FIXED POINT THEOREMS FOR MEIR-KEELER CONDENSING OPERATORS IN PARTIALLY ORDERED BANACH SPACES

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Abstract The purpose of this paper to build up the concept of a Meir-Keeler condensing and integral type condensing operators in partially ordered Banach spaces via the concept of a measure of noncompactness. We also provide a characterization of a Meir-Keeler condensing operators using the notion of L-functions in partially ordered Banach spaces. To attain these results, we relaxed the conditions of boundedness, closeness, and convexity of the set at the expense that the operator is monotone and bounded. In addition, as an application, we apply these results to obtain coupled and tripled fixed theorems. Our results generalize and extend similar literature results.

1. INTRODUCTION

Fixed point theory is certainly a fascinating field of research in mathematics, which offers a wide range of applications in various fields of science and technology. Poincare initiated a study of fixed point theory, after which Brouwer's [19] established influential Brouwer's fixed point theorem for finite dimensional spaces. In addition, Banach has presented its distinguished contraction theorem at 1922 for complete metric space, that provide the assurances of a unique fixed point. Later, Schauder in 1930 prolonged the Brouwer's fixed point results [37] for infinite dimensional spaces by considering the compactness requirement on the operator or on the feasible set. There are a number

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advancements in fixed point theory in several directions, among them single-valued mappings that have been a lot of work in the literature (see, [13, 15, 16, 25, 34, 35, 41–47] and references therein). However, multi valued mappings have several direct applications in the real world compared to single valued mappings (see [32, 33] and others). Furthermore, Kakutani [28] in 1941, presented the Brouwer′s fixed point result for the case of multi-valued mappings. Later in 1969, Nadler [31] extended the Banach contraction theorem from the single-valued mapping to multi-valued mapping by adopting the notion of Hausdorff metric. Thereafter, several authors have modified and generalized the concept of the Banach contraction theorem in various directions (see [25, 26] and others). Among the most key one generalizations of the Banach contraction is a Meir-Keeler type contraction [30] introduced by Meir and Keeler in 1969.

On the other hand, in 1930, Kuratowski [29] suggested the idea of a measure of noncompactness capable of for measuring the degree of noncompactness of a bounded set. The measure of noncompactness is effective in the study of single and multi-valued fixed point theory. The measure of noncompactness, together with some algebraic considerations, is useful in examining the existence of solutions to certain non linear problems under specific conditions. The definitions of a Kuratowski and Hausdorff′s measure of noncompactness are both well-known in literature and, in general, Kuratowski′s measure of noncompactness is very useful in proving Darbo′s famous fixed point theorem for noncompact operators. In order to investigate the fixed point properties of noncompact operators, the Darbo′s fixed point theorem is particularly helpful in the sense that it generalizes the Schauder′s fixed point theorem. The Darbo fixed point theorem is quite helpful for solving differential and integral equations. That is why this research area is truly interested in getting a convenient generalization of Darbo′s fixed point theorem and addressing it to other abstract spaces. To date, numerous research papers have also been published on the generalization of Darbo′s fixed point theorem using a technique of a measure of noncompactness (see [3, 5, 9, 20, 38] and references therein). Furthermore, the application of these fixed point results also established in the field of differential and integral equations [2, 6, 12, 22].

Recently, Aghajani [30] present the notion of Meir-Keleer condensing operator and proposed generalization of Darbo's fixed point theorem comparatively as Meir-Keleer contraction fixed point theorem is generalization of Banach fixed point theorem. Further, also Aghajani [1] introduced integral type condensing operator and proposed Darbo's fixed point theorem for a large class of operators. Inspired from the above papers we seek the validity of these results of above papers in partially ordered Banach space. In this paper, we will extend the results of Theorem 2.10, and Theorem 2.11, into partially ordered Banach spaces. By doing this, we also improve and generalize the works mentioned in [23, 24, 39]. We utilize this notion of measure of noncompactness to prove some fixed point theorems in partially ordered Banach spaces whose positive cone K is normal. To accomplish this result, we relaxed the conditions of boundedness, closeness and convexity of the set at the expense that the operator is monotone and bounded. Furthermore, we apply the obtained fixed point theorems to prove some new coupled fixed point and tripled fixed point results to seek the validity of our results.

The remainder of the paper is written up as follows: We have introductory concepts and approaches to demonstrate our leading results. Section 2 sets out the proposed fixed point theorems for both Meir-Keeler condensing operator and Meir-Keeler condensing operator

through L-function. Section 3 presents the implication of these results in coupled and tripled fixed points results.

2. Preliminaries

During this whole paper, we used the following notions:

- $\mathbb{E}$: Banach space;
- $\|.\|$: norm on a Banach space;
- θ : zero element of a Banach space;
- $B(x, r)$: closed ball having x as a center with radius r;
- λA and $A + B$: the algebraic operations on sets;
- $\overline{A}$: closure of a set A;
- coA : convex hull of a set a set A;
- $\overline{co}A$: closed convex hull of a set A;
- $\mathfrak{M}_{\mathbb{E}}$: set of all bounded subsets of a space $\mathbb{E}$;
- $\mathfrak{N}_{\mathbb{E}}$: set of all relatively compact subsets of a space $\mathbb{E}$;
- Ψ : class of function such that $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing function with $\lim_{n \to \infty} \psi^n(p) = 0$ for all $p \geq 0$;
- Φ : class of function such that $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is a Lebesgue-integrable mapping which is summable on each compact subset of $\mathbb{R}_+$ and for each $\epsilon > 0$, $\int_0^\epsilon \phi(t)dt > 0$.

Now, we recall the axiomatic definition of a measure of noncompactness.

Definition 2.1. [14] A function $\mu : \mathfrak{M}_{\mathbb{E}} \longrightarrow [0, +\infty)$ is called to be a measure of non-compactness on $\mathbb{E}$ if the following conditions have been met:

1. The set $ker\mu = \{A \in \mathfrak{M}_{\mathbb{E}} : \mu(A) = 0\}$ is nonempty and $ker\mu \subset \mathfrak{N}_{\mathbb{E}}$;
2. $A \subset B \Longrightarrow \mu(A) \leq \mu(B)$;
3. $\mu(A) = \mu(\overline{A})$;
4. $\mu(coA) = \mu(A)$;
5. $\mu\big(\kappa A + (1 - \kappa)B\big) \leq \kappa\mu(A) + (1 - \kappa)\mu(B), \quad \forall \kappa \in [0, 1]$;
6. For any sequence A_n of closed sets inside $\mathfrak{M}_{\mathbb{E}}$ so that $A_{n+1} \subset A_n$ for each $n = 1, 2, \cdots$, and if $\lim_{n \to \infty} \mu(A_n) = 0$ then $A_\infty = \bigcap_{n=1}^\infty A_n \neq \phi$.

Remark 2.2. The family $Ker\mu$ defined in the above definition serves as the kernel of the measure of noncompactness of μ. We can also note that intersection set A_∞ is an element of the family $Ker\mu$. Indeed, $\mu(A_\infty) \leq \mu(A_n)$ for all n, we can quickly conclude that $\mu(A_\infty) = 0$. In perspective of this, means that $A_\infty \in Ker\mu$.

Example 2.3. [7] Le a metric space X and the map defined by

$$\phi(A) = \begin{cases} 0, & A \text{ is totally bounded;} \\ 1, & \text{otherwise} \end{cases}$$

or

$$\phi(A) = diam(A)$$

is a measure of noncompactness on X.

Definition 2.4. An operator $F : X \to Y$ is said to be a compact operator if, for every $A \subset X$ bounded, the image set $F(A)$ is relatively compact in the Banach space Y.

Theorem 2.5. *(Darbo's fixed point theorem)[21] Let C be a nonempty, bounded, closed and convex subset of a Banach space $\mathbb{E}$ and $F : C \longrightarrow C$ be a continuous mapping and there is a constant $k \in [0, 1)$ such that*

$$\mu(F(A)) \leq k\mu(A)$$

for every nonempty subset A of C. Then, there exits a fixed a point for mapping F in the set C.

Remark 2.6. The above theorem extends the classical Brouwer′s fixed point theorems to noncompact operators and it has a number of applications in the study of solutions existence of the differential and integral equations.

Definition 2.7. [30] Suppose (X, d) be a metric space and $F : X \longrightarrow X$ is said to be Meir-Keeler contraction (MKC) if for any $\varepsilon > 0$ there exist an $\delta > 0$ such that

$$\varepsilon \leq d(x, y) < \varepsilon + \delta \implies d(Fx, Fy) < \varepsilon, \quad \forall x, y \in X.$$

Theorem 2.8. [30] *Suppose (X, d) be a complete metric space and $F : X \longrightarrow X$ is a Meir-Keeler contraction mapping. Then, there exits a fixed a point for mapping F in the set C.*

By definition of a Meir-Keeler condensing operator for self-mapping $F : C \to C$ has recently been established in [3] and provides the fixed point theorems which are follows.

Definition 2.9. [3] Assume C to be a nonempty subset of a Banach space $\mathbb{E}$ and μ is any measure of noncompactness on $\mathbb{E}$. A continuous mapping $F : C \longrightarrow \mathbb{E}$ is called to be a Meir-Keeler condensing operator if for every $\varepsilon > 0$ there exist an $\delta > 0$ in such a manner that

$$\varepsilon \leq \mu(A) < \varepsilon + \delta \implies \mu(F(A)) < \varepsilon$$

for each bounded subset A of C.

Theorem 2.10. [3] *Assume that $F : C \to C$ is a Meir-Keeler condensing operator on a nonempty, closed, bounded and convex subset C of Banach space $\mathbb{E}$. Then F must have at least one fixed point, as well as the set of all fixed points of F in C is compact set.*

The useful generalization of Darbo′s fixed point theorem given by Aghajani [1] in the form of an integral form as follows.

Theorem 2.11. [1] *Assume C be any nonempty, closed, bounded and convex subset of a Banach space $\mathbb{E}$. Let $F : C \to C$ is a continuous operator such that*

$$\int_0^{\mu(F(A))} \phi(s)ds \leq \psi \left(\int_0^{\mu(A)} \phi(s)ds \right)$$

for every nonempty subset A of C, where μ is any measure of noncompactness and $\psi \in \Psi$ and $\phi \in \Phi$. Then, F has a fixed point.

Subsequently, we let $(\mathbb{E}, ||.||, \preceq)$ be a partially ordered Banach space with the norm $||.||$, having a positive cone K defined by $K = \{x \in \mathbb{E} : x \succeq 0\}$, which one can use to define an order relation on the Banach space. A cone K is called normal cone if there exist a number $M > 0$, such that for every $x, y \in \mathbb{E}$

$$0 \preceq x \preceq y \implies \|x\| \preceq M \|y\|.$$

In this paper, we will extend the results in Theorem 2.10 and Theorem 2.11 into partially ordered Banach spaces. By doing this, we also rectify and generalize the work mentioned in [23, 24]. We use this approach of measure of noncompactness to prove some fixed point theorems in partially ordered Banach spaces whose positive cone K is normal. To accomplish these result, we relaxed the conditions of boundedness, closedness and convexity of the set at the expense that the operator is monotone and bounded. Furthermore, we apply the obtained fixed point theorems to prove some new coupled fixed point and tripled fixed point results to see the effectiveness of our results.

3. AN FIXED POINT THEOREMS FOR MEIR-KEELER CONDENSING OPERATOR

Theorem 3.1. *Assume that $(\mathbb{E}, \|.\|, \preceq)$ is a partially ordered Banach space with a positive normal K cone and μ referred to be any measure of noncompactness on $\mathbb{E}$. If $F : \mathbb{E} \to \mathbb{E}$ is a continuous, non-decreasing, bounded with Meir-Keeler condensing operator and there is an element $x_0 \in \mathbb{E}$ such as $x_0 \preceq Fx_0$. Then F has a fixed point x^* and the monotone successive iterations sequence $\{F^n x_0\}$ converges to x^*. Moreover, the fixed point set of F is compact set.*

Proof. Choose $x_0 \in \mathbb{E}$ and establish a sequence $\{x_n\}$ in $\mathbb{E}$ by $x_{n+1} = F(x_n)$, for all $n \in N^* := N \cup \{0\}$. Due to our hypothesis on F is nondecreasing and $x_0 \preceq F(x_0)$ such that

$$x_0 \preceq x_1 \preceq x_2 \preceq \cdots \preceq x_n \preceq \cdots \tag{3.1}$$

Moreover, we construct a sequence of subsets of $\mathbb{E}$ from the use (3.1) along the following lines i.e. $S_0 = \overline{co}\{x_0, x_1, \ldots\}$, $S_1 = \overline{co}\{x_1, x_2, \ldots\}$ and inductively we have $S_n = \overline{co}\{x_n, x_{n+1}, \ldots\}$. From the construction of S_n it is simply to see that each S_n is closed, bounded and convex with the following inclusion $S_0 \supset S_1 \supset S_2 \supset \cdots \supset S_n \supset \cdots$. Next, we define a sequence of numbers as $\epsilon_n = \mu(S_n)$ and $\delta_n = \delta(\epsilon_n) > 0$. By definition of Meir-Keeler condensing operator we have $\epsilon_n \leq \mu(S_n) < \epsilon_n + \delta_n \implies \mu(S_{n+1}) \leq \mu(F(S_n)) < \epsilon_n = \mu(S_n)$. This implies that $\epsilon_n = \mu(S_n)$ is a positive decreasing sequence of a real numbers and there is a $r \geq 0$ such that $\epsilon_n \longrightarrow r$ as $n \longrightarrow \infty$. Moreover, we show that $r = 0$. If $r \neq 0$ then there is an n_0 such that $r \leq \epsilon_n = \mu(S_n) < r + \delta(r)$ whenever $n > n_0$. By definition of Meir-Keeler condensing operator we have $\mu(F(S_n)) < r$ which is a contradiction, so $r = 0$. So $\mu(S_n) \to 0$ as $n \to \infty$ and $S_{n+1} \subset S_n$, we have

$$S_\infty = \bigcap_{n=1}^{\infty} \neq \phi \quad \text{and} \quad \mu(S_\infty) = 0.$$

By using the following fact

$$\begin{aligned}
\mu(S_0) &= \mu\{\overline{co}\{x_0, x_1, x_2, \cdots\}\} \\
&= \mu\Big\{\overline{co}\big\{\overline{co}\{x_0, x_1, \cdots, x_{n-1}\} \cup \overline{co}(S_n)\big\}\Big\} \\
&= \max\{\overline{co}\{x_0, x_1, \cdots, x_{n-1}\}, \overline{co}(S_n)\} \\
&= \mu(S_n).
\end{aligned} \tag{3.2}$$

We get $\mu(S_0) = \mu(S_n) = 0$, for every $n \in N$. This implies that $\mu(S_0) = 0$, and S_0 forms a compact chain in $\mathbb{E}$. Hence $\{x_n\}$ has a convergent subsequence. Using the monotone

property of F and normality of cone the sequence $\{x_n\} = \{F^n x_0\}$ monotonically converges to a point say $x^* \in S_0$. At the end using the continuity of F we have

$$Fx^* = F(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} Fx_n = \lim_{n\to\infty} x_{n+1} = x^*.$$

Next, let $\Omega = \{x \in \mathbb{E} : Fx = x\}$ and $\mu(\Omega) = b$ then there exist $\delta > 0$ such that $b \leq \mu(\Omega) < b + \delta \implies \mu(F(\Omega)) < b$, but this is impossible as $F(\Omega) = \Omega$, so $\mu(\Omega) = 0$. Now taking into account any convergent sequence $\{x_n\} \subset \Omega$ and $x_n \to x^*$, we have $x^* \in S_0$ because S_0 is closed. Thus, by continuity of F on $x_n = Fx_n \to Fx^*$ and $Fx^* = x^*$ which means that $x^* \in \Omega$, i.e. Ω is a compact set. ∎

Lim [40] introduced the following notion of L-function.

Definition 3.2. [40] Let a function $\theta : [0, \infty) \to [0, \infty)$ is called an L-function if $\theta(0) = 0$, $\theta(s) > 0$ for $s \in (0, \infty)$ and for each $s \in (0, \infty)$ over here is an $\delta > 0$ so that $\theta(t) \leq s$ for every $t \in [s, s + \delta]$.

Example 3.3. Let $\theta(t) = \varrho t$, where $\varrho \in [0, 1)$.

Definition 3.4. A function $\vartheta : [0, \infty) \to [0, \infty)$ is called a strictly L-function if $\vartheta(0) = 0$, $\vartheta(s) > 0$ for $s \in (0, \infty)$ and for each $s \in (0, \infty)$ over here is an $\delta > 0$ so that $\vartheta(t) < s$ for any $t \in [s, s + \delta]$.

Analogous to the proof of Proposition 1 as in [36], we prove the following theorem in the framework of partially ordered Banach spaces.

Theorem 3.5. *Let $(\mathbb{E}, \|.\|, \preceq)$ be a partially ordered Banach space with a positive normal cone K. Suppose that $F : \mathbb{E} \longrightarrow \mathbb{E}$ is a nondecreasing, continuous, and bounded mapping. Then, F is Meir-Keeler condensing operator if and only if there is an L-function θ so that*

$$\mu(F(A)) < \theta(\mu(A)) \tag{3.3}$$

for any bounded subsets A of $\mathbb{E}$.

Proof. For sufficiency: Let θ be an L-function and satisfy expression (3.3) for any bounded subsets $A \subset \mathbb{E}$. For any $\epsilon > 0$ there is an $\delta(\epsilon) > 0$ such that $\theta(t) \leq \epsilon$ for $\epsilon \leq t < \epsilon + \delta(\epsilon)$. Let A be a bounded subset of $\mathbb{E}$ such that $\epsilon \leq \mu(A) < \epsilon + \delta(\epsilon)$ using (3.3), we have $\mu(F(A)) < \theta(\mu(A)) \leq \epsilon$. Thus, F is a Meir-Keeler condensing operator.

For necessity: Suppose that F is a Meir-Keeler condensing operator. Next, we have set out a function $\alpha : (0, \infty) \longrightarrow (0, \infty)$ such that

$$\epsilon \leq \mu(A) < \epsilon + 2\alpha(\epsilon) \Rightarrow \mu(F(A)) < \epsilon \tag{3.4}$$

for $\epsilon \in (0, \infty)$. By using above function α we define a nondecreasing function $\beta : (0, \infty) \longrightarrow [0, \infty)$ as $\beta(t) = \inf \{\epsilon > 0 : t \leq \epsilon + \alpha(\epsilon)\}$, $t \in (0, \infty)$. For $t \leq t + \alpha(t)$, we have $\beta(t) \leq t$. Now define $\theta_1 : [0, \infty) \Longrightarrow [0, \infty)$ by

$$\theta_1(t) = \begin{cases} 0 & \text{if } t = 0, \\ \beta(t) & \text{if } t > 0, \text{ and } \min\{\epsilon > 0 : t \leq \epsilon + \alpha(\epsilon)\} \text{ exists,} \\ \frac{\beta(t) + t}{2} & \text{otherwise.} \end{cases}$$

From above it is clear that $\theta_1(0) = 0$ and $0 < \theta_1(s) \leq s$ for $s \in (0, \infty)$. Fix $s \in (0, \infty)$. In the case of $\theta_1(t) \leq s$ for all $t \in (s, s + \alpha(s)]$, we can put $\delta = \alpha(s)$. In the other case, there

exist $\rho \in (s, s + \alpha(s)]$ with $\theta_1(\rho) > s$. As $\rho \le s + \alpha(s)$, we have $\beta(\rho) \le s$. If $\beta(\rho) = s$, then $\theta_1(\rho) = \beta(\rho) = s < \theta_1(\rho)$, which is a contradiction. Thus, we have

$$\beta(\rho) < s < \theta_1(\rho) = \frac{\beta(\rho) + \rho}{2}.$$

We can choose $u \in (\beta(\rho), s)$ with $\rho \le u + \alpha(u)$, and we put $\delta = s - u > 0$. Fix $t \in [s, s + \delta]$ as

$$t \le s + \delta = 2s - u < 2\frac{\beta(\rho) + \rho}{2} - \beta(\rho) = \delta \le u + \alpha(u),$$

we have $\beta(t) \le u$. Hence

$$\theta_1(t) \le \frac{\beta(t) + t}{2} \le \frac{u + s + \delta}{2} = s.$$

Therefore, θ_1 is an L-function. Let A be any bonded subset of $\mathbb{E}$ with non-zero measure of noncompactness. By definition of L-function θ_1, for every $t \in (0, \infty)$ there exists $\epsilon \in (0, \theta_1(t))$ such that $t \le \epsilon + \alpha(s)$. So, there exists $\epsilon \in (0, \mu(A))$ such that $\mu(A) \le \epsilon + \alpha(\epsilon)$. Therefore $\mu(F(A)) < \epsilon \le \theta_1(\mu(A))$, which complete the proof. ∎

As the obvious consequence of Theorems 3.1 and 3.5, we shall obtain the following fixed point results.

Corollary 3.6. *Let $(\mathbb{E}, \|.\|, \preceq)$ be a partially ordered Banach space with a positive normal cone K and μ is an arbitrary measure of noncompactness on $\mathbb{E}$. Suppose that $F : \mathbb{E} \longrightarrow \mathbb{E}$ is a nondecreasing, continuous and bounded mapping in a such a way that*

$$\mu(F(A)) < \theta(\mu(A))$$

for each bounded subset $A \subset \mathbb{E}$ where θ is an L-function. If there is an element x_0 such that $x_0 \preceq F(x_0)$, then F has a fixed point x^* and monotone sequence $\{F^n(x_0)\}$ of consecutive iterations converges to x^*. In addition, the set of all fixed points of F in $\mathbb{E}$ is compact set.

Corollary 3.7. *Let $(\mathbb{E}, \|.\|, \preceq)$ be a partially ordered Banach space with a positive normal cone K and μ is an arbitrary measure of noncompactness on $\mathbb{E}$. Suppose that $F : \mathbb{E} \longrightarrow \mathbb{E}$ is a nondecreasing, continuous and bounded mapping in a such a way that*

$$\mu(F(A)) \le \vartheta(\mu(A))$$

for any bounded subset $A \subset \mathbb{E}$ where ϑ is a strictly L-function. If there is an element x_0 such that $x_0 \preceq F(x_0)$, then F has a fixed point x^* and monotone sequence $\{F^n(x_0)\}$ of consecutive iterations converges to x^*. In addition, the set of all fixed points F in $\mathbb{E}$ is compact set.

Remark 3.8. We observe that by substituting $\vartheta(t) = \varrho t$ in Corollary 3.7 where $\varrho \in [0, 1)$, provides the Darbo's fixed point theorem.

Corollary 3.9. *Let $(\mathbb{E}, \|.\|, \preceq)$ be a partially ordered Banach space with a positive normal cone K and $T : \mathbb{E} \to \mathbb{E}$ is a nondecreasing, continuous, bounded and also satisfying the following inequality*

$$\|T(x) - T(y)\| \le \vartheta(\|x - y\|),$$

where ϑ is a right continuous nondecreasing and strictly L-function. Assume that $G : \mathbb{E} \to \mathbb{E}$ is a continuous an compact operator. Next, define $F(x) := T(x) + G(x), \forall x \in \mathbb{E}$ and further assume that there is an element $x_0 \in \mathbb{E}$ such that $x_0 \preceq T(x_0) + G(x_0)$. Then, F has a fixed point in $\mathbb{E}$ and additionally the fixed points set of F in $\mathbb{E}$ is compact set.

Proof. Assume $\mu : \mathfrak{M}_{\mathbb{E}} \to \mathbb{R}_+$ be an Kuratowski measure of noncompactness as defined in the paper [29]. In addition, from our assumptions and ϑ to be a nondecreasing, we can write

$$\|T(x) - T(y)\| \leq \sup_{x,y \in A} \vartheta(\|x - y\|) \leq \vartheta(\sup_{x,y \in A} \|x - y\|),$$

implies that

$$diam(T(A)) \leq \vartheta(diam A). \tag{3.5}$$

Due to Kuratowski measure of noncompactness for each $\beta > 0$, there exist $A_1, A_2, \cdots, A_n$ such that $A \subset \bigcup_{i=1}^n A_i$ and $diam A_i \leq \mu(A) + \beta$. As $T(A) \subset \bigcup_{i=1}^n T(A_i)$ and ϑ is a nondecreasing srictly L-function with expression (3.5), we obtain

$$\mu(T(x)) \leq diam(T(A_i)) \leq \vartheta(diam(A_i)) \leq \vartheta(\mu(A) + \beta)$$

above inequality holds for every $\beta > 0$, implies that

$$\mu(T(A)) \leq \vartheta(\mu(A)).$$

Finally, G is compact and with above inequality we get

$$\mu(F(A)) = \mu((T + G)(A)) \leq \mu(T(A) + G(A)) \leq \mu(T(A)) + \mu(G(A)) \leq \vartheta(\mu(A))$$

with hypothesis $x_0 \preceq F(x_0)$, implies the required result as in Corollary 3.7. ∎

3.1. Integral-type Darbo's theorem generalization in partially ordered Banach spaces

Theorem 3.10. *Assume* $(\mathbb{E}, \|.\|, \preceq)$ *be a partially ordered Banach space whose positive normal cone is* K *and* μ *take as an any measure of noncompactness on* $\mathbb{E}$. *Consider* $F : \mathbb{E} \to \mathbb{E}$ *to be a nondecreasing, continuous and bounded with the following*

$$\int_0^{\mu(F(A))} \phi(t)dt \leq \psi\left(\int_0^{\mu(A)} \phi(t)dt\right) \tag{3.6}$$

for every bounded $A \subset \mathbb{E}$ *where* $\phi \in \Phi$ *and* $\psi \in \Psi$. *Moreover, if there is an element* $x_0 \in \mathbb{E}$ *such that* $x_0 \preceq F(x_0)$, *then* F *has a fixed point* x^* *and the sequence* $\{F^n(x_0)\}$ *of successive iterations converges monotonically to* x^*.

Proof. Let choose $x_0 \in \mathbb{E}$ and define a sequence $\{x_n\}$ in $\mathbb{E}$ by $x_{n+1} = F(x_n) \; \forall n \in N^* = N \cup \{0\}$. From given F is nondecreasing and $x_0 \preceq F(x_0)$, we can write as

$$x_0 \preceq x_1 \preceq x_2 \preceq \cdots \preceq x_n \preceq \cdots \tag{3.7}$$

By using (3.7) we construct a sequence of subsets of $\mathbb{E}$ as $A_0 = \overline{co}\{x_0, x_1, ...\}$, $A_1 = \overline{co}\{x_1, x_2, ...\}$, and inductively we can achieve $A_n = \overline{co}\{x_n, x_{n+1}...\}$. From the above discussion we see that A_n is closed, bounded and convex set for all n. We can easily get the following inclusion $A_0 \supset A_1 \supset A_2 \supset \cdots \supset A_n \supset \cdots$, and also we can obtain

$$\mu(A_{n+1}) = \mu(\overline{co}\{x_{n+1}, x_n, \cdots\}) = \mu(F\{x_n, x_{n-1}, \cdots\})$$
$$\leq \mu(F\{\overline{co}\{x_n, x_{n-1}, \cdots\}\}) = \mu(A_n).$$

From expression (3.6) and from above inequality, we have

$$\int_0^{\mu(A_{n+1})} \phi(t)dt \leq \int_0^{\mu(F(A_n))} \phi(t)dt \leq \psi\left(\int_0^{\mu(A_n)} \phi(t)dt\right)$$

$$\leq \psi^2\left(\int_0^{\mu(A_{n-1})} \phi(t)dt\right)$$

$$\vdots$$

$$\leq \psi^n\left(\int_0^{\mu(A_1)} \phi(t)dt\right).$$

So by using that fact that for every every $\delta > 0$, $\int_0^\delta \phi(t)dt > 0$, we conclude that $\mu(A_n) \to 0$, as $n \to \infty$ and $A_{n+1} \subset A_n$ implies that $A_\infty = \bigcap_{n=1}^\infty \neq \phi$ and $\mu(A_\infty) = 0$. Further, form the following fact

$$\mu(A_0) = \mu\left\{\overline{co}\{\overline{co}\{x_0, x_1, \cdots, x_{n-1}\} \cup \overline{co}(A_n)\}\right\} = \mu(A_n),$$

we get $\mu(A_0) = \mu(A_n) = 0$ for every $n \in \mathbb{N}$. This implies that $\mu(A_0) = 0$ and A_0 forms a compact chain in $\mathbb{E}$. Hence $\{x_n\}$ has a convergent subsequence. Using the monotone property of F and normality of cone the sequence $\{x_n\} = \{F^n x_0\}$ converges monotonically to a point say $x^* \in A_0$. At the end, using the continuity of F we get $F(x^*) = F(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} F(x_n) = \lim_{n\to\infty} x_{n+1} = x^*$, which is required.   ∎

Corollary 3.11. *Let $(E, ||.||, \preceq)$ be a partially ordered Banach space with a positive normal cone K and $T : E \to E$ is a continuous, nondecreasing, bounded and satisfying the following inequality*

$$\int_0^{\|Tx-Ty\|} \phi(t)dt \leq \psi\left(\int_0^{\|x-y\|} \phi(t)dt\right), \tag{3.8}$$

for every bounded $S \subset E$ where $\phi \in \Phi$, $\psi \in \Psi$. Moreover, if there exists an element $x_0 \in E$ such that $x_0 \preceq Tx_0$, then T has a fixed point x^* and the sequence $\{T^n x_0\}$ of successive iterations converges monotonically to x^*.

Proof. Define $\mu : \mathfrak{M}_\mathbb{E} \to \mathbb{R}_+$ and by taking $\mu(X) = diam X$, $X \in \mathfrak{M}_\mathbb{E}$, we obtain the required result.   ∎

Remark 3.12. By letting $\phi(t) = 1$ and $\psi(t) = k$, $k \in [0, 1)$, in Theorem 3.10, then we have $\int_0^{\mu(TS)} \phi(t)dt = \mu(TS) \leq k\mu(S) = \int_0^{\mu(S)} \phi(t)dt$. So in that case the Darbo's theorem is obtained.

4. SOME APPLICATIONS

In this section in order to understand the effectiveness of the results presented in the above section, we include two cases where our results can be applied, particularly the existence of a coupled and tripled fixed point in partially ordered Banach spaces.

4.1. Coupled Fixed Point Theorem

Next, we explain some coupled fixed point theorems using Meir-Keeler condensing operator and other Darbo′s type generalization in partially ordered Banach spaces. Before that, let′s recall some necessary definitions and notions.

Definition 4.1. [18] An element (x, y) in $\mathbb{E}^2$ is said to be a coupled fixed point of a mapping $T : \mathbb{E}^2 \to \mathbb{E}$ provided $T(x, y) = x$ and $T(y, x) = y$.

Example 4.2. Let $X = [0, \infty)$ and a mapping $T : X \times X \to X$ can be defined by

$$T(x, y) = x^2 + y^2,$$

for all $x, y \in X$. We can easily see that T has a unique coupled fixed point $(0, 0)$.

Definition 4.3. Assume that $(\mathbb{E}, \|.\|, \preceq)$ is a partially ordered Banach space and a mapping $T : \mathbb{E}^2 \to \mathbb{E}$ is said to be the monotone property on both variables x and y such that for every $x, y \in \mathbb{E}$ provided that the following conditions are satisfied

$$z_1, z_2 \in \mathbb{E}, \ z_1 \preceq z_2 \implies T(z_1, y) \preceq T(z_2, y),$$

$$z_1, z_2 \in \mathbb{E}, \ z_1 \preceq z_2 \implies T(x, z_1) \preceq T(x, z_2).$$

Lemma 4.4. [7] *Let $\mu_1, \mu_2, \cdots, \mu_n$ are the measures of noncomapctness on Banach spaces $\mathbb{E}_1, \mathbb{E}_2, \cdots, \mathbb{E}_n$ respectively. Let a function $F : [0, \infty)^n \longrightarrow [0, \infty)$ is convex and $F(x_1, x_2, \cdots, x_n) = 0$ if and only if each $x_i = 0$ for all $i = 1, 2, \cdots, n$. Next, we define a measure of noncompactness on $\mathbb{E}_1 \times \mathbb{E}_2 \times, \cdots, \times \mathbb{E}_n$ as follows:*

$$\mu(A) = F\big(\mu_1(A_1), \ \mu_2(A_2), \ \cdots, \mu_n(A_n)\big),$$

where A_i denotes the natural projection of A onto $\mathbb{E}_i$ for $i = 1, 2, \cdots, n$.

Now, as a result of Lemma 4.4, we are going to present the following examples.

Example 4.5. Let μ be a measure of noncompactness on a Banach space E, and the function $\Gamma : [0, +\infty)^2 \to [0, +\infty)$ is convex and $\Gamma(x_1, x_2) = 0$ if and only if $x_i = 0$ for $i = 1, 2$. Then $\mu^*(S) = \Gamma(\mu(S_1), \mu(S_2))$ defines a measure of noncompactness on $E \times E$.

Example 4.6. Let μ be a measure of noncompactness on a Banach space E, and consider a map $\Gamma(x, y) = x + y$ for all $(x, y) \in [0, +\infty)^2$. We can see that Γ is convex and $\Gamma(x, y) = 0$ if and only if $x = y = 0$ and all the conditions of Lemma 4.4 are satisfied. Thus, $\mu^*(S) = \mu(S_1) + \mu(S_2)$ define a measure of noncompactness on the space $\mathbb{E} \times \mathbb{E}$.

Example 4.7. Assume μ be a measure of noncompactness on a Banach space E and a map $\Gamma(x, y) = x + y$ for any $(x, y) \in [0, +\infty)^2$. Then we see that Γ is convex and $\Gamma(x, y) = 0$ if and only if $x = y = 0$, hence all the conditions of Lemma 4.4, are satisfied. Thus, $\mu^*(S) = \mu(S_1) + \mu(S_2)$, defines a measure of noncompactness in the space $E \times E$.

Example 4.8. Let μ be a measure of noncompactness on a Banach space E. If we define $J(x, y) = \max\{x, y\}$ for any $(x, y) \in [0, +\infty)^2$, then all the conditions of Lemma 4.4, are satisfied, and $\mu^*(S) = \max\{\mu(S_1), \mu(S_2)\}$ is a measure of noncompactness in the space $E \times E$.

Theorem 4.9. *Let $(\mathbb{E}, \|.\|, \preceq)$ be a partially ordered Banach space with a positive normal cone K and μ be an arbitrary measure of noncompactness on $\mathbb{E}$. Suppose that $T : \mathbb{E}^2 \longrightarrow \mathbb{E}$*

is a continuous and bounded mapping having monotone property and for any $\epsilon > 0$ there exist $\delta > 0$ so that for every bounded subsets S_1, S_2 in $\mathbb{E}$ the following holds:

$$\epsilon \leq \max\{\mu(S_1), \mu(S_2)\} < \epsilon + \delta \implies \mu(T(S_1 \times S_2)) < \epsilon. \tag{4.1}$$

If there exists two elements $x_0, y_0 \in \mathbb{E}$ such that $x_0 \preceq T(x_0, y)$ and $y_0 \preceq T(y_0, x)$ for all $x, y \in \mathbb{E}$. Then T has atleast one coupled fixed point.

Proof. Define a mapping $G : \mathbb{E}^2 \to \mathbb{E}^2$ by $G(x, y) = (T(x, y), T(y, x))$, since T is continuous, bounded with monotone property, it follows that G is also a continuous, bounded and monotone mapping. Let the measure of noncompactness on $\mathbb{E}^2$ be as $\mu^*(S) = \max\{\mu(S_1, \mu(S_2)\}$, where S_1 and S_2 are the natural projections of S on $\mathbb{E}$. Let S be a nonempty bounded subset of $\mathbb{E}^2$ and for $\epsilon > 0$ there exist $\delta(\epsilon) > 0$, such that $\epsilon \leq \mu^*(S) = \max\{\mu(S_1), \mu(S_2)\} < \epsilon + \delta(\epsilon)$, we have

$$\begin{aligned}
\mu^*\big(G(S)\big) \leq \mu^*\big(G(S_1 \times S_2)\big) &= \mu^*\big(T(S_1 \times S_2) \times T(S_2 \times S_1)\big) \\
&= \max\big\{\mu\big(T(S_1 \times S_2)\big), \mu(T(S_2 \times S_1))\big\} < \epsilon.
\end{aligned} \tag{4.2}$$

Further, an element $x^* = (x_0, y_0) \in \mathbb{E}^2$ such that

$$x^* = (x_0, y_0) \preceq \big(T(x_0, y_0), T(y_0, x_0)\big) = G(x_0, y_0) = Gx^*.$$

At last G satisfy all conditions of Theorem 3.1, therefore G has a fixed point which is actually the coupled fixed point of the mapping T. ∎

Corollary 4.10. *Let $(\mathbb{E}, \|.\|, \preceq)$ be a partially ordered Banach space with a positive normal cone K and μ be an arbitrary measure of noncompactness on $\mathbb{E}$. Suppose that $T : \mathbb{E}^2 \longrightarrow \mathbb{E}$ is a continuous and bounded mapping having monotone property. Moreover, for any L-function θ and every bounded subsets S_1, S_2 in $\mathbb{E}$ the following holds*

$$\mu\big(T(S_1 \times S_2)\big) < \frac{1}{2}\theta\big(\mu(S_1) + \mu(S_2)\big),$$

or

$$\mu\big(T(S_1 \times S_2)\big) < \theta\big(\max\{\mu(S_1), \mu(S_2)\}\big).$$

If there exists two elements $x_0, y_0 \in \mathbb{E}$ such that $x_0 \preceq T(x_0, y)$ and $y_0 \preceq T(y_0, x)$ for all $x, y \in \mathbb{E}$. Then T has atleast one coupled fixed point.

Corollary 4.11. *Let $(\mathbb{E}, \|.\|, \preceq)$ be a partially ordered Banach space with a positive normal cone K and μ be an arbitrary measure of noncompactness on $\mathbb{E}$. Suppose that $T : \mathbb{E}^2 \longrightarrow \mathbb{E}$ is a continuous and bounded mapping having monotone property. Moreover, for any strictly L-function ϑ and every bounded subsets S_1, S_2 in $\mathbb{E}$ the following holds*

$$\mu\big(T(S_1 \times S_2)\big) \leq \frac{1}{2}\vartheta\big(\mu(S_1) + \mu(S_2)\big),$$

or

$$\mu\big(T(S_1 \times S_2)\big) \leq \vartheta\big(\max\{\mu(S_1), \mu(S_2)\}\big).$$

If there exists two elements $x_0, y_0 \in \mathbb{E}$ such that $x_0 \preceq T(x_0, y)$ and $y_0 \preceq T(y_0, x)$ for all $x, y \in \mathbb{E}$. Then T has atleast one coupled fixed point.

Corollary 4.12. *Let* $(\mathbb{E}, \|.\|, \preceq)$ *be a partially ordered Banach space with a positive normal cone* K *and* $F : \mathbb{E} \times \mathbb{E} \to \mathbb{E}$ *is a continuous, nondecreasing, bounded and also satisfy the following inequality*

$$\|F(x,y) - F(u,v)\|\frac{1}{2} \leq \vartheta(\|x-u\|, \|y-v\|),$$

where ϑ *is a nondecreasing and right continuous strictly L-function. Assume that* $G : \mathbb{E} \times \mathbb{E} \to \mathbb{E}$ *is a compact and continuous operator. Define* $T(x,y) := F(x,y) + G(x,y)$, $\forall\ (x,y) \in \mathbb{E} \times \mathbb{E}$ *and also assume that there exists an elements* $x_0, y_0 \in \mathbb{E}$ *such that* $x_0 \leq F(x_0, y) + G(x_0, y)$ *and* $y_0 \leq F(y_0, x) + G(y_0, x)$, $\forall\ x, y \in \mathbb{E}$. *Then* T *has a coupled fixed point.*

Proof. Let $\mu : \mathfrak{M}_{\mathbb{E}} \to \mathbb{R}_+$ be the Kuratowski measure of noncompactness defined in [29]. Moreover, assume that S_1 and S_2 be nonempty subsets of $\mathbb{E}$ and from above hypothesis we have

$$\|F(x,y) - F(u,v)\| \leq \frac{1}{2}\vartheta(\|x-u\|, \|y-v\|) \leq \frac{1}{2}\vartheta(diam\|x-u\|, diam\|y-v\|),$$

implies that

$$diam(F(S_1 \times S_2)) \leq \frac{1}{2}\vartheta(diam(S_1) + diam(S_2)).$$

As ϑ is right continuous and by definition of Kuratowski measure of noncompactness, similar to the proof of Corollary 3.9, we have

$$\mu(F(S_1 \times S_2)) \leq \frac{1}{2}\vartheta(\mu(S_1) + \mu(S_2)). \tag{4.3}$$

Also as G is compact and from (4.3) we obtain

$$\begin{aligned}
\mu(T(S_1 \times S_2)) = \mu((F+G)(S_1 \times S_2)) &\leq \mu(F(S_1 \times S_2) + G(S_1 \times S_2)) \\
&\leq \mu(F(S_1 \times S_2)) + \mu(G(S_1 \times S_2)) \\
&\leq \frac{1}{2}\vartheta(\mu(S_1) + \mu(S_2)).
\end{aligned} \tag{4.4}$$

Further, we also have $x_0 \preceq T(x_0, y)$ and $y_0 \preceq T(y_0, x)$ for all $x, y \in \mathbb{E}$. Finally, from Corollary 4.11, completes the proof. ∎

Theorem 4.13. *Let* $(\mathbb{E}, \|.\|, \preceq)$ *be a partially ordered Banach space with a positive normal cone* K *and* μ *be an arbitrary measure of noncompactness on* $\mathbb{E}$. *Suppose that* $T : \mathbb{E}^2 \longrightarrow \mathbb{E}$ *is a continuous, bounded and monotone mapping, which for all bounded subsets* $S_1, S_2 \subset \mathbb{E}$ *satisfy the following inequality*

$$\int_0^{\mu\left(T(S_1 \times S_2)\right)} \phi(t)dt \leq \psi\left(\int_0^{\max\{\mu(S_1), \mu(S_2)\}} \phi(t)dt\right), \tag{4.5}$$

where $\phi \in \Phi$ *and* $\psi \in \Psi$. *If there exists two elements* $x_0, y_0 \in \mathbb{E}$ *such that* $x_0 \preceq T(x_0, y)$ *and* $y_0 \preceq T(y_0, x)$ *for all* $x, y \in \mathbb{E}$. *Then* T *has atleast one coupled fixed point.*

Proof. Define a mapping $G : \mathbb{E}^2 \to \mathbb{E}^2$ by $G(x,y) = (T(x,y), T(y,x))$. Since T is continuous and bounded with monotone property, it follows that G is also continuous, bounded and monotone mapping. Let the measure of noncompactness on $\mathbb{E}^2$ be

$\mu^*(S) = \max\{\mu(S_1), \mu(S_2)\}$, where S_1 and S_2 are the natural projections of S on $\mathbb{E}$. By taking S be a nonempty bounded subset of $\mathbb{E}^2$

$$\int_0^{\mu^*(G(S))} \phi(t)dt \leq \int_0^{\mu^*(G(S_1 \times S_2)} \phi(t)dt = \int_0^{\mu^*(T(S_1 \times S_2) \times T(S_2 \times S_1))} \phi(t)dt$$
$$= \int_0^{\max\{\mu(T(S_1 \times S_2)),\, \mu(T(S_2 \times S_1))\}} \phi(t)dt.$$
(4.6)

Now by hypothesis $\int_0^{\mu(T(S_1 \times S_2))} \phi(t)dt \leq \psi\big(\int_0^{\max\{\mu(S_1),\mu(S_2)\}} \phi(t)dt\big)$ and also we have $\int_0^{\mu(T(S_2 \times S_1))} \phi(t)dt \leq \psi\big(\int_0^{\max\{\mu(S_2),\mu(S_1)\}} \phi(t)dt\big)$, therefore we conclude

$$\int_0^{\mu^*(G(S))} \phi(t)dt \leq \psi\big(\int_0^{\mu^*(S)} \phi(t)dt\big).$$

As same from the previous proof we have

$$x^* = (x_0, y_0) \preceq \big(T(x_0, y_0), T(y_0, x_0)\big) = G(x_0, y_0) = Gx^*.$$

Now all conditions of Theorem 3.10, are satisfied and G has a fixed point.   ∎

4.2. Tripled Fixed Point Theorem

In this section, we prove some tripled fixed point theorem using Meir-Keeler condensing operator. Before that, let's recall some basic definitions and notions.

Definition 4.14. [17] An element (x, y, z) in $\mathbb{E}^3$ is called a tripled fixed point of a mapping $T : \mathbb{E}^3 \to \mathbb{E}$ if $T(x, y, z) = x$, $T(y, x, y) = y$ and $T(z, y, x) = z$.

Definition 4.15. Let $(\mathbb{E}, \|.\|, \preceq)$ be a partially ordered Banach space and $T : \mathbb{E}^3 \to \mathbb{E}$ be a mapping. Then T is said to have the monotone property if T is monotone nondecreasing in all three variables $x, y, z \in E$ the following holds:

$$u_1, u_2 \in \mathbb{E}, \ u_1 \preceq u_2 \implies T(u_1, y, z) \preceq T(u_2, y, z),$$
$$v_1, v_2 \in \mathbb{E}, \ v_1 \preceq v_2 \implies T(x, v_1, z) \preceq T(x, v_2, z),$$
$$w_1, w_2 \in \mathbb{E}, \ w_1 \preceq w_2 \implies T(x, y, w_1) \preceq T(x, y, w_2).$$

Theorem 4.16. *Let $(\mathbb{E}, \|.\|, \preceq)$ be a partially ordered Banach space with a positive normal cone K and μ be an arbitrary measure of noncompactness on $\mathbb{E}$. Suppose that $T : \mathbb{E}^3 \longrightarrow \mathbb{E}$ is a continuous and bounded mapping having monotone property. For any $\epsilon > 0$ there exist $\delta > 0$ such that for all bounded subsets S_1, S_2, S_3 in $\mathbb{E}$ the following holds*

$$\epsilon \leq \max\{\mu(S_1), \mu(S_2), \mu(S_3)\} < \epsilon + \delta \implies \mu\big(T(S_1 \times S_2 \times S_3)\big) < \epsilon. \quad (4.7)$$

If there are three elements $x_0, y_0, z_0 \in \mathbb{E}$ such that $x_0 \preceq T(x_0, y, z)$, $y_0 \preceq T(y_0, x, z)$ and $z_0 \preceq T(z_0, y, x)$ for all $x, y, z \in \mathbb{E}$. Then T has atleast one tripled fixed point.

Proof. For proving this theorem we just follow the same procedure as in Theorem 4.9. By define a mapping $G : \mathbb{E}^3 \to \mathbb{E}^3$ by $G(x, y, z) = \big(T(x, y, z), T(y, x, y), T(z, y, z)\big)$, and the measure of noncompactness on $\mathbb{E}^3$ as follows:

$$\mu^*(S_1 \times S_2, \times S_3) = \max\{\mu(S_1), \mu(S_2), \mu(S_3)\}.$$

    ∎

Corollary 4.17. *Let* $(\mathbb{E}, \|.\|, \preceq)$ *be a partially ordered Banach space with a positive normal cone* K *and* μ *be an arbitrary measure of noncompactness on* $\mathbb{E}$*. Suppose that* $T : \mathbb{E}^3 \longrightarrow \mathbb{E}$ *is a continuous and bounded mapping having monotone property. Moreover, for any L-function* θ *and every bounded subsets* S_1, S_2, S_3 *in* $\mathbb{E}$ *the following holds*

$$\mu\big(T(S_1 \times S_2 \times S_3)\big) < \frac{1}{3}\theta\big(\mu(S_1) + \mu(S_2) + \mu(S_3)\big),$$

or

$$\mu\big(T(S_1 \times S_2 \times S_3)\big) < \theta\big(\max\big\{\mu(S_1), \mu(S_2), \mu(S_3)\big\}\big).$$

If there exists three elements $x_0, y_0, z_0 \in \mathbb{E}$ *such that* $x_0 \preceq T(x_0, y, z)$*,* $y_0 \preceq T(y_0, x, z)$ *and* $z_0 \preceq T(z_0, y, x)$ *for all* $x, y, z \in \mathbb{E}$*. Then* T *has atleast one tripled fixed point.*

Corollary 4.18. *Let* $(\mathbb{E}, \|.\|, \preceq)$ *be a partially ordered Banach space with a positive normal cone* K *and* μ *be an arbitrary measure of noncompactness on* $\mathbb{E}$*. Suppose that* $T : \mathbb{E}^3 \longrightarrow \mathbb{E}$ *is a continuous and bounded mapping having monotone property. Moreover, for any strictly L-function* ϑ *and every bounded subsets* S_1, S_2, S_3 *in* $\mathbb{E}$ *the following holds*

$$\mu\big(T(S_1 \times S_2 \times S_3)\big) \leq \frac{1}{3}\vartheta\big(\mu(S_1) + \mu(S_2) + \mu(S_3)\big),$$

or

$$\mu\big(T(S_1 \times S_2 \times S_3)\big) \leq \vartheta\big(\max\big\{\mu(S_1), \mu(S_2), \mu(S_3)\big\}\big).$$

If there exists three elements $x_0, y_0, z_0 \in \mathbb{E}$ *such that* $x_0 \preceq T(x_0, y, z)$*,* $y_0 \preceq T(y_0, x, z)$ *and* $z_0 \preceq T(z_0, y, x)$ *for all* $x, y, z \in \mathbb{E}$*. Then* T *has atleast one tripled fixed point.*

Corollary 4.19. *Let* $(\mathbb{E}, \|.\|, \preceq)$ *be a partially ordered Banach space with a positive normal cone* K *and* $F : \mathbb{E} \times \mathbb{E} \times \mathbb{E} \to \mathbb{E}$ *is a continuous, nondecreasing, bounded and also satisfy the following inequality*

$$\|F(x, y, z) - F(u, v, w)\|\frac{1}{3} \leq \vartheta(\|x - u\|, \|y - v\|), \|z - w\|,$$

where ϑ *is a nondecreasing and upper semi continuous strictly L-function. Assume that* $G : \mathbb{E} \times \mathbb{E} \times \mathbb{E} \to \mathbb{E}$ *is a compact and continuous operator. Define* $T(x, y, z) := F(x, y, z) + G(x, y, z)$*,* $\forall\ (x, y, z) \in \mathbb{E} \times \mathbb{E} \times \mathbb{E}$ *and also assume that there exists an elements* $x_0, y_0, z_0 \in \mathbb{E}$ *such that* $x_0 \leq F(x_0, y, z) + G(x_0, y, z)$*,* $y_0 \leq F(y_0, x, z) + G(y_0, x, z)$*, and* $z_0 \leq F(z_0, y, z) + G(z_0, y, z)$*,* $\forall\ x, y, z \in \mathbb{E}$*. Then* T *has a tripled fixed point.*

Proof. Follow the same steps as in corollary 4.12. ∎

Theorem 4.20. *Let* $(\mathbb{E}, \|.\|, \preceq)$ *be a partially ordered Banach space with a positive normal cone* K *and* μ *be an arbitrary measure of noncompactness on* $\mathbb{E}$*. Suppose that* $T : \mathbb{E}^3 \longrightarrow \mathbb{E}$ *is a continuous, bounded and monotone mapping, which for all bounded subsets* $S_1, S_2, S_3 \subset \mathbb{E}$ *satisfy the following inequality*

$$\int_0^{\mu\big(T(S_1 \times S_2 \times S_3)\big)} \phi(t)dt \leq \psi\left(\int_0^{\max\big\{\mu(S_1), \mu(S_2), \mu(S_3)\big\}} \phi(t)dt\right), \quad (4.8)$$

where $\phi \in \Phi$ *and* $\psi \in \Psi$*. If there exists three elements* $x_0, y_0, z_0 \in \mathbb{E}$ *such that* $x_0 \preceq T(x_0, y, z)$*,* $y_0 \preceq T(y_0, x, z)$ *and* $z_0 \preceq T(z_0, y, x)$ *for all* $x, y, z \in \mathbb{E}$*. Then* T *has atleast one tripled fixed point.*

Proof. Follow the proof of Theorem 4.13. ∎

5. Conclusion

This paper has suggested some results about the generalization of the Darbo's fixed point theorem which is obtained by using Meir-Keeler condensing operator in partially ordered Banach spaces. The main advantage of these results that these results attain without the conditions of boundedness, closeness, and convexity of the set, but by taking condition of boundedness and monotonocity on the operator. Further, we also discuss a characterization of a Meir-Keeler condensing operator using the notion of L-functions and strictly L-function in partially ordered Banach spaces. At the last section, we apply these results to obtain a few coupled and tripled fixed theorems.

Acknowledgements

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