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SOME FIXED POINT RESULTS ON M_b-METRIC SPACES VIA SIMULATION FUNCTIONS

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Abstract In this research, theorems related with the fixed point were extended to be considered on M_b-metric spaces. The concept of an extension was based on the simulation functions introduced by Khojasteh et al. [10] and some results of MLAIKI et al. [13]. This article provides contents of the fixed point theory developed by many mathematicians, and our discovered result, the uniqueness theorem of a fixed point in complete M_b-metric space.

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1. Introduction and Preliminaries

The existence of the fixed point theorem in Banach space was first investigated by Banach himself who established the well known Banach contraction principle in 1922 [6]. Applications of the discovery play a major role in the existence theory of differential, integral, partial differential and functional equations [11]. This theorem is a principle tool for providing the existence of solutions in games theory, mathematical economic and some biological models [3, 11]. Ever since the idea of the fixed point theorem was proposed, many mathematicians have developed and extended a number of theories related to it.

In 1989, Bakhtin[5] (see also Czerwik [7]) introduced the concept of a b-metric space and proved some fixed point theorems for some contraction mapping in b-metric spaces. This apprehension generalizes Banach's contraction principle in metric space. After that Matthews[12] introduced the notion of a partial metric space and prolonged the contraction principle of Banach in that new framework in 1994. Shukla[20] combined both concepts of $b-$metric and partial metric spaces and proposed the partial $b-$metric space in 2014. The Kannan type fixed point theorem in partial $b-$metric spaces, which is an analog of Banach contraction principle, was also suggested as well.

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In 2014, Asadi et al.[2] introduced M-metric space, which extends the partial metric space and certain fixed point theorems obtained therein. In the later year, Khojasteh et al.[10] established the concept of a simulation functions with a view to consider a new class of the contractions, called $\mathcal{Z}$-contractions. In 2016, Mlaiki et al. generalized concept of M-metric spaces to M_b-metric spaces. The properties of M_b-metric space and the fixed point results based on the space were presented [13]. In 2017, Mongkolkeha et al.[14] proved some fixed point theorems for simulation functions in complete b-metric spaces with partially ordered by using wt-distance. very recently, Abdullahi and Kumam [1] introduced the concept of a partial $b_v(s)$-metric space as a generalization of a partial metric space and a $b_v(s)$-metric space and established some topological properties of the space and some fixed point results. Such understandings generalized, extended and improved several results that have been developed in the previous years.

The main ambition proposed in this article is to extend some results of Mlaiki[13]. The concept of $\mathcal{Z}_{mb}$-contraction was introduced. The theorems which have results on the existence and uniqueness of the fixed point in M_b-metric spaces were also explored.

2. Preliminaries

We begin with giving some notations and preliminaries that we shall need to state our results. This section provides definitions, examples and some theorems related with metric space, b-metric space, partial metric space, partial b-metric space, M-metric space, M_b metric space and $\mathcal{Z}$-contraction. From now on the letters $\mathbb{R}$ and $\mathbb{N}$ denote the set of all real numbers and the set of all natural numbers, respectively.

Definition 2.1. (Metric space)[8] Let X be a nonempty set. A function $d : X \times X \to [0, \infty)$ is said to be a metric on X if it satisfies the following conditions for all $x, y, z \in X$.

(m1) $d(x, y) = 0$ if and only if $x = y$;
(m2) $d(x, y) = d(y, x)$;
(m3) $d(x, y) \leq d(x, z) + d(z, y)$.

Here the pair (X, d) is called a *metric space.*

Definition 2.2. (b-Metric space)[5] Let X be a nonempty set and let a real number $s \geq 1$ be given. A function $d : X \times X \to [0, \infty)$ is said to be a *b-metric* if for all $x, y, z \in X$ the following conditions are satisfied:

(b1) $d(x, y) = 0$ if and only if $x = y$;
(b2) $d(x, y) = d(y, x)$;
(b3) $d(x, y) \leq s[d(x, z) + d(z, y)]$.

The pair (X, d) is called a *b-metric space.*

The given definition provides that every metric space is b-metric for $s = 1$ but not vice versa.

Example 2.3. [18] Let the function $d : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ be defined by $d(x, y) = |x - y|^2$. It is quite easy to figure out that d is a b-metric on $\mathbb{R}$ with $s = 2$. However it is not a metric on $\mathbb{R}$, as

$$d(1, 3) = 4 > 2 = d(1, 2) + d(2, 3).$$

Definition 2.4. (Partial metric space)[12] Let X be a nonempty set. A function $p : X \times X \to [0, \infty)$ is said to be a *partial metric* if for all $x, y, z \in X$ the following conditions are satisfied:

(p1) $p(x,x) = p(y,y) = p(x,y)$ if and only if $x = y$;
(p2) $p(x,x) \leq p(x,y)$;
(p3) $p(x,y) = p(y,x)$;
(p4) $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$.

The pair (X,p) is called a *partial metric space*.

Remark 2.5. [12] Any metric space is always a partial metric space.

Example 2.6. Let function $p : [0,\infty)\times[0,\infty) \to [0,\infty)$ be defined by $p(x,y) = \max\{x,y\}$ for all $x,y \in [0,\infty)$. One can verify that p is a partial metric on $[0,\infty)$. However for any $x > 0$ we have $p(x,x) = x \neq 0$ then p is not a metric.

Definition 2.7. (Partial b-metric space)[20] Let X be a nonempty set. A function $p_b : X\times X \to [0,\infty)$ is said to be a *partial b-metric* if for all $x,y,z \in X$ the following conditions are satisfied:

(pb1) $p_b(x,x) = p_b(y,y) = p_b(x,y)$ if and only if $x = y$;
(pb2) $p_b(x,x) \leq p_b(x,y)$;
(pb3) $p_b(x,y) = p_b(y,x)$;
(pb4) there exists a real number $s \geq 1$ such that

$$p_b(x,y) \leq s\,[p_b(x,z) + p_b(z,y)] - p_b(z,z).$$

The pair (X,p_b) is called a *partial b-metric space*. Number s is called the coefficient of (X,p_b).

Remark 2.8. [20]

(1) For a partial b-metric space (X,p_b), if $x,y \in X$ and $p_b(x,y) = 0$ then $x = y$ but the converse may not be true.
(2) Every partial metric space is a particular case of a partial b-metric space with coefficient $s = 1$.
(3) Every b-metric space is a partial b-metric space with the same coefficient and zero self-distance but not vice versa.

Example 2.9. Let $X = [0,\infty)$, $q > 1$ be a constant and $p_b : X \times X \to [0,\infty)$ be defined by

$$p_b(x,y) = \left(\frac{x+y}{2}\right)^q \quad \text{for all} \quad x,y \in X.$$

Even though (X,p_b) is a partial b-metric space with coefficient $s = 2^{q-1} > 1$, the following statements show that it is neither a b-metric nor a partial metric space.

Since we have $p_b(x,x) = x^q \neq 0$ for any $x > 0$, then p_b is not a b-metric on X. Moreover, for the case that $x = 5, y = 7, z = 1$, we have $p_b(x,y) = p_b(5,7) = 6^q$ and $p_b(x,z) + p_b(y,z) - p_b(z,z) = 3^q + 4^q - 1$. That is $p_b(x,y) > p_b(x,z) + p_b(y,z) - p_b(z,z)$ for all $q > 1$, which implies that p_b is not a partial metric on X.

Notation 2.10. For the simplicity, the following notations are introduced.

- $m_{x,y} := \min\{m(x,x), m(y,y)\}$
- $M_{x,y} := \max\{m(x,x), m(y,y)\}$
- $m_{b\ x,y} := \min\{m_b(x,x), m_b(y,y)\}$
- $M_{b\ x,y} := \max\{m_b(x,x), m_b(y,y)\}$

Definition 2.11. (M-Metric space)[2] Let X be a nonempty set. A function $m : X\times X \to [0,\infty)$ is called an *$M-$metric* if for all $x,y,z \in X$ the following conditions are satisfied:

(m1) $m(x,x) = m(y,y) = m(x,y)$ if and only if $x = y$;
(m2) $m_{x,y} \leq m(x,y)$;
(m3) $m(x,y) = m(y,x)$;
(m4) $m(x,y) - m_{x,y} \leq [m(x,z) - m_{x,z}] + [m(z,y) - m_{z,y}]$.

The pair (X,m) is called an *M-metric space.*

Example 2.12. [2] Let $X = [0,\infty)$ and $m : X \times X \to [0,\infty)$ be defined by

$$m(x,y) = \frac{x+y}{2} \text{ for all } x,y \in X.$$

Then m is an M-metric.

Example 2.13. [2] Let $X = \{1,2,3\}$ and $m : X \times X \to [0,\infty)$ be defined by

$$m(1,1) = 1,\ m(2,2) = 9,\ m(3,3) = 4,$$
$$m(1,2) = m(2,1) = 10,\ m(1,3) = m(3,1) = 7,\ m(3,2) = m(2,3) = 7.$$

The given m is an M-metric but it is not a partial metric because $m(2,2) > m(3,2)$.

Lemma 2.14. [2] *Every partial metric is an M-metric.*

Definition 2.15. (M_b-Metric space)[13] Let X be a nonempty set. A function $m_b : X \times X \to [0,\infty)$ is called an M_b-*metric* if for all $x,y,z \in X$ the following conditions are satisfied:

(mb1) $m_b(x,x) = m_b(y,y) = m_b(x,y)$ if and only if $x = y$;
(mb2) $m_{b\ x,y} \leq m_b(x,y)$;
(mb3) $m_b(x,y) = m_b(y,x)$;
(mb4) There exists a real number $s \geq 1$ such that for all $x,y,z \in X$ we have

$$m_b(x,y) - m_{b\ x,y} \leq s[(m_b(x,z) - m_{b\ x,z}) + (m_b(z,y) - m_{b\ z,y})] - m_b(z,z).$$

Number s is called the coefficient of the M_b-*metric space* (X,m_b).

Example 2.16. [13] Let $X = [0,\infty)$, $p > 1$ be constant and $m_b : X \times X \to [0,\infty)$ be defined by

$$m_b(x,y) = (\max\{x,y\})^p + |x-y|^p \text{ for all } x,y \in X.$$

Then (X,m_b) is an M_b-metric space with coefficient $s = 2^p$, which is not an M-metric space.

Definition 2.17. [13] Each M_b-metric generates a topology τ_{mb} on X whose base is the family of open m_b-balls $\{B_{m_b}(x,\varepsilon) | x \in X, \varepsilon > 0\}$, where $B_{m_b}(x,\varepsilon) = \{y \in X | m_b(x,y) - m_{b\ x,y} < \varepsilon\}$.

Definition 2.18. [13] (Convergence, Cauchy sequence and Completeness) Let (X,m_b) be an M_b-metric space with coefficient $s \geq 1$. Let $\{x_n\}$ be any sequence in X and $x \in X$.

- The sequence $\{x_n\}$ is said to be convergent with respect to τ_{mb} and converges to x, if and only if

$$\lim_{n\to\infty} [m_b(x_n,x) - m_{b\ x_n,x}] = 0.$$

- The sequence $\{x_n\}$ is said to be m_b-Cauchy sequence in (X,m_b) if and only if both

$$\lim_{m,n\to\infty} [m_b(x_n,x_m) - m_{b\ x_n,x_m}] \quad \text{and} \quad \lim_{m,n\to\infty} [M_{b\ x_n,x_m} - m_{b\ x_n,x_m}]$$

exist and are finite.

- An M_b-metric space is said to be complete if for every m_b-Cauchy sequence $\{x_n\}$ in X converges to a point $x \in X$, i.e.

$$\lim_{n\to\infty} [m_b(x_n, x) - m_{b\ x_n,x}] = 0 \quad \text{and} \quad \lim_{n\to\infty} [M_{b\ x_n,x} - m_{b\ x_n,x}] = 0.$$

Definition 2.19. (Simulation function)[10] Let $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$. Then ζ is called a *simulation function* if it satisfies the following conditions:

(ζ1) $\zeta(0,0) = 0$;
(ζ2) $\zeta(t,s) < s - t$ for all $t, s > 0$;
(ζ3) if $\{t_n\}, \{s_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$ then $\limsup_{n\to\infty} \zeta(t_n, s_n) < 0$.

We denote the set of all simulation functions by $\mathcal{Z}$.

Example 2.20. [10] Let $\lambda \in [0,1)$ be given and $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ be defined by

$$\zeta(t,s) = \lambda s - t,$$

for all $t, s \in [0,\infty)$. We can see that ζ satisfies all conditions in definition 2.19. Then ζ is a simulation function.

Example 2.21. [10](Generalization of example 2.20) Let $\zeta_1 : [0,\infty) \times [0,\infty) \to \mathbb{R}$ be defined by

$$\zeta_1(t,s) = \psi(s) - \phi(t),$$

for all $t, s \in [0,\infty)$, where $\psi, \phi : [0,\infty) \to [0,\infty)$ are two continuous functions such that

- $\psi(t) = \phi(t) = 0$ if and only if $t = 0$; and
- $\psi(t) < t \leq \phi(t)$ for all $t > 0$.

ζ in example 2.20 is a particular case of ζ_1, where ζ_1 is also a simulation function.

Example 2.22. [10] Let $\zeta_2 : [0,\infty) \times [0,\infty) \to \mathbb{R}$ be defined by

$$\zeta_2(t,s) = s - \frac{f(t,s)}{g(t,s)},$$

for all $t, s \in [0,\infty)$, where $f, g : [0,\infty) \times [0,\infty) \to [0,\infty)$ are two continuous functions with respect to each variable such that $g(t,s) \neq 0$ and $f(t,s) > g(t,s)$ for all $t, s > 0$. ζ_2 is also a simulation function.

Definition 2.23. [10]($\mathcal{Z}$-contraction) Let (X,d) be a metric space, $T : X \to X$ be a mapping and $\zeta \in \mathcal{Z}$. T is called a *$\mathcal{Z}$-contraction* with respect to ζ if

$$\zeta\left(d(Tx,Ty), d(x,y)\right) \geq 0, \text{ for all } x, y \in X.$$

If T is a $\mathcal{Z}$-contraction with respect to $\zeta \in \mathcal{Z}$, then $d(Tx,Ty) < d(x,y)$ for all distinct $x, y \in X$.

Theorem 2.24. *[10]Let (X,d) be a complete metric space and $T : X \to X$ be a $\mathcal{Z}$-contraction with respect to $\zeta \in \mathcal{Z}$. Then*

- *T has a unique fixed point u in X and*
- *for every $x_0 \in X$, the Picard sequence $\{x_n\}$, defined by $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$, converges to the fixed point u of T.*

3. Main results

In this section, we define the $\mathcal{Z}_{mb}$-contraction and prove an existence of a fixed point for such mapping in a complete M_b-metric space.

Definition 3.1. Let (X, m_b) be an M_b-metric space with a constant $s \geq 1$, $T : X \to X$ be a mapping, and $\zeta \in \mathcal{Z}$. Mapping T is called $\mathcal{Z}_{mb}$-contraction with respect to ζ if the following condition is satisfied

$$\zeta\left(m_b(Tx, Ty), m_b(x, y)\right) \geq 0 \text{ for all } x, y \in X. \tag{3.1}$$

Remark 3.2. If T is a $\mathcal{Z}_{mb}$-contraction with respect to ζ, then $m_b(Tx, Ty) < m_b(x, y)$, for all $x, y \in X$ and $m_b(x, y) > 0$.

Lemma 3.3. *Let (X, m_b) be an M_b-metric space with a constant $s \geq 1$ and $T : X \to X$ be a $\mathcal{Z}_{mb}$-contraction with respect to $\zeta \in \mathcal{Z}$. If $\{x_n\}$ is a Picard sequence with an initial point $x_0 \in X$ then*

$$\lim_{n\to\infty} m_b(x_n, x_{n+1}) = 0.$$

Proof. Let $x_0 \in X$ be arbitrary and $\{x_n\}$ be a Picard sequence in X, i.e $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$.
For the case that there exists $m_b(x_{n_0}, x_{n_0+1}) = 0$ for some $n_0 \in \mathbb{N}$, then $x_{n_0} = x_{n_0+1} = Tx_{n_0}$ so x_0 is a fixed point of T. If we continue the process like $x_{n_0+2} = Tx_{n_0+1} = Tx_{n_0} = x_{n_0}$, $x_{n_0+3} = Tx_{n_0+2} = x_{n_0+2}$ and so on, we have

$$x_{n_0} = x_{n_0+1} = x_{n_0+2} = \cdots = x_{n_0+k} = \cdots, \quad \forall k \in \mathbb{N}.$$

Suppose to the contrary that $m_b(x_{n_0+1}, x_{n_0+2}) > 0$. By (3.1) and property ($\zeta 2$), we have

$$\begin{aligned} 0 &\leq \zeta\left(m_b(Tx_{n_0+1}, Tx_{n_0+2}), m_b\left(x_{n_0+1}, x_{n_0+2}\right)\right) \\ &< m_b\left(x_{n_0+1}, x_{n_0+2}\right) - m_b\left(Tx_{n_0+1}, Tx_{n_0+2}\right) \\ &= m_b\left(x_{n_0}, x_{n_0+1}\right) - m_b\left(Tx_{n_0+1}, Tx_{n_0+2}\right) \\ &= -m_b\left(Tx_{n_0+1}, Tx_{n_0+2}\right). \end{aligned} \tag{3.2}$$

The obtained inequality provides that $m_b(Tx_{n_0+1}, Tx_{n_0+2}) < 0$ which is a contradiction. Hence we must have $m_b(x_n, x_{n+1}) = 0, \ \ \forall n \geq n_0$.

Consequently, we shall assume that $m_b(x_n, x_{n+1}) > 0, \ \forall n \in \mathbb{N}$. By (3.1) and property ($\zeta 2$), we have

$$\begin{aligned} 0 &\leq \zeta\left(m_b(Tx_{n-1}, Tx_n), m_b(x_{n-1}, x_n)\right) \\ &= \zeta\left(m_b(x_n, x_{n+1}), m_b(x_{n-1}, x_n)\right) \\ &< m_b(x_{n-1}, x_n) - m_b(x_n, x_{n+1}). \end{aligned}$$

This implies that $m_b(x_n, x_{n+1}) < m_b(x_{n-1}, x_n)$, $\forall n \in \mathbb{N}$, and the sequence $\{m_b(x_n, x_{n+1})\}$ is a decreasing sequence of nonnegative real numbers. Thus there exists $r \geq 0$ such that $\lim_{n\to\infty} m_b(x_n, x_{n+1}) = r$. Assume that $r > 0$, applying the property ($\zeta 3$) with $t_n = m_b(x_n, x_{n+1})$ and $s_n = m_b(x_{n-1}, x_n)$ provides that

$$0 \leq \limsup_{n\to\infty} \zeta(m_b(x_n, x_{n+1}), m_b(x_{n-1}, x_n)) < 0,$$

which contradicts to the assumption $r > 0$. So $\lim_{n\to\infty} m_b(x_n, x_{n+1}) = 0$. ∎

Lemma 3.4. *Let (X, m_b) be an M_b-metric space with constant $s \geq 1$ and $T : X \to X$ be a $\mathcal{Z}_{mb}$-contraction with respect to $\zeta \in \mathcal{Z}$. If $\{x_n\}$ is a Picard sequence with initial point $x_0 \in X$ then $\{x_n\}$ is an M_b-Cauchy sequence in (X, m_b).*

Proof. Recall that

- $0 \leq m_{b\ x_n,x_{n+1}} \leq m_b(x_n, x_{n+1})$ for all $n \in \mathbb{N}$;
- $\lim_{n\to\infty} m_b(x_n, x_{n+1}) = 0$; $\lim_{n\to\infty} m_{b\ x_n,x_{n+1}} = 0$;
- $m_{b\ x_n,x_{n+1}} = \min\{m_b(x_n, x_n), m_b(x_{n+1}, x_{n+1})\}$; $\lim_{n\to\infty} m_b(x_n, x_n) = 0$;
- $m_{b\ x_n,x_m} = \min\{m_b(x_n, x_n), m_b(x_m, x_m)\}$; $\lim_{m,n\to\infty} m_{b\ x_n,x_m} = 0$;
- $\lim_{m,n\to\infty} (M_{b\ x_n,x_m} - m_{b\ x_n,x_m}) = \lim_{m,n\to\infty} |m_b(x_n, x_n) - m_b(x_m, x_m)| = 0$.

Next, we will show that $\lim_{m,n\to\infty} (m_b(x_n, x_m) - m_{b\ x_n,x_m}) = 0$.
Define

$$M_b^*(x, y) = m_b(x, y) - m_{b\ x,y}, \quad \forall x, y \in X.$$

If $\lim_{m,n\to\infty} M_b^*(x_n, x_m) \neq 0$, then there exist $\varepsilon > 0$ and $\{l_k\} \subset \mathbb{N}$ such that

$$M_b^*(x_{l_k}, x_{n_k}) \geq \varepsilon. \tag{3.3}$$

Suppose that l_k is the smallest integer which satisfies (3.3) such that

$$M_b^*(x_{l_k-1}, x_{n_k}) < \varepsilon.$$

Now, we split the consideration into the following two cases:
Case (i): If $s = 1$, property (mb4) provides

$$\begin{aligned}
\varepsilon &\leq M_b^*(x_{l_k}, x_{n_k}) = m_b(x_{l_k}, x_{n_k}) - m_{b\ x_{l_k},x_{n_k}} \\
&\leq [m_b(x_{l_k}, x_{l_k-1}) - m_{b\ x_{l_k},x_{l_k-1}}] + [m_b(x_{l_k-1}, x_{n_k}) - m_{b\ x_{l_k-1},x_{n_k}}] \\
&\quad -m_b(x_{l_k-1}, x_{l_k-1}) \\
&< M_b^*(x_{l_k}, x_{l_k-1}) + \varepsilon - m_b(x_{l_k-1}, x_{l_k-1}).
\end{aligned}$$

Since $\lim_{k\to\infty} M_b^*(x_{l_k}, x_{n_k}) = \varepsilon$, therefore $\lim_{k\to\infty} (m_b(x_{l_k}, x_{n_k}) - m_{bx_{l_k},x_{n_k}}) = \varepsilon$. On the other hand $\lim_{k\to\infty} m_{b\ x_{l_k},x_{n_k}} = 0$, so we have

$$\lim_{k\to\infty} m_b(x_{l_k}, x_{n_k}) = \varepsilon. \tag{3.4}$$

Again by (mb4), we have

$$\begin{aligned}
M_b^*(x_{l_k}, x_{n_k}) \leq &M_b^*(x_{l_k}, x_{l_k+1}) + M_b^*(x_{l_k+1}, x_{n_k+1}) + M_b^*(x_{n_k+1}, x_{n_k}) \\
&- m_b(x_{l_k+1}, x_{l_k+1}) - m_b(x_{n_k+1}, x_{n_k+1}),
\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}
M_b^*(x_{l_k+1}, x_{n_k+1}) \leq &M_b^*(x_{l_k+1}, x_{l_k}) + M_b^*(x_{l_k}, x_{n_k}) + M_b^*(x_{n_k}, x_{n_k+1}) \\
&- m_b(x_{l_k}, x_{l_k}) - m_b(x_{n_k}, x_{n_k}).
\end{aligned} \tag{3.6}$$

From (3.5) and (3.6), we get

$$\begin{aligned} M_b^*(x_{l_k}, x_{n_k}) \leq & M_b^*(x_{l_k}, x_{l_k+1}) + M_b^*(x_{l_k+1}, x_{n_k+1}) + M_b^*(x_{n_k+1}, x_{n_k}) \\ & - m_b(x_{l_k+1}, x_{l_k+1}) - m_b(x_{n_k+1}, x_{n_k+1}) \\ \leq & M_b^*(x_{l_k}, x_{l_k+1}) + M_b^*(x_{l_k+1}, x_{l_k}) + M_b^*(x_{l_k}, x_{n_k}) \\ & + M_b^*(x_{n_k}, x_{n_k+1}) - m_b(x_{l_k}, x_{l_k}) - m_b(x_{n_k}, x_{n_k}) \\ & + M_b^*(x_{n_k+1}, x_{n_k}) - m_b(x_{l_k+1}, x_{l_k+1}) - m_b(x_{n_k+1}, x_{n_k+1}). \end{aligned} \tag{3.7}$$

Letting $k \to \infty$ in (3.7) and using lemma 3.3 and (3.4), we have

$$\lim_{k\to\infty} m_b(x_{l_k+1}, x_{n_k+1}) = \varepsilon. \tag{3.8}$$

Using property ($\zeta 3$) with $t_k = m_b(x_{l_k+1}, x_{n_k+1})$ and $s_k = m_b(x_{l_k}, x_{n_k})$, we have

$$0 \leq \limsup_{k\to\infty} \zeta(m_b(x_{l_k+1}, x_{n_k+1}), m_b(x_{l_k}, x_{n_k})) < 0,$$

which is a contradiction. Therefore $\{x_n\}$ is an M_b-Cauchy sequence.
Case (ii): If $s > 1$, property (mb4) provides

$$\begin{aligned} \varepsilon \leq & M_b^*(x_{l_k}, x_{n_k}) \\ = & m_b(x_{l_k}, x_{n_k}) - m_{b\ x_{l_k}, x_{n_k}} \\ \leq & s[(m_b(x_{l_k}, x_{l_k-1}) - m_{b x_{l_k}, x_{l_k-1}}) + (m_b(x_{l_k-1}, x_{n_k}) - m_{b\ x_{l_k-1}, x_{n_k}})] \\ & - m_b(x_{l_k-1}, x_{l_k-1}) \\ = & s M_b^*(x_{l_k-1}, x_{n_k}) + s[m_b(x_{l_k}, x_{l_k-1}) - m_{b\ x_{l_k}, x_{l_k-1}}] - m_b(x_{l_k-1}, x_{l_k-1}) \\ < & s\varepsilon + s[m_b(x_{l_k}, x_{l_k-1}) - m_{b\ x_{l_k}, x_{l_k-1}}] - m_b(x_{l_k-1}, x_{l_k-1}). \end{aligned}$$

As $k \to \infty$, the limit is

$$\varepsilon \leq \lim_{k\to\infty} M_b^*(x_{l_k}, x_{n_k}) \leq s\varepsilon. \tag{3.9}$$

Since $\lim_{k\to\infty} m_{b\ x_{l_k}, x_{n_k}} = 0$, thus

$$\varepsilon \leq \lim_{k\to\infty} m_b(x_{l_k}, x_{n_k}) \leq s\varepsilon. \tag{3.10}$$

Again by (mb4), we have

$$\begin{aligned} M_b^*(x_{l_k}, x_{n_k}) = & m_b(x_{l_k}, x_{n_k}) - m_{b x_{l_k}, x_{n_k}} \\ \leq & s[(m_b(x_{l_k}, x_{l_k+1}) - m_{b x_{l_k}, x_{l_k+1}}) + (m_b(x_{l_k+1}, x_{n_k}) - m_{b\ x_{l_k+1}, x_{n_k}})] \\ & - m_b(x_{l_k+1}, x_{l_k+1}) \\ \leq & s[(m_b(x_{l_k}, x_{l_k+1}) - m_{b\ x_{l_k}, x_{l_k+1}}) \\ & + s[(m_b(x_{l_k+1}, x_{n_k+1}) - m_{b\ x_{l_k+1}, x_{n_k+1}}) + (m_b(x_{n_k+1}, x_{n_k}) - m_{b\ x_{n_k+1}, x_{n_k}})] \\ & - m_b(x_{n_k+1}, x_{n_k+1})] - m_b(x_{l_k+1}, x_{l_k+1}) \\ = & s[M_b^*(x_{l_k}, x_{l_k+1}) + s[M_b^*(x_{l_k+1}, x_{n_k+1}) + M_b^*(x_{n_k+1}, x_{n_k})]] \\ & - s m_b(x_{n_k+1}, x_{n_k+1})] - m_b(x_{l_k+1}, x_{l_k+1}) \\ = & s M_b^*(x_{l_k}, x_{l_k+1}) + s^2 M_b^*(x_{l_k+1}, x_{n_k+1}) + s^2 M_b^*(x_{n_k+1}, x_{n_k}) \\ & - s m_b(x_{n_k+1}, x_{n_k+1}) - m_b(x_{l_k+1}, x_{l_k+1}). \end{aligned} \tag{3.11}$$

Similar to the above, we find that

$$M_b^*(x_{l_k+1}, x_{n_k+1}) \leq sM_b^*(x_{l_k+1}, x_{l_k}) + s^2M_b^*(x_{l_k}, x_{n_k}) + s^2M_b^*(x_{n_k}, x_{n_k+1}) - sm_b(x_{n_k}, x_{n_k}) - m_b(x_{l_k}, x_{l_k}). \tag{3.12}$$

Using (3.11) and (3.12), then

$$\begin{aligned}\varepsilon \leq& M_b^*(x_{l_k}, x_{n_k}) \\ \leq& sM_b^*(x_{l_k}, x_{l_k+1}) + s^2M_b^*(x_{l_k+1}, x_{n_k+1}) + s^2M_b^*(x_{n_k+1}, x_{n_k}) \\ &- sm_b(x_{n_k+1}, x_{n_k+1}) - m_b(x_{l_k+1}, x_{l_k+1}) \\ \leq& sM_b^*(x_{l_k}, x_{l_k+1}) + s^2(sM_b^*(x_{l_k+1}, x_{l_k}) + s^2M_b^*(x_{l_k}, x_{n_k}) \\ &+ s^2M_b^*(x_{n_k}, x_{n_k+1}) - sm_b(x_{n_k}, x_{n_k}) - m_b(x_{l_k}, x_{l_k})) \\ &+ s^2M_b^*(x_{n_k+1}, x_{n_k}) - sm_b(x_{n_k+1}, x_{n_k+1}) - m_b(x_{l_k+1}, x_{l_k+1}).\end{aligned} \tag{3.13}$$

As $k \to \infty$ in (3.13), we have

$$\begin{aligned}&\varepsilon \leq \lim_{k\to\infty} s^2M_b^*(x_{l_k+1}, x_{n_k+1}) \leq s^4\varepsilon \\ &\frac{\varepsilon}{s^2} \leq \lim_{k\to\infty} M_b^*(x_{l_k+1}, x_{n_k+1}) \leq s^2\varepsilon.\end{aligned} \tag{3.14}$$

Since $\lim_{k\to\infty} m_{b\ x_{l_k+1}, x_{n_k+1}} = 0$, (3.14) is derived to

$$\frac{\varepsilon}{s^2} \leq \lim_{k\to\infty} m_b(x_{l_k+1}, x_{n_k+1}) \leq s^2\varepsilon. \tag{3.15}$$

From (3.1) and property (ζ2), we obtain

$$\begin{aligned}0 &\leq \zeta(m_b(Tx_{l_k}, Tx_{n_k}), m_b(x_{l_k}, x_{n_k})) \\ &= \zeta(m_b(x_{l_k+1}, x_{n_k+1}), m_b(x_{l_k}, x_{n_k})) \\ &< m_b(x_{l_k}, x_{n_k}) - m_b(x_{l_k+1}, x_{n_k+1}).\end{aligned}$$

Hence

$$\begin{aligned}0 &\leq \limsup_{k\to\infty}(m_b(x_{l_k}, x_{n_k}) - m_b(x_{l_k+1}, x_{n_k+1})) \\ &\leq \limsup_{k\to\infty} m_b(x_{l_k}, x_{n_k}) - \liminf_{k\to\infty} m_b(x_{l_k+1}, x_{n_k+1}) \\ &\leq s\varepsilon - s^2\varepsilon \\ &< 0,\end{aligned}$$

which is a contradiction. This shows that $\{x_n\}$ is an M_b-Cauchy sequence. ∎

Lemma 3.5. *Let (X, m_b) be an M_b-metric space with constant $s \geq 1$ and $T : X \to X$ be a $\mathcal{Z}_{mb}$-contraction with respect to $\zeta \in \mathcal{Z}$. If $\{x_n\}$ is a Picard sequence with initial point $x_0 \in X$ and $x_n \to x$ as $n \to \infty$, then $Tx_n \to Tx$ as $n \to \infty$.*

Proof. If $m_b(Tx_n, Tx) = 0$, then $m_{b\ Tx_n, Tx} \leq m_b(Tx_n, Tx) = 0$ which implies that $\lim_{n\to\infty}(m_b(Tx_n, Tx) - m_{bTx_n, Tx}) = 0$. This means $Tx_n \to Tx$ as $n \to \infty$. Otherwise, if $m_b(Tx_n, Tx) > 0$ then $m_b(Tx_n, Tx) < m_b(x_n, x)$ and $m_b(x_n, x) > 0$. Here, we consider in two cases as the following:

Case (i) Because $\lim_{n\to\infty} m_b(x_n, x_n) = 0$, $m_b(x, x) < m_b(x_n, x_n)$ implies $m_b(x, x) = 0$. By $m_{b\ x_n, x} = \min\{m_b(x_n, x_n), m_b(x, x)\}$, $\lim_{n\to\infty} m_{b\ x_n, x} = 0$, and $\lim_{n\to\infty} m_b(x_n, x) = 0$, then

$\lim_{n\to\infty} m_b(Tx_n, Tx) \leq \lim_{n\to\infty} m_b(x_n, x) = 0$. Therefore $\lim_{n\to\infty}(m_b(Tx_n, Tx) - m_{b\ Tx_n,Tx}) = 0$ and thus $Tx_n \to Tx$ as $n \to \infty$.

Case (ii) If $m_b(x, x) \geq m_b(x_n, x_n)$, then again $\lim_{n\to\infty} m_b(x_n, x_n) = 0$ which implies that $\lim_{n\to\infty} m_{b\ x_n,x} = 0$. Thus $\lim_{n\to\infty} m_b(x_n, x) = 0$.

As $\lim_{n\to\infty} m_b(Tx_n, Tx) \leq \lim_{n\to\infty} m_b(x_n, x) = 0$, we have $\lim_{n\to\infty}(m_b(Tx_n, Tx) - m_{bTx_n,Tx}) = 0$ so that $Tx_n \to Tx$ as $n \to \infty$. ∎

Theorem 3.6. *Let (X, m_b) be a complete M_b-metric space with constant $s \geq 1$ and $T : X \to X$ be a $\mathcal{Z}_{mb}$-contraction with respect to $\zeta \in \mathcal{Z}$. T has a unique fixed point $u \in X$ such that $m_b(u, u) = 0$.*

Proof. Let $x_0 \in X$ and $\{x_n\}$ be a Picard sequence with initial point x_0. Now by lemma 3.4, the sequence $\{x_n\}$ is an M_b-Cauchy. Here (X, m_b) is complete, then there exists some $u \in X$ such that

$$\lim_{n\to\infty}(m_b(x_n, u) - m_{bx_n,u}) = 0 \ \text{ and } \ \lim_{m,n\to\infty}(M_{bx_n,u} - m_{bx_n,u}) = 0.$$

Since

$$\begin{aligned} M_{b\ x_n,u} - m_{b\ x_n,u} &= \max\{m_b(x_n, x_n), m_b(u, u)\} - \min\{m_b(x_n, x_n), m_b(u, u)\} \\ &= |m_b(x_n, x_n) - m_b(u, u)| \end{aligned}$$

and $\lim_{n\to\infty} m_b(x_n, x_n) = 0$, so $m_b(u, u) = 0$.

From $x_n \to u$ as $n \to \infty$ and lemma 3.5, we have

$$\lim_{n\to\infty}(m_b(Tx_n, Tu) - m_{b\ Tx_n,Tu}) = 0 \text{ and } \lim_{n\to\infty}(m_b(x_{n+1}, Tu) - m_{b\ x_{n+1},Tu}) = 0.$$

By the properties

$$\begin{aligned} &M_{b\ x_{n+1},Tu} - m_{b\ x_{n+1},Tu} = |m_b(x_{n+1}, x_{n+1}) - m_b(Tu, Tu)|, \\ &\lim_{n\to\infty} m_b(x_{n+1}, x_{n+1}) = 0, \lim_{n\to\infty} m_{b\ x_{n+1},Tu} = 0 \text{ and } \lim_{n\to\infty} m_b(x_{n+1}, Tu) = 0, \end{aligned}$$

now we get $m_b(Tu, Tu) = 0$.

Next we will show that $m_b(u, Tu) = 0$. Since $x_n \to u$ as $n \to \infty$ and

$$\begin{aligned} &|(m_b(x_n, Tu) - m_{b\ x_n,Tu}) - (m_b(u, Tu) - m_{b\ u,Tu})| \\ &\qquad \leq |s[(m_b(x_n, u) - m_{b\ x_n,u}) + (m_b(u, Tu) - m_{b\ u,Tu})] - m_b(u, u) \\ &\qquad - (s[(m_b(u, u) - m_{b\ u,u}) + (m_b(u, Tu) - m_{b\ u,Tu})] - m_{b\ u,u})|, \end{aligned}$$

so $\lim_{n\to\infty} |(m_b(x_n, Tu) - m_{b\ x_n,Tu}) - (m_b(u, Tu) - m_{b\ u,Tu})| \leq 0$ and

$$\lim_{n\to\infty}(m_b(x_n, Tu) - m_{b\ x_n,Tu}) = m_b(u, Tu) - m_{b\ u,Tu} = m_b(u, Tu).$$

By lemma 3.5 , we have $\lim_{n\to\infty}(m_b(x_n, Tu) - m_{bx_n,Tu}) = 0$, hence $m_b(u, Tu) = 0$. Therefore, $m_b(u, u) = m_b(Tu, Tu) = m_b(u, Tu) = 0$. Property (mb1) gives $Tu = u$.

Finally, we will show that a fixed point of T is unique. Suppose that $u, v \in X$ are two fixed points of T. Then $m_b(u, u) = m_b(v, v) = 0$.

From property (mb3), we get

$$\begin{aligned} m_b(u, v) - m_{b\ u,v} &\leq s[(m_b(u, Tu) - m_{b\ u,Tu}) + (m_b(Tu, v) - m_{b\ Tu,v})] - m_b(Tu, Tu) \\ &= s m_b(Tu, v). \end{aligned}$$

If $m_b(u,v) > 0$ then

$$0 \leq \zeta(m_b(Tu,Tv), m_b(u,v)) < m_b(u,v) - m_b(Tu,Tv) = 0,$$

which is a contradiction. Therefore $m_b(u,v) = 0$, which means $u = v$. ∎

Corollary 3.7. *[13] Let (X, m_b) be a complete M_b-metric space with a constant $s \geq 1$ and $T : X \to X$ be a mapping. Suppose that there exists $\lambda \in [0,1)$ such that*

$$m_b(Tx,Ty) \leq \lambda m_b(x,y) \text{ for all } x,y \in X.$$

Then T has a unique fixed point $u \in X$ and $m_b(u,u) = 0$.

Proof. The result follows from theorem 3.6 by using simulation function

$$\zeta(t,s) = \lambda s - t,$$

for all $t, s \geq 0$. ∎

Example 3.8. Let $X = [0,1]$ and $m_b : X \times X \to \mathbb{R}$ be defined by

$$m_b(x,y) = \left(\frac{x+y}{2}\right)^2.$$

Then (X, m_b) is a complete M_b-metric space with $s = 2$. Define $T : X \to X$ by

$$Tx = \frac{x}{x+1} \text{ for all } x \in X.$$

Let $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ be defined by $\zeta(t,s) = \frac{s}{s+1} - t$. Then ζ is a simulation function. Indeed, we obtain

$$\begin{aligned}
\zeta\left(m_b(Tx,Ty), m_b(x,y)\right) &= \zeta\left(m_b\left(\frac{x}{x+1}, \frac{y}{y+1}\right), m_b(x,y)\right) \\
&= \frac{m_b(x,y)}{m_b(x,y)+1} - m_b\left(\frac{x}{x+1}, \frac{y}{y+1}\right) \\
&= \frac{\left(\frac{x+y}{2}\right)^2}{\left(\frac{x+y}{2}\right)^2 + 1} - \left(\frac{\frac{x}{x+1} + \frac{y}{y+1}}{2}\right)^2.
\end{aligned}$$

Since $0 \leq x \leq 1$ and $0 \leq y \leq 1$, $\frac{x}{x+1} \leq \frac{x}{2}$ and $\frac{y}{y+1} \leq \frac{y}{2}$. Then

$$\begin{aligned}
\zeta(m_b(Tx,Ty), m_b(x,y)) &\geq \frac{\left(\frac{x+y}{2}\right)^2}{\left(\frac{x+y}{2}\right)^2 + 1} - \frac{1}{4}\left(\frac{x+y}{2}\right)^2 \\
&= \frac{\left(\frac{x+y}{2}\right)^2 - \frac{1}{4}\left(\frac{x+y}{2}\right)^4 - \frac{1}{4}\left(\frac{x+y}{2}\right)^2}{\left(\frac{x+y}{2}\right)^2 + 1} \\
&= \frac{\frac{3}{4}\left(\frac{x+y}{2}\right)^2 - \frac{1}{4}\left(\frac{x+y}{2}\right)^4}{\left(\frac{x+y}{2}\right)^2 + 1} \\
&= \frac{\frac{1}{4}\left(\frac{x+y}{2}\right)^2\left(3 - \left(\frac{x+y}{2}\right)^2\right)}{\left(\frac{x+y}{2}\right)^2 + 1} \\
&\geq 0.
\end{aligned}$$

Thus all the conditions of theorem 3.6 are satisfied. Hence T has a fixed point $x = 0$ and $m_b(0, 0) = 0$.

Example 3.9. Let $X = [0, 1]$ and $m_b : X \times X \to \mathbb{R}$ be defined by

$$m_b(x, y) = \max\{x, y\}^p + |x - y|^p \quad \text{where} \quad p > 1.$$

Then (X, m_b) is a complete M_b-metric space with $s = 2^p$.

Define $T : X \to X$ by

$$Tx = \frac{x}{3} \quad \text{for all } x \in X.$$

Let $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ be defined by $\zeta(t, s) = \frac{s}{2} - t$. Then ζ is a simulation function. Indeed, we obtain

$$\begin{aligned}
\zeta(m_b(Tx, Ty), m_b(x, y)) &= \frac{m_b(x, y)}{2} - m_b(Tx, Ty) \\
&= \frac{\max\{x, y\}^p + |x - y|^p}{2} - \max\left\{\frac{x}{3}, \frac{y}{3}\right\}^p - |\frac{x}{3} - \frac{y}{3}|^p \\
&= \frac{\max\{x, y\}^p}{2} + \frac{|x - y|^p}{2} - \max\left\{\frac{x}{3}, \frac{y}{3}\right\}^p - \frac{|x - y|^p}{3^p} \\
&\geq 0.
\end{aligned}$$

Thus all the conditions of theorem 3.6 are satisfied. Hence T has a fixed point $x = 0$ and $m_b(0, 0) = 0$.

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