)\|x_n-x_{n-1}\|\Big)\\
&\le \alpha_n\|x_n-x_{n-1}\|M^*,
\end{aligned} \tag{3.7}$$

where $M^* := \sup_{n\ge 1}\Big((1+\alpha_n)\|x_n-x_{n-1}\|\Big)$. Observe that M^* exists since $\{x_n\}$ is bounded. Hence,

$$\sum_{n=1}^{\infty}\alpha_n(1+\alpha_n)\|x_n-x_{n-1}\|^2 \le \sum_{n=1}^{\infty}\alpha_n\|x_n-x_{n-1}\|M^*<\infty. \tag{3.8}$$

Therefore,

$$\lim_{n\to\infty} \alpha_n(1+\alpha_n)\|x_n - x_{n-1}\| = 0. \tag{3.9}$$

From (3.6), we obtain

$$\begin{aligned} &\delta_n\beta_n\|w_n - Tw_n\|^2 \\ &\le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \alpha_n(\|x_n - x^*\|^2 - \|x_{n-1} - x^*\|^2) \\ &\quad + \alpha_n(1+\alpha_n)\|x_n - x_{n-1}\|^2 + \varphi_n. \end{aligned} \tag{3.10}$$

Observe that $\lim_{n\to\infty}\varphi_n = 0$ in view of $\sum_{n=1}^{\infty}(1-\beta_n-\delta_n) < \infty$, $\sum_{n=1}^{\infty}\|r_n\| < \infty$ and $\{x_n\}$ is bounded. Therefore, we get from (3.10) that

$$\sum_{n=1}^{\infty}\delta_n\beta_n\|w_n - Tw_n\|^2 < \infty.$$

Therefore,

$$\liminf_{n\to\infty}\|w_n - Tw_n\| = 0.$$

Furthermore,

$$\|w_n - x_n\| = \alpha_n\|x_n - x_{n-1}\| \to 0,\ n\to\infty,$$

since $\sum_{n=1}^{\infty}\alpha_n\|x_n - x_{n-1}\| < \infty$.

We show that $\lim_{n\to\infty}\|w_n - Tw_n\| = 0$. Observe from (3.1), we get

$$\begin{aligned} &w_n - Tw_n \\ &= x_n + (w_n - x_n) - Tw_n \\ &= \beta_{n-1}(w_{n-1} - Tw_{n-1}) + (\delta_{n-1}+\beta_{n-1})Tw_{n-1} + r_{n-1} + (w_n - x_n) - Tw_n \\ &= \beta_{n-1}(w_{n-1} - Tw_{n-1}) + (w_n - x_n) + r_{n-1} + (\delta_{n-1}+\beta_{n-1})Tw_{n-1} - Tw_n \\ &= \beta_{n-1}(w_{n-1} - Tw_{n-1}) + (w_n - x_n) + r_{n-1} + (\delta_{n-1}+\beta_{n-1})Tw_{n-1} \\ &\quad - (\delta_{n-1}+\beta_{n-1})Tx_n + (\delta_{n-1}+\beta_{n-1})Tx_n - (\delta_{n-1}+\beta_{n-1})Tw_n \\ &\quad + (\delta_{n-1}+\beta_{n-1})Tw_n - Tw_n. \end{aligned} \tag{3.11}$$

Hence,

$$\begin{aligned} &\|w_n - Tw_n\| \\ &\le \beta_{n-1}\|w_{n-1} - Tw_{n-1}\| + (\delta_{n-1}+\beta_{n-1})\|x_n - w_{n-1}\| + \|w_n - x_n\| \\ &\quad + (\delta_{n-1}+\beta_{n-1})\|w_n - x_n\| + (1-\delta_{n-1}-\beta_{n-1})\|Tw_n\| + \|r_{n-1}\| \\ &\le \beta_{n-1}\|w_{n-1} - Tw_{n-1}\| + (\delta_{n-1}+\beta_{n-1})\|x_n - w_{n-1}\| + 2\|w_n - x_n\| \\ &\quad + (1-\delta_{n-1}-\beta_{n-1})\|Tw_n\| + \|r_{n-1}\|. \end{aligned} \tag{3.12}$$

Observe that

$$\begin{aligned} x_n - w_{n-1} &= (\beta_{n-1}-1)w_{n-1} + \delta_{n-1}Tw_{n-1} + r_{n-1} \\ &= (\delta_{n-1}+\beta_{n-1}-1)w_{n-1} + \delta_{n-1}(Tw_{n-1} - w_{n-1}) + r_{n-1}. \end{aligned} \tag{3.13}$$

So,

$$\begin{aligned}&\|x_n - w_{n-1}\|\\&\le (1-\delta_{n-1}-\beta_{n-1})\|w_{n-1}\| + \delta_{n-1}\|w_{n-1} - Tw_{n-1}\| + \|r_{n-1}\| \qquad (3.14)\\&\le (1-\delta_{n-1}-\beta_{n-1})\|w_{n-1}\| + (1-\beta_{n-1})\|w_{n-1} - Tw_{n-1}\| + \|r_{n-1}\|\end{aligned}$$

Also,

$$\|w_n - x_n\| = \theta_n\|x_n - x_{n-1}\|. \qquad (3.15)$$

Putting (3.14) and (3.15) into (3.12), we have for some $\bar{M} > 0$,

$$\begin{aligned}&\|w_n - Tw_n\|\\&\le \beta_{n-1}\|w_{n-1} - Tw_{n-1}\| + (1-\delta_{n-1}-\beta_{n-1})\|w_{n-1}\|\\&\quad + (1-\beta_{n-1})\|w_{n-1} - Tw_{n-1}\| + 2\|r_{n-1}\| + 2\theta_n\|x_n - x_{n-1}\|\\&\quad + (1-\delta_{n-1}-\beta_{n-1})\|Tw_n\|\\&= \|w_{n-1} - Tw_{n-1}\| + 2\theta_n\|x_n - x_{n-1}\| + (1-\delta_{n-1}-\beta_{n-1})(\|Tw_n\| + \|w_{n-1}\|)\\&\quad + 2\|r_{n-1}\|\\&\le \|w_{n-1} - Tw_{n-1}\| + 2\theta_n\|x_n - x_{n-1}\| + (1-\delta_{n-1}-\beta_{n-1})\bar{M} + 2\|r_{n-1}\|.\end{aligned} \qquad (3.16)$$

Take $\zeta_n := 2\theta_n\|x_n - x_{n-1}\| + (1-\delta_{n-1}-\beta_{n-1})\bar{M} + 2\|r_{n-1}\|$, we see that $\sum_{n=1}^{\infty}\zeta_n < \infty$. Using Lemma 2.2, we obtain that $\lim_{n\to\infty}\|w_n - Tw_n\|$ exists. Since $\liminf_{n\to\infty}\|w_n - Tw_n\| = 0$, we obtain that $\lim_{n\to\infty}\|w_n - Tw_n\| = 0$.

We conclude by using the result of Opial given in Lemma 2.4. We have proven in (i) above that for an arbitrary $x^* \in F(T)$, $\lim_{n\to\infty}\|x_n - x^*\|$ exists and $\{x_n\}$ is bounded. On the other hand, let x be a sequential weak cluster point of $\{x_n\}$, that is, the latter has a subsequence $\{x_{n_k}\}$ which converge weakly to x. Since $\lim_{n\to\infty}\|w_n - x_n\| = 0$, we get $w_{n_k} \rightharpoonup x$ as $k \to \infty$. Furthermore, we obtain $Tw_{n_k} - w_{n_k}$ as $k \to \infty$. Applying now Lemma 2.5 for the sequence $\{w_{n_k}\}$, we conclude that $x \in F(T)$. From Lemma 2.4, it follows that the sequence $\{x_n\}$ converges weakly to a point in $F(T)$. ■

4. Application to the Douglas-Rachford splitting method

Let us first recall the basics that are required to derive and analyze the Douglas-Rachford splitting method; for the corresponding details, we refer, for example, to the monograph by Bauschke and Combettes [18].

Let $\gamma > 0$ be a fixed parameter, and let us denote by

$$J_\gamma^A := (I + \gamma A)^{-1} \quad \text{and} \quad J_\gamma^B := (I + \gamma B)^{-1}$$

the resolvents of A and B, respectively, which are known to be firmly nonexpansive. Furthermore, let us write

$$R_\gamma^A := 2J_\gamma^A - I \quad \text{and} \quad R_\gamma^A := 2J_\gamma^B - I$$

for the corresponding reflections (also called Cayley operators), and note that the firm nonexpansiveness of the resolvents implies immediately that these reflections are nonexpansive operators.

Since one can show that $0 \in Tx$ for $T = A + B$ if and only if $x = J_\gamma^B(y)$, where y is a fixed point of the nonexpansive mapping $R_\gamma^A R_\gamma^B$, a natural way to find a zero of $T = A+B$ is therefore to apply the Krasnoselskii-Mann iteration to this operator, which yields the iteration

$$x_{n+1} := (1-\lambda_n)y_n + \lambda_n R_\gamma^A R_\gamma^B y_n, \quad n \geq 1, \tag{4.1}$$

which in turn gives an approximation in the original variables by setting $x_n := J_\gamma^B y_n$. Note that this iteration requires only the evaluation of the resolvents of A and B separately, not of their sum $T = A+B$. Recall that equation (4.1) is known as the Douglas-Rachford splitting method, whereas the special case $\lambda_n = 1$ for all $n \geq 1$ gives the PeacemanRachford splitting method.

Using the definitions of the reflection operators, we can rewrite the iteration equation (4.1) as

$$\begin{aligned} y_{n+1} :&= (1-\lambda_n)y_n + \lambda_n(2J_\gamma^A(2J_\gamma^B y_n - y_n) - 2J_\gamma^B y_n + y_n) \\ &= y_n + 2\lambda_n(J_\gamma^A(2J_\gamma^B y_n - y_n) - J_\gamma^B y_n). \end{aligned} \tag{4.2}$$

Following Combettes [8], we also allow errors a_n and b_n in the evaluation of the resolvents J_γ^A and J_γ^B, which generalized Douglas-Rachford splitting method:

$$y_{n+1} := y_n + 2\lambda_n(J_\gamma^A(2(J_\gamma^B y_n + b_n) - y_n) + a_n - (J_\gamma^B y_n + b_n)). \tag{4.3}$$

Next, we want to investigate the weak convergence property for inertial generalized Douglas-Rachford splitting algorithm in the following theorem.

Theorem 4.1. *Let H be a real Hilbert space. Let $\gamma \in (0,\infty)$, let $\{\beta_n\}$ and $\{\delta_n\}$ be real sequences in $[0,1]$ such that $\beta_n + \delta_n \leq 1$ for all $n \geq 1$. Suppose $\{a_n\}$ and $\{b_n\}$ are sequences in H. Assume that $0 \in ran(A+B)$. Let the sequence $\{y_n\}$ in H be generated by choosing $y_0 = y_1 \in H$ and using the recursion*

$$\begin{cases} w_n = y_n + \alpha_n(y_n - y_{n-1}) \\ x_{n+1} = \beta_n w_n + 2\delta_n(J_\gamma^A(2(J_\gamma^B y_n + b_n) - w_n) + a_n) - 2\delta_n(J_\gamma^B y_n + b_n) + \delta_n w_n \end{cases} \tag{4.4}$$

where $\{\alpha_n\}$ is chosen such that given $\alpha \in [0,1)$, we have $0 \leq \alpha_n \leq \bar{\alpha}_n$ with $\bar{\alpha}_n$ defined by

$$\bar{\alpha}_n := \begin{cases} \min\left\{\alpha, \ \frac{\epsilon_n}{\|y_n - y_{n-1}\|}\right\} & \text{if} \quad y_n \neq y_{n-1}, \\ \alpha & \text{otherwise}, \end{cases} \tag{4.5}$$

$\{\beta_n\}, \{\delta_n\}$ are sequences in $[0,1]$ and $\{\epsilon_n\}$ is a positive sequence satisfying

(a) $\beta_n + \delta_n \leq 1, \ \sum_{n=1}^{\infty} \delta_n \beta_n = \infty$;
(b) $\sum_{n=1}^{\infty}(1-\beta_n-\delta_n) < \infty$;
(c) $\sum_{n=1}^{\infty} \delta_n\big(\|a_n\| + \|b_n\|\big) < \infty$;
(d) $\sum_{n=1}^{\infty} \epsilon_n < \infty$.

Then the sequence $\{y_n\}$ generated by (4.4) converges weakly to some point $y \in H$ such that $J_\gamma^B y \in (A+B)^{-1}(0)$, i.e. $x := J_\gamma^B y$ is a solution of the monotone inclusion problem for the operator $T := A+B$.

Proof. Using the notation of the reflection operator, we set and define $T := R^A_\gamma R^B_\gamma$. Now, following the same line of arguments given in the proof of Theorem 25.6 of [18] and Corollary 5.2 of [8] and applying Theorem 3.1, it is not difficult to show that $\{y_n\}$ converges weakly to a fixed point of T. ∎

5. Numerical results on Image recovery

Let us suppose that we have a noisy image of dimension $n \times n$ with missing pixels. Our goal is to find the closest image to the original one. General image recovery problem can be formulated by the inversion of the following observation model:

$$g = Ky + b,$$

where g is the observed image, y is the unknown image, matrix K is a linear operation and b is the noise.

A regularization method should be used in the image restoration process. In the literature, there is the growing interest in using l_1 norm for solving these types of problems. The l_1 regularization can remove noise in the restoration process that it is given by

$$\min_y \frac{1}{2}\|ky - g\|^2 + \mu\|y\|_1, \tag{5.1}$$

where $\|\cdot\|$ denotes the Euclidean norm, μ is a positive regularization parameter which measures the trade-off between a good fit and a regularized solution, and $\|\cdot\|_1$ is the l_1 regularization term.

We are considering the real Hilbert spaces H. For a function $f : H \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$, we denote by $domf := \{x \in H : f(x) < +\infty\}$ its effective domain and say that f is proper if $domf \neq \emptyset$ and $f(x) > -\infty$ for all $x \in H$. We denote by $\Gamma(H)$ the family of proper, convex and lower semi-continuous extended real-valued functions defined on H.

The subdifferential of f at $x \in H$, with $f(x) \in \mathbb{R}$, is the set

$$\partial f(x) := \{u \in H | \forall z \in H, \langle y - x, u\rangle + f(x) \leq f(z)\}.$$

The Moreau envelope of index $\gamma \in (0, +\infty)$ of a function $f \in \Gamma(H)$ is the continuous convex function

$$\gamma f : H \to \overline{\mathbb{R}} \quad \text{such that} \quad x \mapsto \inf_{z \in H}\{f(z) + \frac{1}{2\gamma}\|z - x\|^2\}.$$

The operator

$$\text{prox}_{\gamma f} : H \to H \quad \text{such that} \quad x \mapsto \arg\min_{z \in H}\{f(z) + \frac{1}{2\gamma}\|z - x\|^2\}$$

is called the proximity operator of γf. We know that the notion of a proximity operator was introduced by Moreau in 1962 [20]. Notice that $J^{\partial f}_\gamma = (I + \gamma\partial f)^{-1} = \text{prox}_{\gamma f}$ and $\text{prox}_{\gamma f}$ is nonexpansive: $(\forall x \in H)(\forall y \in H)\ \|\text{prox}_{\gamma f}x - \text{prox}_{\gamma f}y\| \leq \|x - y\|$.

From (5.1), we set $f(y) = \frac{1}{2}\|Ky - g\|^2$ and $g = \mu\|y\|_1$. Let $B = \partial f$ and $A = \partial g$. In this case the proximal mapping with respect to f is

$$\begin{aligned} J^{\partial f}_\gamma y &= \text{prox}_{\gamma f}(y) \\ &= \arg\min_{z \in H}\{\frac{1}{2}\|Kz - g\|^2 + \frac{1}{2\gamma}\|z - y\|^2\} \\ &= (K^T K + \gamma^{-1} I)^{-1}(K^T g + \gamma^{-1} y), \end{aligned}$$

while $J^{\partial g}_{\gamma} y = \text{prox}_{\gamma g}(y)$ is the following soft-thresholding operator,

$$[\text{prox}_{\gamma g}(y)]_i = \text{sign}(y_i) \cdot \max\{|y_i| - \gamma\mu, 0\}, \quad i = 1, 2, 3, \ldots, n.$$

we consider two blurring functions from MATLAB: a motion blur (Matlab function is, "fspecial('motion',15,60)") and a Gaussian blur (Matlab function is, "fspecial('gaussian',5,5)") respectively, then add random noise. We compare our algorithm (IMDR) in Theorem 4.1 with $\alpha \in \{0.01, 0.05, 0.6, 0.9\}$, $\varepsilon_n = 100n^{-\frac{3}{2}}$, $\beta_n = 0.61$, $\delta_n = 0.39$, $\gamma = 14, \mu = 10^{-62}$ and Douglas-Rachford splitting (DRS) method [8] with $\lambda_n = 0.29$, $\gamma = 14, \mu = 10^{-62}$, since sequence $\{\text{prox}_{\gamma f}(y_n)\}$ converges to a solution and $b_n \to 0$ in (4.3) and (4.4),), we obtain the following result, which is immediately relevant to digital image recovery. In our paper, the comparison is done in terms of the relative error (relerr) defined as

$$\frac{\|x_n - y\|^2}{\|y\|^2},$$

the quality of image recovery is measured by the improvement in signal to noise ratio (ISNR). Note that ISNR defined as

$$\text{ISNR} = 10 \log \frac{\|y - g\|^2}{\|y - x_n\|^2},$$

where y, g, and x_n are the original image, the observed image, and estimated image at iteration n, respectively. All algorithms are implemented under Windows 10 and MATLAB 2017b running on a Dell laptop with Intel(R) Core(TM) i5 CPU and 4 GB of RAM. The stopping criterion of the algorithm is

$$\frac{\|x_{n+1} - x_n\|}{\|x_{n+1}\|} < 10^{-4}.$$

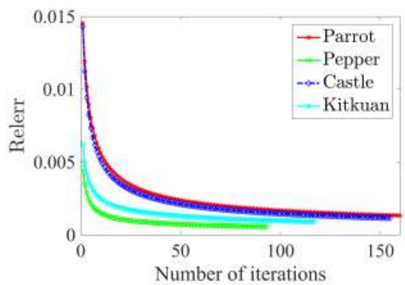
The test images are Parot (256×256), Castle (512×512), Pepper (256×256) and Kitkuan (222×222),which show in Fig. 1. We can see from Table 1 and the Figures 1-6 that our proposed algorithm (IMDR) in Theorem 4.1 is competitive with Douglas-Rachford splitting (DRS) method in [8] in terms of relative error (relerr) and signal to noise ratio (ISNR). We see from Table 1 that our proposed algorithm (IMDR) behaves better as the initial factor α_n approaches 1.

(A) Parrot (B) Castle (C) Pepper (D) Kitkuan

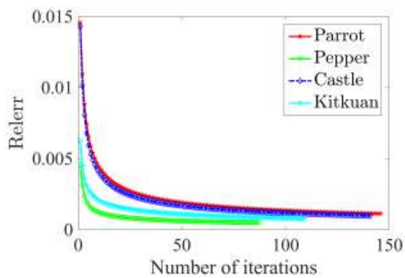
FIGURE 1. Test images

Table 1. Restoration comparison

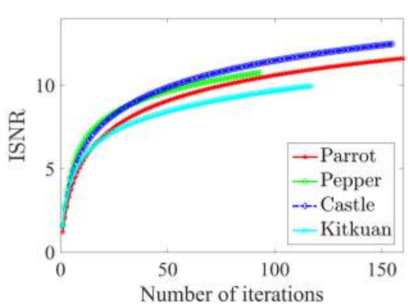
Image	Blur	DRS			IMDR			
		Time	ISNR	Relerr	α_n	Time	ISNR	Relerr
Parrot	motion	2.8556	11.6268	1.3257×10^{-3}	0.01	2.7314	11.6354	1.3230×10^{-3}
					0.05	2.7234	11.6573	1.3164×10^{-3}
					0.6	2.6513	12.0903	1.1915×10^{-3}
					0.9	2.5279	12.3824	1.1140×10^{-3}
Castle	Gaussian	7.2906	10.7766	5.6223×10^{-4}	0.01	7.8061	10.7834	5.6135×10^{-4}
					0.05	7.6866	10.8135	5.5748×10^{-4}
					0.6	7.4510	11.1924	5.1090×10^{-4}
					0.9	7.2898	11.4593	4.8045×10^{-4}
Pepper	motion	2.6250	12.4976	1.1623×10^{-3}	0.01	2.7144	12.5073	1.1597×10^{-3}
					0.05	2.6241	12.5284	1.1540×10^{-3}
					0.6	2.5235	12.9858	1.0387×10^{-3}
					0.9	2.5033	13.2938	9.6759×10^{-4}
Kitkuan	Gaussian	2.8338	9.9770	9.0492×10^{-4}	0.01	2.7971	9.9848	9.0328×10^{-4}
					0.05	2.7886	10.0140	8.9723×10^{-4}
					0.6	2.6593	10.4018	8.2059×10^{-4}
					0.9	2.5978	10.6692	7.7159×10^{-4}



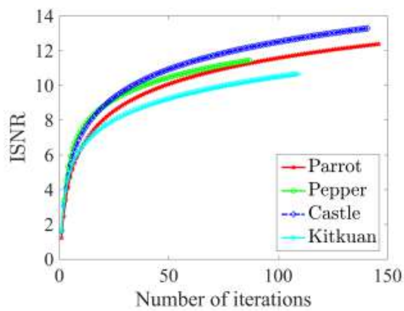
(A) Relerr for DRS



(B) Relerr for IMDR

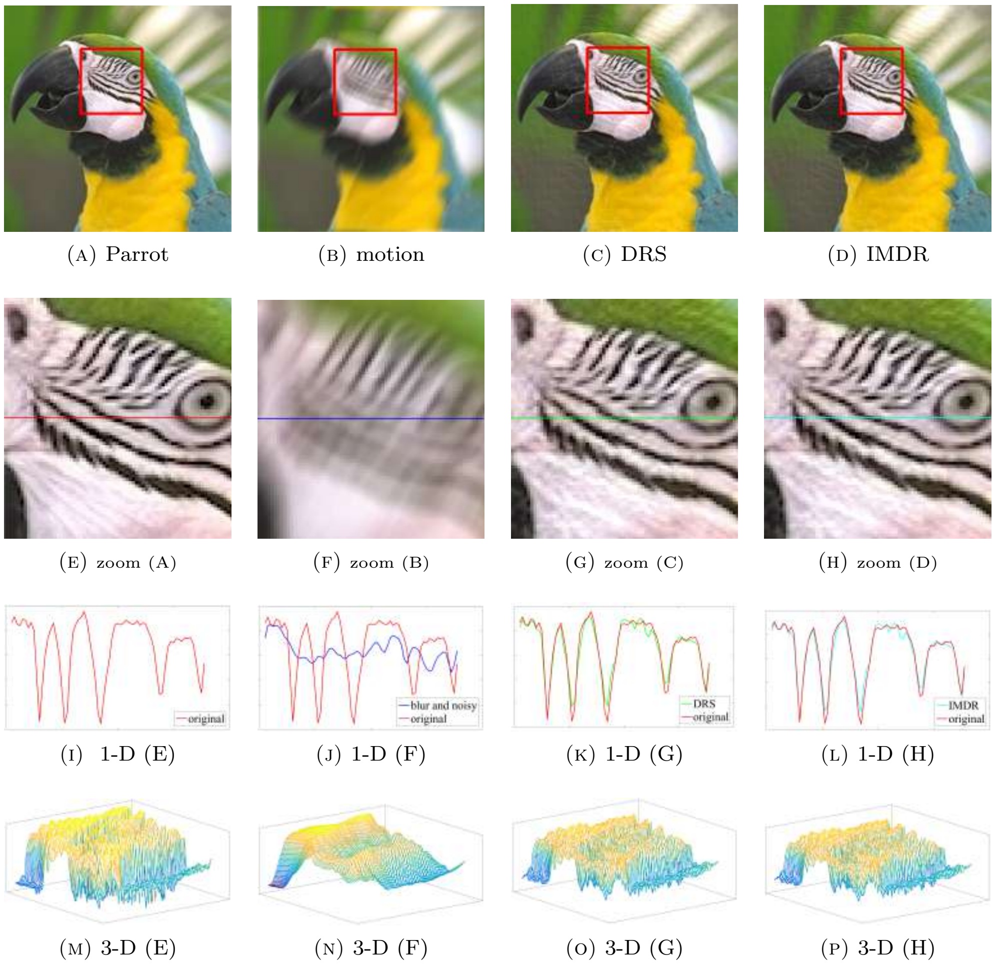


(C) ISNR for DRS



(D) ISNR for IMDR

Figure 2. Result for DRS and IMDR with $\alpha_n = 0.9$.



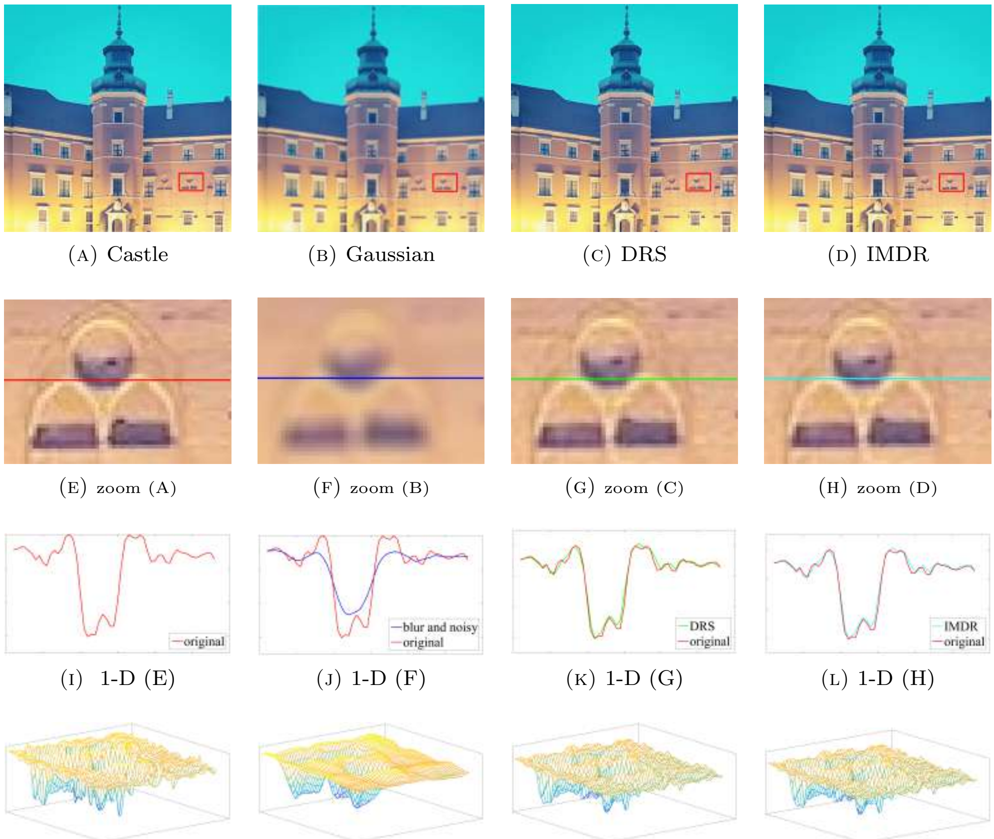
(A) Parrot (B) motion (C) DRS (D) IMDR

(E) zoom (A) (F) zoom (B) (G) zoom (C) (H) zoom (D)

(I) 1-D (E) (J) 1-D (F) (K) 1-D (G) (L) 1-D (H)

(M) 3-D (E) (N) 3-D (F) (O) 3-D (G) (P) 3-D (H)

FIGURE 3. Result for Parrot image with motion blur and random noise (IMDR with $\alpha_n = 0.9$).



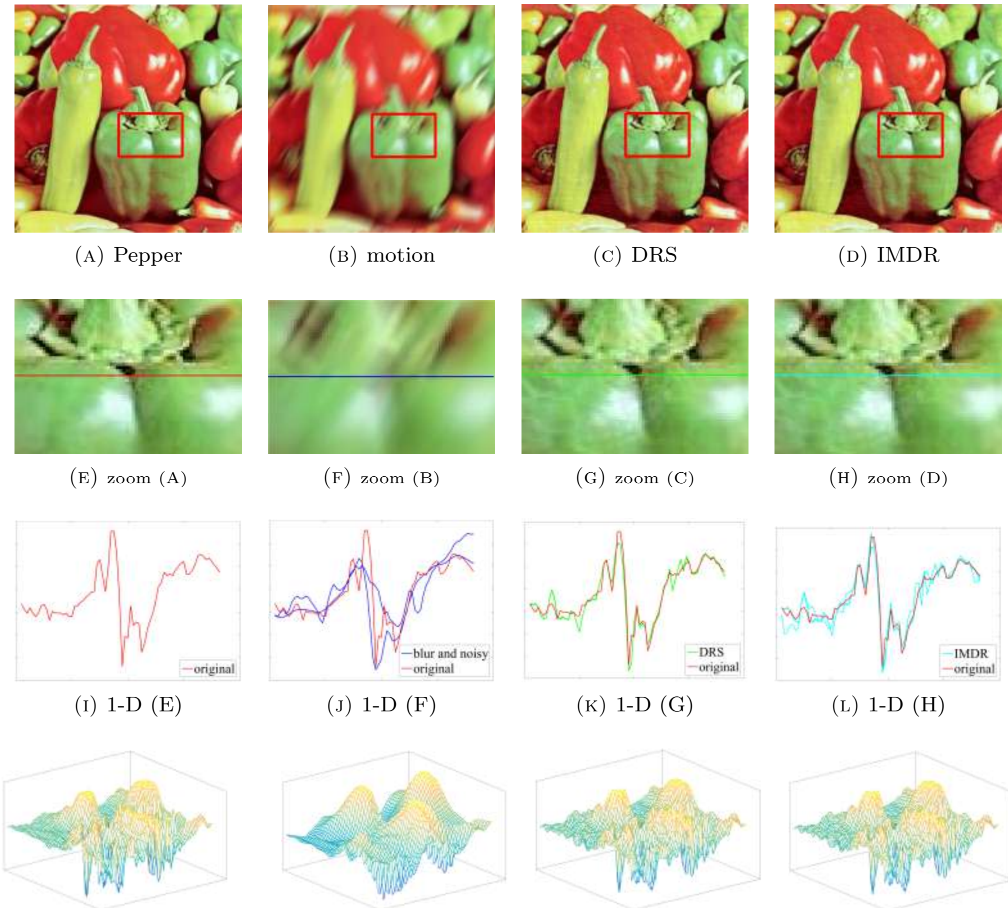
(A) Castle (B) Gaussian (C) DRS (D) IMDR

(E) zoom (A) (F) zoom (B) (G) zoom (C) (H) zoom (D)

(I) 1-D (E) (J) 1-D (F) (K) 1-D (G) (L) 1-D (H)

(M) 3-D (E) (N) 3-D (F) (O) 3-D (G) (P) 3-D (H)

FIGURE 4. Result for Castle image with Gaussian blur and random noise (IMDR with $\alpha_n = 0.9$).



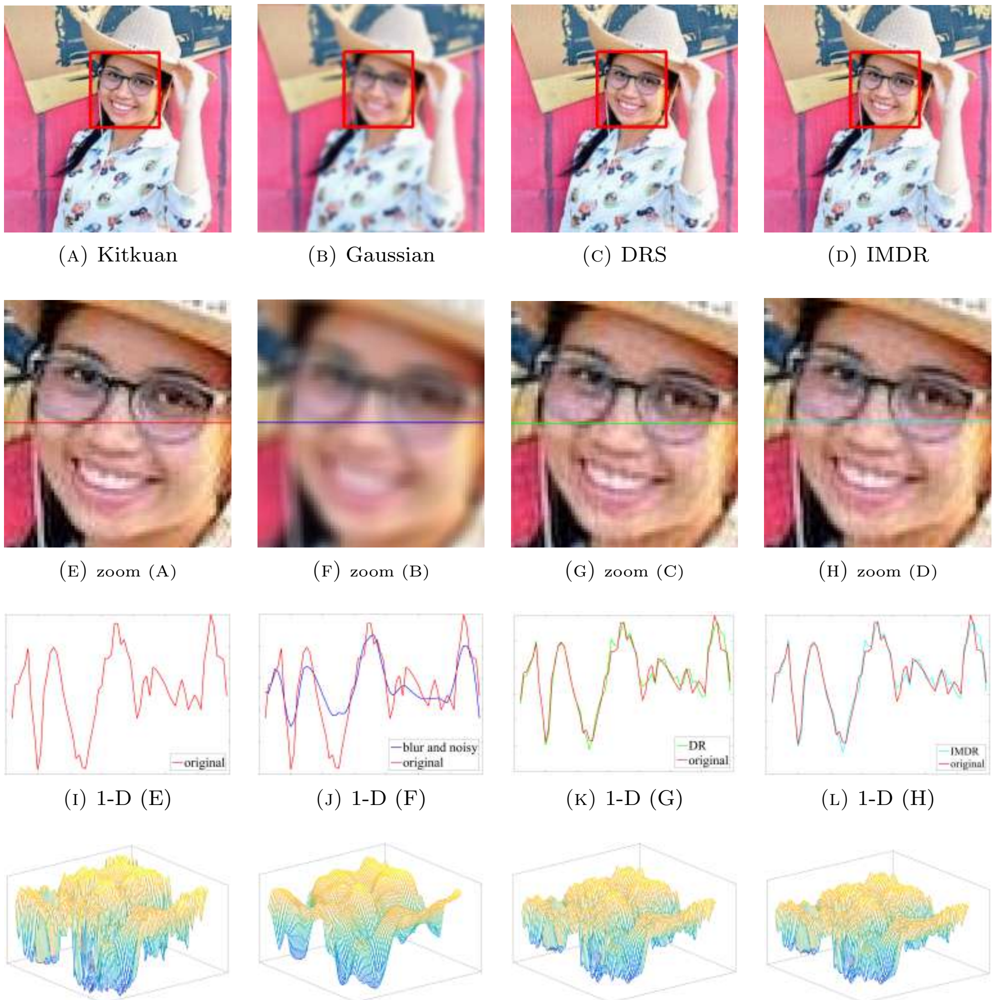
(A) Pepper (B) motion (C) DRS (D) IMDR

(E) zoom (A) (F) zoom (B) (G) zoom (C) (H) zoom (D)

(I) 1-D (E) (J) 1-D (F) (K) 1-D (G) (L) 1-D (H)

(M) 3-D (E) (N) 3-D (F) (O) 3-D (G) (P) 3-D (H)

FIGURE 5. Result for Pepper image with motion blur and random noise (IMDR with $\alpha_n = 0.9$).



(A) Kitkuan (B) Gaussian (C) DRS (D) IMDR

(E) zoom (A) (F) zoom (B) (G) zoom (C) (H) zoom (D)

(I) 1-D (E) (J) 1-D (F) (K) 1-D (G) (L) 1-D (H)

(M) 3-D (E) (N) 3-D (F) (O) 3-D (G) (P) 3-D (H)

FIGURE 6. Result for Kitkuan image with Gaussian blur and random noise (IMDR with $\alpha_n = 0.9$).

6. CONCLUSIONS

In this paper, we propose a generalized Krasnoselskii-Mann iteration with inertial extrapolation term for approximation of fixed point of nonexpansive mapping and obtain weak convergence result in real Hilbert spaces. We give some application of our result to image recovery problem and compare our proposed method with the Douglas-Rachford Algorithm. The numerical implementations and comparisons of our proposed method and Douglas-Rachford Algorithm show that our proposed method outperform the Douglas-Rachford Algorithm in terms of relative error (relerr) and signal to noise ratio (ISNR). In the next project, we give some rates of convergence of the proposed method and its variant with applications to inverse problems and comparison with the forward-backward method and Nesterov accelerations.

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