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# JUNGCK-TYPE FIXED POINT THEOREM IN 0-COMPLETE PARTIAL $b_v(s)$-METRIC SPACES

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**Abstract** We present a variant of Jungck common fixed point theorem in a new and more generalized space, namely; 0-complete partial $b_v(s)$-metric space and deduce some well-known results as corollaries. We also give some illustrative examples to support our results.

**MSC:** 47H10; 47H09; 54E25

**Keywords:** Fixed points; $b$-metric space; $b_v(s)$-metric space; $v$-generalized metric space; Partial metric space; Partial $b_v(s)$-metric space; Contractive conditions

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## 1. Introduction and Preliminaries

Many generalizations of metric space and metric fixed point theory are given in different direction with a sole goal in mind; to promote and developed science and technology (See [1–3, 5, 6, 11–14]). The concept of 0-complete partial metric spaces is one among such generalizations and was established by Romaguera [12] and was further studied by Shukla and Radenovic [13] and others.

In this article, we will continue in this direction to broaden the applicability of fixed point results. We will present the concept of 0-complete partial $b_v(s)$-metric spaces and study some fixed point results involving Jungck type contraction in the introduced space

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and finally give some corollaries as consequences of our main theorems. The obtained results in this paper generalize many results in the literature as will be highlighted later.

From now on, $\mathbb{R}, \mathbb{R}^+, \mathbb{R}_+$ and $\mathbb{N}$, will denote the set of real numbers, positive real numbers, non-negative real numbers and natural numbers respectively. Let us recall the definitions of the $b$-metric spaces, the partial $b$-metric spaces, $b_v(s)$-metric spaces and partial $b_v(s)$-metric spaces.

**Definition 1.1.** [5, 6]. Let $X$ be a nonempty set. A $b$-metric on X is a function $d : X \times X \longrightarrow \mathbb{R}_+$, if there exists a real number $s \geq 1$ such that the following conditions hold

$b_1$. $d(x,y) = 0 \iff x = y$ for all $x, y \in X$;
$b_2$. $d(x,y) = d(y,x)$ for all $x, y \in X$;
$b_3$. $d(x,y) \leq s[d(x,z) + d(z,y)]$ for all $x, y, z \in X$.

Then $d$ is called a $b$-metric on $X$ and $(X, d)$ is called a $b$-metric space.

**Definition 1.2.** [14]. Let $X$ be a nonempty set. A partial $b$-metric on a nonempty set $X$ is a function $b : X \times X \longrightarrow \mathbb{R}_+$, such that

$Pb_1$. $x = y \iff b(x,x) = b(x,y) = b(y,y)$ for all $x, y \in X$;
$Pb_2$. $b(x,x) \leq b(x,y)$ for all $x, y \in X$;
$Pb_3$. $b(x,y) = b(y,x)$ for all $x, y \in X$;
$Pb_4$. there exists a real number $s \geq 1$ such that for all $x, y, z \in X$, $b(x,y) \leq s[b(x,z) + d(z,y)] - b(z,z)$.

Then $b$ is called a partial $b$-metric on $X$ and $(X, b)$ is called a partial $b$-metric space.

**Definition 1.3.** [11]. Let $X$ be a nonempty set, $b_v : X \times X \longrightarrow \mathbb{R}_+$ and $v \in \mathbb{N}$ such that if for all $x, y \in X$ and for all distinct points $u_1, u_2, \ldots, u_v \in X \backslash \{x, y\}$ the following hold:

$bv_1$. $b_v(x,y) = 0 \iff x = y$;
$bv_2$. $b_v(x,y) = b_v(y,x)$;
$bv_3$. there is $s \in \mathbb{R}$ with $s \geq 1$ such that
$b_v(x,y) \leq s[b_v(x,u_1) + b_v(u_1,u_2) + \cdots + b_v(u_v,y)]$.

Then $b_v$ is called a $b_v(s)$-metric on $X$ and $(X, b_v)$ is called a $b_v(s)$-metric space with a coefficient $s$.

**Definition 1.4.** [3]. Let $X$ be a nonempty set, $p_{b_v} : X \times X \longrightarrow \mathbb{R}_+$ and $v \in \mathbb{N}$ such that if for all $x, y \in X$ and for all distinct points $u_1, u_2, \ldots, u_v \in X \backslash \{x, y\}$ the following hold:

$Pbv_1$. $x = y \iff p_{b_v}(x,x) = p_{b_v}(x,y) = p_{b_v}(y,y)$;
$Pbv_2$. $p_{b_v}(x,x) \leq p_{b_v}(x,y)$;
$Pbv_3$. $p_{b_v}(x,y) = p_{b_v}(y,x)$;
$Pbv_4$. there is $s \in \mathbb{R}$ with $s \geq 1$ such that
$p_{b_v}(x,y) \leq s[p_{b_v}(x,u_1) + p_{b_v}(u_1,u_2) + \cdots + p_{b_v}(u_v,y)] - \sum_{i=1}^{v} p_{b_v}(u_i,u_i)$.

Then $p_{b_v}$ is called a partial $b_v(s)$-metric on $X$ and $(X, p_{b_v})$ is called a partial $b_v(s)$-metric space with a coefficient $s$.

**Example 1.5.** [3]. Let $X = \{a, b, c, d\}$ and $p_{b_v} : X \times X \longrightarrow \mathbb{R}_+$ be defined by:

$$p_{b_v}(x,y) = \begin{cases} 0, & \text{if } x = y = a; \\ 2, & \text{if } x, y \in \{a, b\}, x \neq y; \\ 1, & \text{otherwise.} \end{cases}$$

for all $x, y \in X$.

Then $(X, p_{b_v})$ is a partial $b_2(\frac{4}{3})$-metric space which is neither a $b_2(\frac{4}{3})$-metric space nor a partial $b_2(1)$-metric space, due to the fact that, $p_{b_v}(b,b) \neq 0$ and $p_{b_v}(a,b) = 2 > 1 = p_{b_v}(a,c) + p_{b_v}(c,d) + p_{b_v}(d,b) - p_{b_v}(c,c) - p_{b_v}(d,d)$ respectively.

**Remark 1.6.**

$R_1$. A partial $b_1(1)$-metric space is the partial metric space of [9];

$R_2$. A partial $b_1(s)$-metric space is the partial $b$-metric space with coefficient $s$ of [14];

$R_3$. A partial $b_2(1)$-metric space is the partial rectangular metric space of [15];

$R_4$. A partial $b_2(s)$-metric space is the partial rectangular $b$-metric space with coefficient $s$;

$R_5$. A partial $b_v(1)$-metric space is the partial $v$-generalized metric space.

In the following, we illustrate that in agreement with the other generalized metric the topology on partial $b_v(s)$-metric is not compatible with the usual metric topology [16].

**Example 1.7.** [3] Let $X = A \cup B$, where $A = \{\frac{1}{n} : n \in \mathbb{N}\}$ and $B = \{0, 2, 4\}$. If $p_{b_v} : X \times X \longrightarrow \mathbb{R}_+$ is defined by:

$$p_{b_v}(x,y) = \begin{cases} \frac{1}{3}, & \text{if } x = y; \\ 3, & \text{if } x \neq y \text{ and } x, y \in A; \\ \frac{2}{3}, & \text{if } x \neq y \text{ and } x, y \in B; \\ \frac{1}{3} + \frac{1}{n}, & \text{if } x \neq y, x \in A \text{ and } y \in B. \end{cases}$$

for all $x, y \in X$.
Then $(X, p_{b_v})$ is a partial $b_3(2)$-metric space which is neither a $b_3(2)$-metric space nor a partial $b_3(1)$-metric space, due to the fact that, $p_{b_v}(0,0) \neq 0$ and $p_{b_v}(\frac{1}{2}, \frac{1}{4}) = 3 > \frac{7}{4} = p_{b_v}(\frac{1}{2}, 0) + p_{b_v}(0,2) + p_{b_v}(2,4) + p_{b_v}(4, \frac{1}{4}) - p_{b_v}(0,0) - p_{b_v}(2,2) - p_{b_v}(4,4)$, respectively.

In what follows, we give definitions of the convergence of a sequence, Cauchy sequence, completeness, 0-Cauchy sequence and 0-completeness in partial $b_v(s)$-metric spaces.

**Definition 1.8.** Let $(X, p_{b_v})$ be a partial $b_v(s)$-metric space with coefficient $s \geq 1$. Let $\{x_n\}$ be a sequence in $(X, p_{b_v})$ and $x \in X$. Then,

i.) $\{x_n\}$ is said to converge to $x$ with respect to $\tau_{p_{b_v}}$ if and only if $\lim_{n\to\infty} p_{b_v}(x_n, x) = p_{b_v}(x,x) = 0$. Moreover, $x$ is called the limit point of $\{x_n\}$;

ii.) $\{x_n\}$ is called Cauchy if $\lim_{n,m\to\infty} p_{b_v}(x_n, x_m)$ exists (and is finite);

iii.) $\{x_n\}$ is called 0-Cauchy if $\lim_{n,m\to\infty} p_{b_v}(x_n, x_m) = 0$;

iv.) $(X, p_{b_v})$ is called 0-complete if for every 0-Cauchy sequence $\{x_n\}$ in $X$ converges with respect to $\tau_{p_{b_v}}$ to a point $x \in X$ such that $p_{b_v}(x,x) = 0$.

**Lemma 1.9.** [3]. *Let $(X, p_{b_v})$ be a partial $b_v(s)$-metric space with coefficient $s \geq 1$, then for any $n \in \mathbb{N}$ the couple $(X, p_{b_v})$ is a partial $b_{nv}(s^n)$-metric space.*

**Lemma 1.10.** *Let $(X, p_{b_v})$ be a partial $b_v(s)$-metric space with coefficient $s \geq 1$ and $T, I : X \longrightarrow X$. If $\{Ix_n\}$ is a sequence in $X$ defined by $Ix_{n+1} = Tx_n$ for all $n \geq 0$ with $Ix_n \neq Ix_{n+1}$. Let $k \in [0,1)$ such that*

$$p_{b_v}(Ix_{n+1}, Ix_n) \leq k\, p_{b_v}(Ix_n, Ix_{n-1}) \text{ for all } n \in \mathbb{N}. \tag{1.1}$$

*Then, either $T$ and $I$ have a common fixed point or $Ix_n \neq Ix_m$ for all distinct $n, m \in \mathbb{N}$.*

*Proof.* If $Ix_n = Ix_{n+1}$ then $x_n$ is a common fixed point for $T$ and $I$. Now it suffices to prove the case when $Ix_n \neq Ix_{n+t}$ for all $n, m \in \mathbb{N}$ with $n \geq 0$ and $t \geq 1$. Let $Ix_n = Ix_{n+t}$

for some $n \geq 0$ and $t \geq 1$, then $Ix_{n+1} = Ix_{n+t+1}$ and $Tx_n = Tx_{n+t}$. Inequality (1.1) implies that

$$p_{b_v}(Ix_{n+1}, Ix_n) = p_{b_v}(Ix_{n+t+1}, Ix_{n+1}) \leq k^t\, p_{b_v}(Ix_{n+1}, Ix_n) < p_{b_v}(Ix_{n+1}, Ix_n)$$

which is a contradiction. Therefore, $Ix_n \neq Ix_m$ for all distinct $n, m \in \mathbb{N}$. ∎

The next result is a generalization of Lemma 1.12 given in [10].

**Lemma 1.11.** *Let $(X, p_{b_v})$ be a partial $b_v(s)$-metric space with coefficient $s \geq 1$ and $\{Ix_n\}$ be a sequence in $X$ such that for all $n \geq 0$, $Ix_n \neq Ix_{n+1}$. Let $\kappa \in [0,1)$ and $\alpha, \beta, \tau, \delta \in \mathbb{R}^+$ such that*

$$p_{b_v}(Ix_n, Ix_m) \leq \kappa\, p_{b_v}(Ix_{n-1}, Ix_{m-1}) + (\alpha + s\tau)\kappa^n + (\beta + s\delta)\kappa^m \tag{1.2}$$

*for all $n, m \in \mathbb{N}$. Then $\{Ix_n\}$ is 0-Cauchy.*

*Proof.* It is easy to see that the proof holds if $\kappa = 0$. For $\kappa \in (0,1)$, since $\lim_{n\to\infty} \kappa^n = 0$, there exists $n_0 \in \mathbb{N}$ such that $0 < \kappa^{n_0} s < 1$ holds. Using (1.2) we have

$$\begin{aligned}
p_{b_v}(Ix_{n+1}, Ix_n) &\leq \kappa\, p_{b_v}(Ix_n, Ix_{n-1}) + (\alpha + s\tau)\kappa^n + (\beta + s\delta)\kappa^m \\
&\leq \kappa^2\, p_{b_v}(Ix_{n-1}, Ix_{n-2}) + 2(\alpha + s\tau)\kappa^{n+1} + 2(\beta + s\delta)\kappa^n \\
&\ \ \vdots \\
&\leq \kappa^n\, p_{b_v}(Ix_1, Ix_0) + n[(\alpha + s\tau)\kappa^{n+1} + (\beta + s\delta)\kappa^n] \\
&= \kappa^n\, p_{b_v}(Ix_1, Ix_0) + n\kappa^n\Theta_1.
\end{aligned}$$

where $\Theta_1 = (\alpha + s\tau)\kappa + (\beta + s\delta)$. Similarly,

$$\begin{aligned}
p_{b_v}(Ix_{n+r}, Ix_{m+r}) &\leq \kappa\, p_{b_v}(Ix_{n+r-1}, Ix_{m+r-1}) + (\alpha + s\tau)\kappa^{n+r} + (\beta + s\delta)\kappa^{m+r} \\
&\leq \kappa^2\, p_{b_v}(Ix_{n+r-2}, Ix_{m+r-2}) + 2(\alpha + s\tau)\kappa^{n+r} + 2(\beta + s\delta)\kappa^{m+r} \\
&\ \ \vdots \\
&\leq \kappa^r\, p_{b_v}(Ix_n, Ix_m) + r[(\alpha + s\tau)\kappa^{n+r} + (\beta + s\delta)\kappa^{m+r}] \\
&= \kappa^r\, p_{b_v}(Ix_n, Ix_m) + r\kappa^r\Theta_2.
\end{aligned}$$

where $\Theta_2 = (\alpha + s\tau)\kappa^n + (\beta + s\delta)\kappa^m$ and $r \geq 1$.
Now, we will consider the followings two cases:

- $v \geq 2$ and
- $v = 1$

For $v \geq 2$, we have

$$\begin{aligned}
p_{b_v}(Ix_n, Ix_m) &\leq s[p_{b_v}(Ix_n, Ix_{n+1}) + p_{b_v}(Ix_{n+1}, Ix_{n+2}) + \cdots + p_{b_v}(Ix_{n+v-2}, Ix_{n+n_0}) \\
&+ p_{b_v}(Ix_{n+n_0}, Ix_{m+n_0}) + p_{b_v}(Ix_{m+n_0}, Ix_m)] - \sum_{k=1}^{v-2} p_{b_v}(Ix_{n+k}, Ix_{n+k}) \\
&- p_{b_v}(Ix_{n+n_0}, Ix_{n+n_0}) - p_{b_v}(Ix_{n+m_0}, Ix_{n+m_0}) \\
&\leq s[p_{b_v}(Ix_n, Ix_{n+1}) + p_{b_v}(Ix_{n+1}, Ix_{n+2}) + \cdots + p_{b_v}(Ix_{n+v-3}, Ix_{n+v-2}) \\
&+ p_{b_v}(Ix_{n+v-2}, Ix_{n+n_0}) + p_{b_v}(Ix_{n+n_0}, Ix_{m+n_0}) + p_{b_v}(Ix_{m+n_0}, Ix_m)] \\
&\leq s[(\kappa^n + \kappa^{n+1} + \cdots + \kappa^{n+v-3}) p_{b_v}(Ix_1, Ix_0) \\
&+ (n\kappa^n + (n+1)\kappa^{n+1} + \cdots + (n+v-3)\kappa^{n+v-3})\Theta_1 \\
&+ \kappa^n p_{b_v}(Ix_{v-2}, Ix_{n_0}) + n\kappa^n \Theta_2 + \kappa^{n_0} p_{b_v}(Ix_n, Ix_m) + n_0 \kappa^{n_0} \Theta_2 \\
&+ \kappa^m p_{b_v}(Ix_{n_0}, Ix_0) + m\kappa^m \Theta_2].
\end{aligned} \tag{1.3}$$

As $n, m \to \infty$ in (1.3), we obtain $p_{b_v}(Ix_n, Ix_m) \to 0$. Therefore, $\{Ix_n\}$ is a 0-Cauchy sequence in $X$.
For $v = 1$, the proof holds using Lemma 1.9. ∎

## 2. Fixed Point Theorems

**Definition 2.1.** Let $(X, p_{b_v})$ be a 0-complete partial $b_v(s)$-metric space with coefficient $s \geq 1$ and $S, T : X \longrightarrow X$. Then, a point $x \in X$ is called a common fixed point of $S$ and $T$ if $x = Sx = Tx$.

In the sequel we present a variant of Jungck fixed point result [8] in a 0-complete partial $b_v(s)$-metric space.

**Theorem 2.2.** *Let $(X, p_{b_v})$ be a 0-complete partial $b_v(s)$-metric space with coefficient $s \geq 1$ and $T, I : X \longrightarrow X$ be commuting mappings satisfying the following inequality*

$$p_{b_v}(Tx, Ty) \leq k\, p_{b_v}(Ix, Iy) \tag{2.1}$$

*for all $x, y \in X$, where $k \in (0,1)$ and $s \geq 1$. If $I(X) \subseteq T(X)$ and $I$ is continuous then $T$ and $I$ have a unique common fixed point.*

*Proof.* **(Existence:)** Let $x_0 \in X$ be arbitrary, since $Tx_0 \in I(X)$ there exists an $x_1 \in X$ such that $Ix_1 = Tx_0$. Thus, generally for any $x_n \in X$ chosen, we have $x_{n+1} \in X$ such that $Ix_{n+1} = Tx_n$. Now, we show that $\{Ix_n\}$ is a 0-Cauchy sequence in $X$. By (2.1), we have

$$p_{b_v}(Ix_{n+1}, Ix_n) = p_{b_v}(Tx_n, Tx_{n-1}) \leq k\, p_{b_v}(Ix_n, Ix_{n-1}). \tag{2.2}$$

It follows from (2.2) that

$$p_{b_v}(Ix_{n+1}, Ix_n) \leq k^n\, p_{b_v}(Ix_1, Ix_0), \quad \text{for all } n \in \mathbb{N}.$$

If $Ix_n = Ix_{n+1}$ then $x_n$ is a fixed point of $T$ and we have nothing more to prove. So, we shall suppose that $Ix_n \neq Ix_{n+1}$ for all $n \geq 0$. Then from Lemma 1.10 we obtain $Ix_n \neq Ix_m$ for all distinct $n, m \in N$. From (2.1), we have

$$p_{b_v}(Ix_n, Ix_m) = p_{b_v}(Tx_{n-1}, Tx_{m-1}) \leq k\, p_{b_v}(Ix_{n-1}, Ix_{m-1}).$$

First method: Similar to one in [11].

Using Lemma 1.11, $\{Ix_n\}$ is a 0-Cauchy sequence in $X$. Since $X$ is 0-complete then there exists $x^* \in X$ such that

$$\lim_{n\to\infty} Ix_n = \lim_{n\to\infty} Tx_{n-1} = x^*. \tag{2.3}$$

By (2.1) and the continuity of $I$, $T$ is also continuous. Since $T$ and $I$ commute, we have

$$Ix^* = I(\lim_{n\to\infty} Tx_n) = \lim_{n\to\infty} ITx_n = \lim_{n\to\infty} TIx_n = T(\lim_{n\to\infty} Ix_n) = Tx^*. \tag{2.4}$$

Now we suppose that $y^* = Ix^* = Tx^*$, then

$$Ty^* = TIx^* = ITx^* = Iy^*.$$

If $Tx^* \neq Ty^*$, using (2.1) we have

$$p_{b_v}(Tx^*, Ty^*) \leq kp_{b_v}(Ix^*, Iy^*) = kp_{b_v}(Tx^*, Ty^*) < p_{b_v}(Tx^*, Ty^*) \tag{2.5}$$

which is a contradiction. Hence $Tx^* = Ty^*$, therefore $y^* = Iy^* = Ty^*$ implying that $y^*$ is a unique common fixed point for $T$ and $I$.

Second Method: Without using the continuity of $I$ and commuting property of the mappings.

Using Lemma 1.11, $\{Ix_n\}$ is a 0-Cauchy sequence in $X$. Since $X$ is 0-complete then there exists $x^* \in X$ such that

$$\lim_{n,m\to\infty} p_{b_v}(Ix_n, Ix_m) = \lim_{n\to\infty} p_{b_v}(Ix_n, Ix^*) = 0. \tag{2.6}$$

Now we show that $x^*$ is a common fixed point for $T$ and $I$.

$$\begin{aligned} p_{b_v}(Ix^*, Tx^*) &\leq s[p_{b_v}(Ix^*, Ix_n) + p_{b_v}(Ix_n, Ix_{n+1}) + \cdots + p_{b_v}(Ix_{n+v-1}, Ix_{n+v}) \\ &\quad + p_{b_v}(Ix_{n+v}, Tx^*)] - \sum_{k=1}^{v} p_{b_v}(Ix_{n+k}, Ix_{n+k}) \\ &= s[p_{b_v}(Ix^*, Ix_n) + p_{b_v}(Ix_n, Ix_{n+1}) + \cdots + p_{b_v}(Ix_{n+v-1}, Ix_{n+v}) \\ &\quad + p_{b_v}(Tx_{n+v-1}, Tx^*)] - \sum_{k=1}^{v} p_{b_v}(Ix_{n+k}, Ix_{n+k}) \\ &\leq s[p_{b_v}(Ix^*, Ix_n) + p_{b_v}(Ix_n, Ix_{n+1}) + \cdots + p_{b_v}(Ix_{n+v-1}, Ix_{n+v}) \\ &\quad + kp_{b_v}(Ix_{n+v-1}, Ix^*)] - \sum_{k=1}^{v} p_{b_v}(Ix_{n+k}, Ix_{n+k}). \end{aligned} \tag{2.7}$$

From (2.6), as $n \longrightarrow \infty$ in (2.7), we get $p_{b_v}(Ix^*, Tx^*) = 0$ i.e. $Ix^* = Tx^*$. Hence $x^*$ is a common fixed point for $T$ and $I$.

**(Uniqueness:)** Let $x^*, y^* \in X$ be two distinct common fixed points of $T$ and $I$ i.e. $x^* \neq y^*$, such that $Tx^* = Ix^* = x^*$ and $Ty^* = Iy^* = y^*$. Then, it follows from (2.1) that

$$p_{b_v}(x^*, y^*) = p_{b_v}(Tx^*, Ty^*) \leq k\, p_{b_v}(Ix^*, Iy^*) = k\, p_{b_v}(x^*, y^*) < p_{b_v}(x^*, y^*)$$

which is a contradiction. Hence $p_{b_v}(x^*, y^*) = 0$ implying $x^* = y^*$. ∎

From Theorem 2.2, we obtain the following variant of Banach fixed point theorem in partial $b_v(s)$-metric space.

**Corollary 2.3.** *Let $(X, p_{b_v})$ be a 0-complete partial $b_v(s)$-metric space with coefficient $s \geq 1$ and $T : X \longrightarrow X$ be a mappings satisfying the following inequality*

$$p_{b_v}(Tx, Ty) \leq k\, p_{b_v}(x, y) \tag{2.8}$$

*for all $x, y \in X$, where $k \in (0, 1)$ and $s \geq 1$. Then $T$ has a unique fixed point.*

**Remark 2.4.** Corollary 2.3 provides a complete solution to an open problem 1 raised by George et al. [7].

We obtain the following result as a consequence of Theorem 2.2 if we put $c = \frac{1}{k}$ and let $T$ be the identity map.

**Corollary 2.5.** *Let $(X, p_{b_v})$ be a 0-complete partial $b_v(s)$-metric space with coefficient $s \geq 1$ and $I : X \longrightarrow X$. Let there exists a $c > 1$ such that*

$$p_{b_v}(Ix, Iy) \geq c\, p_{b_v}(x, y) \tag{2.9}$$

*for all $x, y \in X$. If $I$ is continuous and onto then $I$ has a unique fixed point.*

**Remark 2.6.** Corollary 2.5 is a generalization of Theorem 3.4 given in [11].

**Lemma 2.7.** *Let $(X, b)$ be a partial b-metric space with coefficient $s \geq 1$. Then, $(X, p_{b_v})$ is partial $b_v(s)$-metric space with coefficient $s^v > 1$.*

*Proof.* Suppose that $(X, p_{b_v})$ is a partial $b$-metric space with coefficient $s \geq 1$. Let $u_1, u_2, \ldots, u_v$ be distict points in $X$ such that $u_1, u_2, \ldots, u_v \in X \backslash \{x, y\}$. Then

$$\begin{aligned} p_{b_v}(x, y) &\leq s[p_{b_v}(x, u_1) + p_{b_v}(u_1, y)] - p_{b_v}(u_1, u_1) \\ &\leq s[p_{b_v}(x, u_1) + s[p_{b_v}(u_1, u_2) + p_{b_v}(u_2, y)]] - \sum_{i=1}^{2} p_{b_v}(u_i, u_i) \\ &\vdots \\ &\leq s^v[p_{b_v}(x, u_1) + p_{b_v}(u_1, u_2) + \cdots + p_{b_v}(u_v, y)] - \sum_{i=1}^{v} p_{b_v}(u_i, u_i). \end{aligned} \tag{2.10}$$

Hence $(X, p_{b_v})$ is a partial $b_v(s)$-metric space with coefficient $s^v > 1$. ∎

* Let $(X, p_{b_v})$ be a 0-complete partial $b$-metric space with coefficient $s \geq 1$ and $T, I : X \longrightarrow X$ be commuting mappings satisfying the following inequality

$$p_{b_v}(Tx, Ty) \leq k\, p_{b_v}(Ix, Iy) \tag{2.11}$$

for all $x, y \in X$, where $k \in (0, 1)$. If $I(X) \subseteq T(X)$ and $I$ is continuous then $T$ and $I$ have a unique common fixed point.

* Let $(X, p_{b_v})$ be a 0-complete partial $b$-metric space with coefficient $s \geq 1$ and $T : X \longrightarrow X$ be mapping satisfying the following inequality

$$p_{b_v}(Tx, Ty) \leq k\, p_{b_v}(x, y) \tag{2.12}$$

for all $x, y \in X$, where $k \in (0, 1)$ and $s \geq 1$. Then $T$ have a unique fixed point.

* Let $(X, p_{b_v})$ be a 0-complete partial $b$-metric space with coefficient $s \geq 1$ and $I : X \longrightarrow X$. Let there exists a $k > 1$ such that

$$p_{b_v}(Ix, Iy) \geq k\, p_{b_v}(x, y) \tag{2.13}$$

for all $x, y \in X$. If $I$ is continuous and onto then $I$ has a unique fixed point.

## Conclusion

In this paper, the notion of 0-complete partial $b_v(s)$-metric space has been introduced and used to establish a variant of Jungck common fixed point theorem. Consequently the obtained theorem generalized various results due to Aleksić et al. [4].

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