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AN EXTRAGRADIENT METHOD WITHOUT MONOTONICITY

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Abstract In this paper, we present an iterative process for finding a common element of solution sets of the variational inequality problem, without a monotone mapping, and fixed point of a nonexpansive mapping. Moreover, we prove a weak convergence theorem by using an our process and give a numerical experiment.

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1. Introduction

The concept of the variational inequality problem, denoted $VI(C, A)$, is to find $x^* \in C$ such that

$$\langle A(x^*), y - x^*\rangle \geq 0, \quad \forall y \in C \tag{1.1}$$

where C is a nonempty closed convex subset of a real Hilbert space H, $A : C \to H$ is a continuous, and $\langle\cdot,\cdot\rangle$ denotes the inner product in H. Let $SOL(C, A)$ be the solution set of $VI(C, A)$ and $SOL(C, A)_D$ be the solution set of the dual variational inequality:

$$SOL(C, A)_D := \{x \in C \mid \langle A(y), y - x\rangle \geq 0, \forall y \in C\}. \tag{1.2}$$

There are a lot of iterative processes for finding $SOL(C, A)$ such as Goldstein-Levitin-Polyak projection methods [1, 2]; proximal point methods [18]; extragradient projection methods [5, 7, 10–15]; double projection methods [9, 16, 21]. These iterative processes assume the assumption either the monotonicity of A or $SOL(C, A) \subset SOL(C, A)_D$ which means that

$$\forall x^* \in SOL(C, A),\ \langle A(y), y - x^*\rangle \geq 0, \forall y \in C. \tag{1.3}$$

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Karamardian [19] showed that the assumption $SOL(C,A) \subset SOL(C,A)_D$ is a straightway consequence of pseudomonotonicity of A on C, and indicated that a monotone mapping can imply a pseudomonone mapping. In 2015, Ye and He [9], suggested a new method which is called a double projection method under the only assumption $SOL(C,A)_D \neq \emptyset$. This assumption is equivalent to the following inequality :

$$\exists \hat{x} \in SOL(C,A),\ \langle A(y), y - \hat{x}\rangle \geq 0, \forall y \in C. \tag{1.4}$$

Clearly, if $SOL(C,A) \subset SOL(C,A)_D$ then $SOL(C,A)_D \neq \emptyset$, but not converse. They proved that their method can find $SOL(C,A)$ without the monotonicity of A, and gave some numerical experiments.

The problem for finding a common element of the set of fixed point of a nonexpansive mapping and the solution set of the variational inequality problem for an inverse strongly-monotone mapping was presented by Takahashi and Toyoda [20]. A mapping S of C into itself is called nonexpansive if

$$\|Sx - Sy\| \leq \|x - y\| \quad \forall x, y \in C. \tag{1.5}$$

We denote by $F(S)$ the set of fixed point of S. By the way, their process obtained a weak convergence theorem for two sequences. Later, mathematicians were interested to establish iterative processes for solving a previous problem (see in [4, 22]). In 2006, Nadezhkina and Takahashi [17] introduced an iterative process for finding a common element of $F(S) \cap SOL(C,A)$ in a real Hilbert space H, and showed that any inverse strongly-monotone is monotone and k-Lipschitz continuous. Their process follows the idea of an extragradient method [7] by setting $A : C \to H$ is k-Lipschitz continuous monotone, and $S : C \to C$ is nonexpansive. They constructed

$$\begin{aligned} &x_0 = x \in C \\ &y_n = P_C(x_n - \lambda_n A x_n) \\ &x_{n+1} = \alpha_n x_n + (1-\alpha_n) S P_C(x_n - \alpha_n A y_n) \end{aligned}$$

for every $n = 0, 1, 2, ...$, where $\{\lambda_n\} \subset [a,b]$ for some $a, b \in (0, \frac{1}{k})$, and proved that $\{x_n\}$ and $\{y_n\}$ converge weakly to a point in $F(S) \cap SOL(C,A)$.

In this article, we establish an iterative process for finding a common element of $F(S)$ for a nonexpansive mapping S, and $SOL(C,A)$ without the monotonicity of A by setting $SOL(C,A)_D \neq \emptyset$, and $A : C \to H$ is only k-Lipschitz continuous. Furthermore, we prove a weak convergence theorem, and give a numerical experiment for support in an our result.

2. Preliminaries

This section contains definitions that will be used in this work. Note that H is a real Hilbert space, $C \subset H$ is a nonempty closed and convex set, and $A : C \to H$ is a continuous operator. The projection from $x \in H$ onto C is defined by $P_C := \arg\min\{\|y - x\| \mid y \in C\}$. The natural residual function $r_\mu(\cdot)$ is defined by $r_\mu(x) := x - P_C(x - \mu A(x))$, where $\mu > 0$ is a parameter. If $\mu = 1$, we write $r(x)$ for $r_\mu(x)$.

Lemma 2.1. [3] *For any $x \in H$ and $z \in C$,*
(A) $\|P_C(x) - z\|^2 \leq \|x - z\|^2 - \|P_C(x) - x\|^2$;
(B) $\langle P_C(x) - x, z - P_C(x)\rangle \geq 0$.

Lemma 2.2. [8] *Let H be a real Hilbert space, h be a real-valued function on H, and $K := \{x \in H : h(x) \leq 0\}$. If K is nonempty and h is Lipschitz continuous with modulus*

$\theta > 0$, *then*

$$dist(x, K) \geq \theta^{-1}h(x),\ \forall x \in H, \tag{2.1}$$

where $dist(x, K)$ *denotes the distance from* x *to* K*.*

Remark 2.3. If we set $K := K \cap C$ and $K \cap C \neq \emptyset$, then (2.1) holds. Note that C and $K \cap C$ are closed, so there exist $\min_{y \in K\cap C} \|x - y\|$ and $\min_{y\in K} \|x - y\|$ which $\min_{y\in K\cap C} \|x - y\| \leq \min_{y\in K} \|x - y\|$, that is, $dist(x, K) \leq dist(x, C \cap K)$

Lemma 2.4. $x^* \in SOL(C, A)$ *if and only if* $\|r_\mu(x^*)\| = 0$.

Proof. The proof is similar with Proposition 1.5.8 in [6]. ∎

Lemma 2.5. *For every* $x \in C$*,*

$$\langle A(x), r_\mu(x)\rangle \geq \mu^{-1}\|r_\mu(x)\|^2. \tag{2.2}$$

Proof. The proof is similar with Lemma 2.1 in [16]. ∎

Lemma 2.6. *Let the function* h_n *be defined by (3.2) and* $\{x_n\}$ *be generated by Algorithm in Theorem 3.1. If* $SOL(C, A)_D \neq \emptyset$*, then* $h_n(x_n) \geq (1 - \sigma)\|r_{\lambda_n}(x_n)\|^2 > 0$ *for every* n*. If* $x^* \in SOL(C, A)_D$*, then* $h_n(x^*) \leq 0$ *for every* n*.*

Proof. The proof is similar with Lemma 2.8 in [9]. ∎

Lemma 2.7. *If* $\{x_n\}$ *is an infinite sequence generated by Algorithm in Theorem 3.1 and* $\tilde{x}$ *is any accumulation point of* $\{x_n\}$*, then* $\tilde{x} \in \cap_{n=1}^{\infty} H_n$*.*

Proof. Let l be a nonnegative integer and $\tilde{x}$ be an accumulation point of $\{x_n\}$. There is a subsequence $\{x_{n_m}\}$ of $\{x_n\}$, so $\lim_{m\to\infty} x_{n_m} = \tilde{x}$. By the definition of $x_{n_m} = \alpha_{n_m-1}x_{n_m-1}+(1-\alpha_{n_m-1})SP_{C\cap\tilde{H}_{n_m-1}}(x_{n_m-1}-\lambda_{n_m-1}A(y_{n_m-1}))$ and $\tilde{H}_{n_m-1} = \cap_{j=1}^{j=n_m-1} H_j$, we obtain $x_{n_m} \in H_l$ for every $m \geq l + 1$. Since H_l is closed and $\lim_{m\to\infty} x_{n_m} = \tilde{x}$, we have $\tilde{x} \in H_l$. ∎

Lemma 2.8. [17] *Let* $\{\alpha_n\}$ *be a sequence of real numbers such that* $0 < a \leq \alpha_n \leq b < 1$ *for every* $n = 0, 1, 2, \ldots,$ *and* $\{v_n\}, \{w_n\}$ *sequences in* H *such that*

$$\limsup_{n\to\infty} \|v_n\| \leq c, \quad \limsup_{n\to\infty} \|w_n\| \leq c, \text{ and } \lim_{n\to\infty} \|\alpha_n v_n + (1 - \alpha_n)w_n\| = c, \tag{2.3}$$

for some $c \geq 0$*. Then* $\lim_{n\to\infty} \|v_n - w_n\| = 0$.

Lemma 2.9. [17] *Let* $\{x_n\}$ *be a sequence in* H*. Suppose that for each* $u \in C$*,*

$$\|x_{n+1} - u\| \leq \|x_n - u\| \tag{2.4}$$

for every $n = 0, 1, 2, \ldots$*. Then, the sequence* $\{P_C x_n\}$ *converges strongly to some* $z \in C$*.*

3. Main results

In this section, we propose the algorithm for finding a common point of $F(S) \cap SOL(C, A)$ under the assumption $SOL(C, A)_D \neq \emptyset$, A is only a k-Lipschitz continuous mapping, and S is a nonexpansive mapping. We call the following method that an extragradient method without monotonicity.

Theorem 3.1. *Let C be a closed convex subset of a real Hilbert space H. Let A be a k-Lipschitz continuous mapping of C onto H, $SOL(C,A)_D \neq \emptyset$ and S be a nonexpansive mapping of C into itself such that $F(S) \cap SOL(C,A) \neq \emptyset$. Let $\{x_n\}, \{y_n\}$ be sequences generated by*
Algorithm *For every $n = 0, 1, 2, \dots,$ choose $x_0 \in C$ as an initial point, $\sigma \in (0,1), \gamma \in (0,1), \{\lambda_n\} \subset [a,b]$, $\exists\, a, b \in (0, \frac{1}{k})$ and $\{\alpha_n\} \in [c,d]$, $\exists\, c, d \in (0,1)$. Compute*

$$z_n := P_C(x_n - \lambda_n A(x_n))$$

Step 1. *Compute $r_{\lambda_n}(x_n) = x_n - z_n$. If $r_{\lambda_n}(x_n) = 0$, stop. Other, go to Step 2.*
Step 2. *Compute $y_n = x_n - \eta_n r_{\lambda_n}(x_n)$, where $\eta_n = \gamma_{m_n}$, with m_n being the smallest nonnegative integer satisfying*

$$\langle A(x_n) - A(x_n - \gamma_m r_{\lambda_n}(x_n)), r_{\lambda_n}(x_n)\rangle \leq \sigma \|r_{\lambda_n}(x_n)\|^2. \tag{3.1}$$

Step 3. *Compute $x_{n+1} = \alpha_n x_n + (1-\alpha_n) S P_{C\cap \tilde{H}_n}(x_n - \lambda_n A(y_n))$, where $\tilde{H}_n := \bigcap_{j=0}^{j=n} H_j$ with $H_j := \{v : h_j(v) \leq 0\}$ is a halfspace defined by*

$$h_j(v) := \langle A(y_j), v - y_j\rangle. \tag{3.2}$$

Let $n = n+1$ and return to Step 1.
Then, the sequences $\{x_n\}, \{y_n\}$ converge weakly to the point $z \in F(S) \cap SOL(C,A)$, where $z = \lim_{n\to\infty} P_{F(S)\cap SOL(C,A)}(x_n)$.

Proof. Let $b_n = P_{C\cap\tilde{H}_n}(x_n - \lambda_n A(y_n))$ for every $n = 0, 1, 2, \dots$. Let $u \in F(S) \cap SOL(C,A)$. From Lemma 2.1 (A), we have

$$\begin{aligned}
\|b_n - u\|^2 &\leq \|x_n - \lambda_n A(y_n) - u\|^2 - \|x_n - \lambda_n A(y_n) - b_n\|^2 \\
&= \|x_n - u\|^2 - \|x_n - b_n\|^2 + 2\lambda_n \langle A(y_n), u - b_n\rangle \\
&= \|x_n - u\|^2 - \|x_n - b_n\|^2 + 2\lambda_n (\langle A(y_n) - A(u), u - y_n\rangle \\
&\quad + \langle A(u), u - y_n\rangle + \langle A(y_n), y_n - b_n\rangle) \\
&\leq \|x_n - u\|^2 - \|x_n - b_n\|^2 + 2\lambda_n \langle A(y_n), y_n - b_n\rangle \\
&= \|x_n - u\|^2 - \|x_n - y_n\|^2 - 2\langle x_n - y_n, y_n - b_n\rangle - \|y_n - b_n\|^2 \\
&\quad + 2\lambda_n \langle A(y_n), y_n - b_n\rangle \\
&= \|x_n - u\|^2 - \|x_n - y_n\|^2 - \|y_n - b_n\|^2 + 2\langle x_n - \lambda_n A(y_n) - y_n, b_n - y_n\rangle.
\end{aligned}$$

Thank to Lemma 2.1 (B), we receive

$$\begin{aligned}
\langle x_n - \lambda_n A(y_n) - y_n, b_n - y_n\rangle &= \langle x_n - \lambda_n A(x_n) - y_n, b_n - y_n\rangle \\
&\quad + \langle \lambda_n A(x_n) - \lambda_n A(y_n), b_n - y_n\rangle \\
&\leq \langle \lambda_n A(x_n) - \lambda_n A(y_n), b_n - y_n\rangle \\
&\leq \lambda_n k \|x_n - y_n\| \, \|b_n - y_n\|.
\end{aligned}$$

This implies that

$$\begin{aligned}\|b_n-u\|^2 &\leq \|x_n-u\|^2-\|x_n-y_n\|^2-\|y_n-z_n\|^2\\ &\quad +2\lambda_n k\|x_n-y_n\|\,\|z_n-y_n\|\\ &\leq \|x_n-u\|^2-\|x_n-y_n\|^2-\|y_n-z_n\|^2\\ &\quad +\lambda_n^2k^2\|x_n-y_n\|^2+\|y_n-z_n\|^2\\ &\leq \|x_n-u\|^2+(\lambda_n^2k^2-1)\|x_n-y_n\|t^2\\ &\leq \|x_n-u\|^2.\end{aligned}$$

We also have

$$\begin{aligned}\|x_{n+1}-u\|^2 &= \|\alpha_n x_n+(1-\alpha_n)Sb_n-u\|^2\\ &= \|\alpha_n(x_n-u)+(1-\alpha_n)(Sb_n-u)\|^2\\ &\leq \alpha_n\|x_n-u\|+(1-\alpha_n)\|Sb_n-u\|^2\\ &\leq \alpha_n\|x_n-u\|^2+(1-\alpha_n)\|b_n-u\|^2\\ &\leq \alpha_n\|x_n-u\|^2+(1-\alpha_n)(\|x_n-u\|^2+(\lambda_n^2k^2-1)\|x_n-y_n\|^2)\\ &= \|x_n-u\|^2+(1-\alpha_n)(\lambda_n^2k^2-1)\|x_n-y_n\|^2\\ &\leq \|x_n-u\|^2.\end{aligned}$$

So, there is $c=\lim_{n\to\infty}\|x_n-u\|$ and the sequence $\{x_n\}$, $\{b_n\}$ are bounded. By the previuos relations, we observe that

$$\|x_n-y_n\|^2\leq\frac{1}{(1-\alpha_n)(1-\lambda_n^2k^2)}(\|x_n-u\|^2-\|x_{n+1}-u\|^2). \tag{3.3}$$

Therefore, $x_n-y_n\to 0,\ n\to\infty$. Moreover, we have

$$\begin{aligned}\|y_n-b_n\|^2 &\leq \|y_n-\eta_n b_n\|^2\\ &= \|x_n-\eta_n(x_n-P_C(x_n-\lambda_nA(x_n)))-\eta_nP_{C\cap\tilde{H}_n}(x_n-\lambda_nA(y_n))\|^2\\ &\leq \|x_n-\eta_n(x_n-P_C(x_n-\lambda_nA(x_n)))-\eta_nP_C(x_n-\lambda_nA(y_n))\|^2\\ &\leq \|x_n-\eta_nx_n\|^2+(\eta_n\lambda_nk)^2\|y_n-x_n\|^2\\ &= (1-\eta_n^2)\|x_n\|^2+(\eta_n\lambda_nk)^2\|y_n-x_n\|^2\\ &\leq (1-\eta_n^2)\|x_n\|^2+\frac{(\eta_n\lambda_nk)^2}{(1-\alpha_n)(1-\lambda_n^2k^2)}(\|x_n-u\|^2-\|x_{n+1}-u\|^2).\end{aligned}$$

Thus $y_n-b_n\to 0, n\to\infty$. It obtains that $x_n-b_n\to 0, n\to\infty$. Since A is k-Lipschitz continuous, it follows that

$$A(y_n)-A(b_n)\to 0\quad as\quad n\to\infty. \tag{3.4}$$

We know that $\{x_n\}$ is bounded. There is a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which converges weakly to some $z\in F(S)\cap SOL(C,A)$. We are going to show that $z\in SOL(C,A)$. Let $x^*\in\cap_{n=1}^{\infty}(H_n\cap C)$. We follow the previous proof of $\{\|x_n-u\|^2\}$ by setting $x^*=u$. It can imply that the sequence $\{\|x_n-x^*\|^2\}$ is nonincreasing and convergent. Therefore

$$\lim_{n\to\infty} dist(x_n, C\cap\tilde{H}_n)=0. \tag{3.5}$$

By the hypothesis, $A(x)$ and $r_{\lambda_n}(x)$ are continuous.
So the sequence $\{z_n\},\{r_{\lambda_n}(x_n)\}$, and $\{y_n\}$ are bounded.

Likewise, $\{A(y_n)\}$ is bounded by the continuity of A. For some $W > 0$,

$$\|A(y_n)\| \leq W, \ \forall n. \tag{3.6}$$

By the definition of $\tilde{H}_n$, we obtain that $\tilde{H} \subseteq H_n$ for every n. Thus

$$dist(x_n, C \cap H_n) \leq dist(x_n, C \cap \tilde{H}_n). \tag{3.7}$$

From (3.5), it follows that

$$\lim_{n\to\infty} dist(x_n, C \cap H_n) = 0. \tag{3.8}$$

Obviously, every function h_n is Lipschitz continuous on C with modulus W.
By Lemmas 2.2 and 2.6, it follows

$$dist(x_n, C \cap H_n) \geq W^{-1} h_n(x_n) \geq W^{-1}(1-\sigma)\eta_n \|r_{\lambda_n}(x_n)\|^2. \tag{3.9}$$

In accordance with (3.8) and (3.9), we can obtain that $\lim_{n\to\infty} \eta_n \|r_{\lambda_n}(x_n)\|^2 = 0$. If $\lim_{n\to\infty} \sup \eta_n > 0$, then we must have $\lim_{n\to\infty} \inf \|r_{\lambda_n}(x_n)\| = 0$. Since $\{r_{\lambda_n}(x)\}$ is continuous and $\{x_n\}$ is bounded, there is an accumulation point $\hat{x}$ of $\{x_n\}$ such that $r_{\lambda_n}(\hat{x}) = 0$. From Lemmas 2.4 and 2.7, it implies that $\hat{x} \in \cap_{n=1}^{\infty}(H_n \cap SOL(C, A))$. Replace x^* by $\hat{x}$, it follows that $\{\|x_n - \hat{x}\|^2\}$ is nonincreasing and convergent. Since $\hat{x}$ is an accumulation point of $\{x_n\}$, we have $x_n \to \hat{x} := z \in SOL(C, A)$ If $\lim_{n\to\infty} \sup \eta_n = 0$. then $\lim_{n\to\infty} \eta_n = 0$. Suppose that $\bar{x}$ is an accumulation point of $\{x_n\}$. There is a subsequence $\{x_{n_j}\}$ converges to $\bar{x}$. By the choice of η_n, (3.1) is not satisfied for $m_n - 1$, that is,

$$\langle A(x_{n_j}) - A(x_{n_j} - \gamma^{-1}\eta_{n_j}), r_{\lambda_n}(x_{k_j})\rangle > \sigma \|r_{\lambda_n}(x_{n_j})\|^2. \tag{3.10}$$

Since $r_{\lambda_n}(x)$ and $A(x)$ are continuous, taking the limit in (3.10), we obtain

$$0 \leq \sigma \|r_{\lambda_n}(\bar{x})\|^2 \leq 0. \tag{3.11}$$

By (3.11), it implies that $r_{\lambda_n}(\bar{x}) = 0$. So $\bar{x} \in \cap_{n=1}^{\infty}(H_n \cap SOL(C, A))$. By the primal case, $\{x_n\} \to \bar{x} := z \in SOL(C, A)$. Onwards, we are going to show that $z \in F(S)$. Let $u \in F(S) \cap SOL(C, A)$. Consider

$$\|x_n - u\| \geq \|b_n - u\| \geq \|Sb_n - u\|, \tag{3.12}$$

it follows $\lim_{n\to\infty} \sup \|Sb_n - u\| \leq c$. Moreover,

$$\lim_{n\to\infty} \|\alpha_n(x_n - u) + (1-\alpha_n)(Sb_n - u)\| = \lim_{n\to\infty} \|x_{n+1} - u\| = c. \tag{3.13}$$

Lemma 2.8 yields $\lim_{n\to\infty} \|Sb_n - x_n\| = 0$. Consider

$$\|Sx_n - x_n\| \leq \|Sx_n - Sb_n\| + \|Sb_n - x_n\| \leq \|x_n - b_n\| + \|Sb_n - x_n\|, \tag{3.14}$$

it follows that $\lim_{n\to\infty} \|Sx_n - x_n\| = 0$. The demiclosedness of $I - S$ yields $\{x_{n_i}\} \rightharpoonup z$ and $\lim_{n\to\infty} \|Sx_n - x_n\| = 0$. This implies that $z \in F(S)$. Assume that $\{x_{n_j}\}$ is another subsequence of $\{x_n\}$ such that $\{x_{n_j}\} \rightharpoonup z' \in F(S) \cap SOL(C, A)$. We will show that $z = z'$. Let $z \neq z'$. The Opial condition yields

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - z\| &= \liminf_{n\to\infty} \|x_{n_i} - z\| < \liminf_{n\to\infty} \|x_{n_i} - z'\| \\
&= \lim_{n\to\infty} \|x_n - z'\| = \liminf_{n\to\infty} \|x_{n_j} - z'\| \\
&< \liminf_{n\to\infty} \|x_{n_j} - z\| = \lim_{n\to\infty} \|x_n - z\|.
\end{aligned}$$

It is a contradiction. So $z = z'$. Since $x_n - y_n \to 0$, and $x_n \rightharpoonup z \in F(S) \cap SOL(C, A)$, it receives $y_n \rightharpoonup z \in F(S) \cap SOL(C, A)$. Putting $u_n = P_{F(S)\cap SOL(C,A)} x_n$. We will show that $z = \lim_{n\to\infty} u_n$. Note that $z \in F(S) \cap SOL(C, A)$ and $u_n = P_{F(S)\cap SOL(C,A)} x_n$. It is obvious that $\langle z - u_n, u_n - x_n\rangle \geq 0$. Thus $\{u_n\}$ converges strongly to some $z_0 \in F(S) \cap SOL(C, A)$ by Lemma 2.9. That is $\langle z - z_0, z_0 - z\rangle \geq 0$. Hence $z = z_0$. This proof is complete. ■

4. Applications

In this section, we assume that $C = H$, $SOL(H, A)_D \neq \emptyset$, and $A : H \to H$ is only a k-Lipschitz continuous mapping in Theorems 4.1 and 4.2 in [17]. By using Theorem 3.1, it can obtain two following theorems.

Theorem 4.1. *Suppose that $S : H \to H$ is a nonexpansive mapping such that $F(S) \cap A^{-1}0 \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_0 = x \in H$ and let*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)S(x_n - \lambda_n A(x_n - \lambda_n A(x_n))) \tag{4.1}$$

For every $n = 0, 1, 2, \ldots,$ where $\sigma \in (0, 1), \gamma \in (0, 1), \{\lambda_n\} \subset [a, b], \exists\, a, b \in (0, \frac{1}{k})$ and $\{\alpha_n\} \in [c, d], \exists\, c, d \in (0, 1)$. Then, the sequence $\{x_n\}$ converges weakly to some point $z \in F(S) \cap A^{-1}0$ where $z = \lim_{n\to\infty} P_{F(S)\cap A^{-1}0} x_n$.

Proof. Setting $A^{-1}0 = SOL(H, A)$ and $P_H = I$. According to Theorem 3.1, we receive the wistful result. ■

Theorem 4.2. *Let $B : H \to 2^H$ be a maximal monotone mapping such that $A^{-1}0 \cap B^{-1}0 \neq \emptyset$. Let J^B_r be the resolvent of B for each $r > 0$. Let $\{x_n\}$ be a sequence generated by*

$$x_0 = x \in H$$
$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)J^B_r(x_n - \lambda_n A(x_n - \lambda_n A(x_n)))$$

For every $n = 0, 1, 2, \ldots,$ where $\sigma \in (0, 1), \gamma \in (0, 1), \{\lambda_n\} \subset [a, b], \exists\, a, b \in (0, \frac{1}{k})$ and $\{\alpha_n\} \in [c, d], \exists\, c, d \in (0, 1)$. Then, the sequence $\{x_n\}$ converges weakly to some point $z \in A^{-1}0 \cap B^{-1}0$ where $z = \lim_{n\to\infty} P_{A^{-1}0\cap B^{-1}0} x_n$.

Proof. Setting $A^{-1}0 = SOL(H, A), F(J^B_r) = B^{-1}0$ and $P_H = I$. By Theorem 3.1, we receive the wistful result. ■

5. A numerical experiment

Example 5.1. Let $H = \mathbf{R}, C = [1, 10]$, and $A : C \to H$ be defined by $A(x) = -2x$. Let $S(x) = x$ for every $x \in C$. We are going to show that A is not a monotone mapping. For all $x, y \in C$, it obtains

$$\langle Ax, x - y\rangle = -2(x - y)^2$$

So, we can choose $x, y \in C$ such that $\langle Ax, x - y\rangle < 0$. Hence A is not a monotone mapping. It easy to check that $SOL(C, A)_D \neq \emptyset$. Choose $\sigma = 0.5, \gamma = 0.01, 0.1, 0.5, 0.9, \lambda_n = \frac{1}{2}(\frac{4n+1}{n})$, and $\alpha_n = \frac{2+3n}{6n}$. The assumptions in Theorem 3.1 are satisfied. We have consummated all processes in Matlab R2015 running on a Desktop with Intel(R) Core(TM) i5-7200u CPU 2.50 GHz, and 4 GB RAM. We give the stopping criteria $\|x_{n+1} - x_n\| < \varepsilon$ with $\varepsilon = 10^{-9}$ is a tolerance to cease the algorithms. The commutation results reported in the following table:

Table : The results computed on the algorithm in Theorem 3.1

$N.P.$	γ	Average iteration	Average times
5	0.01	1401	690.5656
5	0.1	229	26.7586
5	0.5	106	8.1406
5	0.9	97	6.4202

where

- N.P: the number of the tested problems.
- Average iteration: the average number of iterations.
- Average times: the average CPU-computation times (in s).

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