



FIXED POINT RESULTS FOR α_s -NONEXPANSIVE MAPPINGS ON PARTIAL b -METRIC SPACES

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Abstract In this paper, we introduce the class of α_s -non-expansive mappings which satisfy a generalized contractive condition. We establish some fixed point theorems for a introduced mappings in a complete partial b -metric space. Our results generalize some fixed point results existing in the current literature. We give an example which explains the main result. As an application, we show the existence of the solution of a boundary valued problem.

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1. INTRODUCTION

Let k be non-negative real number. Suppose that a self-mapping T on a metric space (X, d) satisfies the inequality

$$d(T(x), T(y)) \leq kd(x, y) \text{ for all } x, y \in X.$$

If $k < 1$, then T is called a contractive mapping and for $k = 1$, T is known as non-expansive mapping. There exists plenty of literature for contractive and non-expansive type mappings, where the contractive and non-expansive conditions are replaced with more general conditions (see for example [2–4, 6, 7, 11, 15, 21, 22]).

In 1989, Bakhtin [1] introduced the concept of b -metric space, however, Czerwik [5] initiated study of fixed point of self-mappings in the b -metric space and proved an analogue of Banach's fixed point theorem. Since then, a large number of researchers have presented remarkable fixed point results for various classes of single-valued and multi-valued operators in the b -metric spaces: see for example, Kir and Kiziltunc [13], Parvaneh *et al.* [18],

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Roshan *et al.* [23], Shatanawi *et al.* [24]. Khamsi and Hussain [12] obtained some results on KKM mappings in cone b -metric spaces. Recently, Jovanović *et al.* [10], Hussain and Huang [8, 9] have dealt with spaces of this kind, although under different names (in the spaces called metric-type) and obtained (common) fixed point results.

Matthews [16] presents a symmetric generalized metric which he announced as partial metric; an approach which sheds a new light on how metric tools such as Banach's Theorem can be extended to non-Hausdorff topologies.

Following Matthews and Czerwik, Shukla [25] introduced the concept of the partial b -metric space which generalizes partial metric and established fixed point theorems for Banach contraction, Kannan contraction and Chatterjea contraction defined on a complete partial b -metric space. Recently, Mustafa *et al.* [17], Latif *et al.* [14] and Piri *et al.* [19] have established some fixed point results in complete partial b -metric spaces.

In this paper, we investigate the fixed points of α_s -non-expansive single-valued mappings defined on a partial b -metric space subject to a generalized contractive condition. Our results generalize the results presented in [5, 16, 27]. An example and an application are given to explain the main theorem.

2. PRELIMINARIES

We denote the set of natural numbers, rational numbers, $(-\infty, +\infty)$, $(0, +\infty)$ and $[0, +\infty)$ by \mathbb{N} , \mathbb{Q} , \mathbb{R} , \mathbb{R}^+ and \mathbb{R}_0^+ , respectively.

Shukla generalized the notion of b -metric, as follows:

Definition 2.1. [25] Let X be a nonempty set and $s \geq 1$ be a real number. A mapping $p_b : X \times X \rightarrow \mathbb{R}_0^+$ is said to be a partial b -metric if it satisfies following axioms, for all $x, y, z \in X$

- (p_b1) $x = y$ if and only if $p_b(x, y) = p_b(x, x) = p_b(y, y)$;
- (p_b2) $p_b(x, x) \leq p_b(x, y)$;
- (p_b3) $p_b(x, y) = p_b(y, x)$;
- (p_b4) $p_b(x, y) \leq s[p_b(x, z) + p_b(z, y)] - p_b(z, z)$.

The triplet (X, p_b, s) is called a partial b -metric space.

Remark 2.2. The self distance $p_b(x, x)$, referred to the size or weight of x , is a feature used to describe the amount of information contained in x .

Remark 2.3. Obviously, every partial metric space is a partial b -metric space with coefficient $s = 1$ and every b -metric space is a partial b -metric space with zero self-distance. However, the converse of this fact need not to hold.

Example 2.4. Let $X = \mathbb{R}^+$ and $k > 1$, the mapping $p_b : X \times X \rightarrow \mathbb{R}^+$ defined by

$$p_b(x, y) = \left\{ (x \vee y)^k + |x - y|^k \right\} \text{ for all } x, y \in X$$

is a partial b -metric on X with $s = 2^k$. For $x = y$, $p_b(x, x) = x^k \neq 0$, so, p_b is not a b -metric on X .

Let $x, y, z \in X$ such that $x > z > y$. Then following inequality always holds

$$(x - y)^k > (x - z)^k + (z - y)^k.$$

Since, $p_b(x, y) = x^k + (x - y)^k$ and $p_b(x, z) + p_b(z, y) - p_b(z, z) = x^k + (x - z)^k + (z - y)^k$, therefore,

$$p_b(x, y) > p_b(x, z) + p_b(z, y) - p_b(z, z).$$

This shows that p_b is not a partial metric on X .

Example 2.5. [25] Let X be a nonempty set and p be a partial metric defined on X . The mapping $p_b : X \times X \rightarrow \mathbb{R}^+$ dened by

$$p_b(x, y) = [p(x, y)]^q \text{ for all } x, y \in X \text{ and } q > 1$$

defines a partial b -metric.

Definition 2.6. Let (X, p_b, s) be a partial b -metric space. The mapping $d_{p_b} : X \times X \rightarrow \mathbb{R}_0^+$ defined by

$$d_{p_b}(x, y) = 2p_b(x, y) - p_b(x, x) - p_b(y, y) \text{ for all } x, y \in X$$

defines a metric on X , called induced metric.

In partial b -metric space (X, p_b, s) , we immediately have a natural definition for the open balls:

$$B_{p_b}(x; \epsilon) = \{y \in X | p_b(x, y) < p_b(x, x) + \epsilon\} \text{ for all } x \in X. \quad (2.1)$$

Remark 2.7. The open balls in a partial b -metric space (X, p_b, s) may not be open set.

Proof. Let $X = \{a, b, c\}$ and define p_b as follows: $p_b(a, a) = p_b(c, c) = 1, p_b(b, b) = 1/2, p_b(a, b) = p_b(b, a) = 3, p_b(a, c) = p(c, a) = 3/2, p_b(b, c) = p_b(c, b) = 1$. Then p is a partial b -metric, $c \in B_{p_b}(a; 1)$ but for any $r > 0, B_{p_b}(c; r)$ does not lie in $B_{p_b}(a; 1)$. This implies that $B_{p_b}(a; 1)$ is not an open set in (X, p_b, s) . ■

The following definition and Lemma are taken from Shukla in [25].

Definition 2.8. [25] Let (X, p_b, s) be a partial b -metric space.

- (1) A sequence $\{x_n\}_{n \in \mathbb{N}}$ in (X, p_b, s) is called a Cauchy sequence if $\lim_{n, m \rightarrow \infty} p_b(x_n, x_m)$ exists and is finite.
- (2) A partial b -metric space (X, p_b, s) is said to be complete if every Cauchy sequence $\{x_n\}_{n \in \mathbb{N}}$ in X converges, with respect to $\mathcal{T}[p_b]$, to a point $v \in X$ such that

$$p_b(x, x) = \lim_{n, m \rightarrow \infty} p_b(x_n, x_m).$$

Lemma 2.9. [25] Let (X, p_b, s) be a partial b -metric space. Then

- (1) every Cauchy sequence in (X, d_{p_b}) is also a Cauchy sequence in (X, p_b, s) and vice versa;
- (2) a partial b -metric (X, p_b, s) is complete if and only if the metric space (X, d_{p_b}) is complete;
- (3) a sequence $\{x_n\}_{n \in \mathbb{N}}$ in X converges to a point $v \in X$ with respect to $\mathcal{T}[(d_{p_b})]$ if and only if

$$\lim_{n \rightarrow \infty} p_b(v, x_n) = p_b(v, v) = \lim_{n \rightarrow \infty} p_b(x_n, x_m);$$

- (4) if $\lim_{n \rightarrow \infty} x_n = v$ such that $p_b(v, v) = 0$, then $\lim_{n \rightarrow \infty} p_b(x_n, k) = p_b(v, k)$ for every $k \in X$.

Remark 2.10. Unlike metric space, in a partial b -metric space the limit of a convergent sequence may not be unique. Indeed, if $X = \mathbb{R}^+$ and let $\sigma > 0$ be any constant. Define $p_b : X \times X \rightarrow \mathbb{R}^+$ by $p_b(x, y) = x \vee y + \sigma$ for all $x, y \in X$, then (X, p_b, s) is a partial b -metric space with arbitrary coefficient $s \geq 1$. Define the sequence $\{x_n\}$ in X by $x_n = \rho$ for all $n \in \mathbb{N}$. We note that if $y \geq \rho$ then $p_b(x_n, y) = y + \sigma = p_b(y, y)$, thus $\lim_{n \rightarrow \infty} p_b(x_n, y) = p_b(y, y)$ for all $y \geq \rho$. Hence, the limit of a convergent sequence is not unique.

Recently, Popescu [20] introduced the concept of α -orbital admissible mappings.

Definition 2.11. [20] Let $X \neq \emptyset$ and let $T : X \rightarrow X$; $\alpha : X \times X \rightarrow \mathbb{R}_0^+$ be mappings. We say the mapping T is α -orbital admissible if for all $x \in X$, we have

$$\alpha(x, T(x)) \geq 1 \text{ implies } \alpha(T(x), T^2(x)) \geq 1.$$

3. THE FIXED POINT RESULTS

Definition 3.1. Let $X \neq \emptyset$ and let $T : X \rightarrow X$; $\alpha : X \times X \rightarrow \mathbb{R}_0^+$ be mappings. We say the mapping T is α_s -orbital admissible if for all $x \in X$, we have

$$\alpha_s(x, T(x)) \geq s^2 \text{ implies } \alpha_s(T(x), T^2(x)) \geq s^2.$$

Definition 3.2. Let (X, p_b, s) be a partial b -metric space. We say the mapping $T : X \rightarrow X$ is a non-expansive mapping if

$$p_b(T(x), T(y)) \leq p_b(x, y) \text{ for all } x, y \in X.$$

Definition 3.3. Let (X, p_b, s) be a partial b -metric space. Let $T : X \rightarrow X$; $\alpha_s : X \times X \rightarrow \mathbb{R}_0^+$ be mappings. We say the mapping T is α_s -non-expansive mapping if for all $x, y \in X$, we have

$$\alpha_s(x, y) \geq s^2 \text{ implies } p_b(T(x), T(y)) \leq p_b(x, y).$$

The following example illustrates the concept of α_s -non-expansive mapping.

Example 3.4. Let $X = \mathbb{R}$ be endowed with a partial b -metric

$$p_b(x, y) = \left\{ (x \vee y)^2 + |x - y|^2 \right\} \text{ for all } x, y \in X.$$

Define mapping $T : X \times X$ by

$$T(x) = \frac{x^2 + x}{2} \text{ for all } x \in X.$$

Define $\alpha_4 : X \times X \rightarrow \mathbb{R}_0^+$ by

$$\alpha_4(x, y) = \begin{cases} 16, & \text{if } x \in [0, 1] \\ 0, & \text{otherwise} \end{cases}$$

Let $x, y \in X$ be such that $\alpha_4(x, y) = 16$, then $x, y \in [0, 1]$. In this case, we note that

$$\left(\frac{x^2 + x}{2} \vee \frac{y^2 + y}{2} \right)^2 \leq (x \vee y)^2 \text{ and } \left| \frac{x^2 + x}{2} - \frac{y^2 + y}{2} \right|^2 \leq |x - y|^2.$$

Thus,

$$p_b(T(x), T(y)) = \left\{ (T(x) \vee T(y))^2 + |T(x) - T(y)|^2 \right\} \leq p_b(x, y).$$

Hence, T is an α_s -non-expansive mapping. Observe that T is not a non-expansive mapping. In fact, for $x = 1$, $y = 2$ we have

$$p_b(T(1), T(2)) = 13 > P_b(1, 2) = 5.$$

Lemma 3.5. *Let (X, p_b, s) be a partial b -metric space and $T : X \rightarrow X$ be a non-expansive mapping. If there exists $x_0 \in X$ and $\{x_n\}$ be the Picard iterative sequence with initial point x_0 . Then the sequence*

$$\left\{ \frac{p_b(x_{n-1}, x_n) + p_b(x_n, x_{n+1})}{p_b(x_{n-1}, x_n) + p_b(x_n, x_{n+1}) + 1} \right\}_{n \in \mathbb{N}}$$

is non-increasing.

Proof. Since, the mapping T is non-expansive, so, $p_b(x_{n-1}, x_n) \geq p_b(x_n, x_{n+1})$. We note that

$$\frac{p_b(x_{n-1}, x_n) + p_b(x_n, x_{n+1})}{p_b(x_{n-1}, x_n) + p_b(x_n, x_{n+1}) + 1} \geq \frac{p_b(x_n, x_{n+1}) + p_b(x_{n+1}, x_{n+2})}{p_b(x_n, x_{n+1}) + p_b(x_{n+1}, x_{n+2}) + 1}$$

if and only if

$$p_b(x_{n-1}, x_n) \geq p_b(x_{n+1}, x_{n+2}),$$

this completes the proof. ■

Definition 3.6. Let (X, p_b, s) be a partial b -metric space. The space (X, p_b, s) is said to be α_s -regular if for any sequence $\{x_n\} \subset X$ such that $\alpha_s(x_n, x_{n+1}) \geq s^2$ for all $n \in \mathbb{N}$ and $x_n \rightarrow x$ as $n \rightarrow \infty$, we have $\alpha_s(x_n, x) \geq s^2$ for all $n \in \mathbb{N}$.

The following is our main result.

Theorem 3.7. *Let (X, p_b, s) be a complete partial b -metric space and let $T : X \rightarrow X$ be an α_s -non-expansive and an α_s -orbital admissible mapping satisfying the following contractive condition:*

$$p_b(T(x), T(y)) \leq \left(\frac{p_b(x, T(y)) + p_b(y, T(x))}{s(p_b(x, T(x)) + p_b(y, T(y)) + 1)} + k \right) \mathcal{M}(x, y), \quad (3.1)$$

for all $x, y \in X$ such that $\alpha_s(x, y) \geq s^2$ and $k \in [0, 1)$, where

$$\mathcal{M}(x, y) = \max \left\{ p_b(x, y), p_b(x, Tx), p_b(y, Ty), \frac{p_b(x, Ty) + p_b(y, Tx)}{2s} \right\}.$$

Assume that

(a) there exists $x_0 \in X$ such that $\alpha_s(x_0, T(x_0)) \geq s^2$ and

$$\frac{p_b(x_0, T(x_0)) + p_b(T(x_0), T^2(x_0))}{1 + p_b(x_0, T(x_0)) + p_b(T(x_0), T^2(x_0))} + k < \frac{1}{s}; \quad (3.2)$$

(b) X is α_s -regular,

then T has at least one fixed point in X .

Proof. By assumption (a) there exists $x_0 \in X$ such that $\alpha_s(x_0, T(x_0)) \geq s^2$ and (3.2) holds. Define a sequence $\{x_n\}$ by $x_n = T(x_{n-1}) = T^n(x_0)$ for all $n \geq 1$. Since, T is an α_s -orbital admissible mapping. Thus, $\alpha_s(T(x_0), T^2(x_0)) \geq s^2$. By repeated application of α_s -orbital admissibility of mapping T , we have for all $n \geq 0$

$$\alpha_s(x_n, x_{n+1}) = \alpha_s(T^n(x_0), T^{n+1}(x_0)) \geq s^2.$$

Since $\alpha_s(x_n, x_{n+1}) \geq s^2$ for all $n \geq 0$, thus,

$$p_b(x_n, x_{n+1}) \leq p_b(x_{n-1}, x_n), \text{ for all } n \in \mathbb{N}.$$

We note that,

$$\begin{aligned} \mathcal{M}(x_{n-1}, x_n) &= \max \left\{ \begin{array}{l} p_b(x_{n-1}, x_n), p_b(x_{n-1}, x_n), p_b(x_n, x_{n+1}), \\ \frac{p_b(x_{n-1}, x_{n+1}) + p_b(x_n, x_n)}{2s} \end{array} \right\} \\ &= \max \{p_b(x_{n-1}, x_n), p_b(x_n, x_{n+1})\}. \end{aligned}$$

If $\mathcal{M}(x_{n-1}, x_n) = p_b(x_n, x_{n+1})$, then by substituting $x = x_{n-1}$ and $y = x_n$ in (3.1), we get a contradiction. Thus,

$$\begin{aligned} p_b(x_n, x_{n+1}) &\leq \left(\frac{p_b(x_{n-1}, x_{n+1}) + p_b(x_n, x_n)}{s(p_b(x_{n-1}, x_n) + p_b(x_n, x_{n+1}) + 1)} + k \right) p_b(x_{n-1}, x_n) \\ &\leq \left(\frac{p_b(x_{n-1}, x_n) + p_b(x_n, x_{n+1})}{1 + p_b(x_{n-1}, x_n) + p_b(x_n, x_{n+1})} + k \right) p_b(x_{n-1}, x_n). \end{aligned} \quad (3.3)$$

By Lemma 3.5, we know that the sequence

$$\left\{ \frac{p_b(x_{n-1}, x_n) + p_b(x_n, x_{n+1})}{p_b(x_{n-1}, x_n) + p_b(x_n, x_{n+1}) + 1} \right\}_{n \in \mathbb{N}}$$

is non-increasing. Consequently,

$$\frac{p_b(x_{n-1}, x_n) + p_b(x_n, x_{n+1})}{p_b(x_{n-1}, x_n) + p_b(x_n, x_{n+1}) + 1} \leq \frac{p_b(x_0, x_1) + p_b(x_1, x_2)}{p_b(x_0, x_1) + p_b(x_1, x_2) + 1}.$$

Let $\delta = \frac{p_b(x_0, x_1) + p_b(x_1, x_2)}{p_b(x_0, x_1) + p_b(x_1, x_2) + 1} + k$. By (3.3), we obtain

$$p_b(x_n, x_{n+1}) \leq \delta^n p_b(x_0, x_1), \text{ for all } n \in \mathbb{N}. \quad (3.4)$$

By triangle inequality, for $m > n$, we have

$$\begin{aligned} p_b(x_n, x_m) &\leq s[p_b(x_n, x_{n+1}) + p_b(x_{n+1}, x_m)] - p_b(x_{n+1}, x_{n+1}) \\ &\leq s p_b(x_n, x_{n+1}) + s^2[p_b(x_{n+1}, x_{n+2}) + p_b(x_{n+2}, x_m)] \\ &\quad - s p_b(x_{n+2}, x_{n+2}) \\ &\leq s p_b(x_n, x_{n+1}) + s^2 p_b(x_{n+1}, x_{n+2}) \\ &\quad + s^3[p_b(x_{n+2}, x_{n+3}) + p_b(x_{n+3}, x_m)] - s^2 p_b(x_{n+3}, x_{n+3}) \\ &\leq s p_b(x_n, x_{n+1}) + s^2 p_b(x_{n+1}, x_{n+2}) + s^3 p_b(x_{n+2}, x_{n+3}) \\ &\quad + \dots + s^{m-n} p_b(x_{m-1}, x_m). \end{aligned}$$

By inequality (3.4), we obtain

$$\begin{aligned} p_b(x_n, x_m) &\leq s \delta^n p_b(x_0, x_1) + s^2 \delta^{n+1} p_b(x_0, x_1) + s^3 \delta^{n+3} p_b(x_0, x_1) \\ &\quad + \dots + s^{m-n} \delta^{m-1} p_b(x_0, x_1) \\ &\leq s \delta^n (1 + s \delta + (s \delta)^2 + \dots + (s \delta)^{m-n-1}) p_b(x_0, x_1) \\ &\leq \frac{s \delta^n}{1 - s \delta} p_b(x_0, x_1). \end{aligned}$$

Since, $\delta \in [0, \frac{1}{s})$ and $s > 1$, thus

$$\lim_{n,m \rightarrow \infty} p_b(x_n, x_m) = 0. \quad (3.5)$$

Hence, $\{x_n\}$ is a Cauchy sequence in (X, p_b, s) . Since (X, p_b, s) is a complete partial b -metric space, therefore, by Lemma 2.9(3), there exists $x^* \in X$ (say) such that $\lim_{n \rightarrow \infty} d_{p_b}(x_n, x^*) = 0$ if and only if

$$\lim_{n \rightarrow \infty} p_b(x^*, x_n) = p_b(x^*, x^*) = \lim_{n,m \rightarrow \infty} p_b(x_n, x_m). \quad (3.6)$$

By (3.5) and (3.6), we have

$$\lim_{n \rightarrow \infty} p_b(x^*, x_n) = p_b(x^*, x^*) = 0 \Rightarrow x_n \xrightarrow{p_b} x^*.$$

Since, $\alpha_s(x_n, x_{n+1}) \geq s^2$ and $x_n \xrightarrow{p_b} x^*$, by assumption (b), we have $\alpha_s(x_n, x^*) \geq s^2$. By contractive condition (3.1), we have

$$p_b(x_{n+1}, T(x^*)) \leq \left(\frac{p_b(x_n, T(x^*)) + p_b(x^*, x_{n+1})}{s(p_b(x_n, x_{n+1}) + p_b(x^*, T(x^*) + 1))} + k \right) \mathcal{M}(x_n, x^*).$$

Taking limit as $n \rightarrow \infty$, we have

$$p_b(x^*, T(x^*)) \leq \left(\frac{p_b(x^*, T(x^*))}{s(p_b(x^*, T(x^*)) + 1)} + k \right) p_b(x^*, T(x^*)).$$

Hence, by axioms (p_b1) and (p_b2) , we get $x^* = T(x^*)$ which shows that x^* is a fixed point of T . \blacksquare

Remark 3.8. Let T satisfies the conditions assumed in the statement of Theorem 3.7. If x^* and y^* are two distinct fixed points of T satisfying $\alpha_s(x^*, y^*) \geq s^2$, then

$$p_b(x^*, y^*) \geq \frac{s(1-k)}{2}.$$

Proof. We have proved that set of fixed points of T is nonempty. Let x^* and y^* be two distinct fixed points of T . By contractive condition (3.1), we have

$$p_b(T(x^*), T(y^*)) \leq \left(\frac{p_b(x^*, T(y^*)) + p_b(y^*, T(x^*))}{s(p_b(x^*, T(x^*)) + p_b(y^*, T(y^*)) + 1)} + k \right) \mathcal{M}(x^*, y^*).$$

As $\mathcal{M}(x^*, y^*) = p_b(x^*, y^*)$, thus,

$$p_b(x^*, y^*) \leq \left(\frac{p_b(x^*, y^*) + p_b(y^*, x^*)}{s(p_b(x^*, x^*) + p_b(y^*, y^*) + 1)} + k \right) p_b(x^*, y^*)$$

implies that

$$\frac{s(1-k)}{2} \leq p_b(x^*, y^*).$$

The following example illustrates Theorem 3.7. \blacksquare

Example 3.9. Let $X = \{1, 2, 3, 4\}$ be endowed with the mapping $p_b : X \times X \rightarrow \mathbb{R}_0^+$ defined by

$$p_b(x, y) = \begin{cases} |x - y|^2 + (x \vee y)^2, & \text{if } x \neq y, \\ x, & \text{if } x = y \neq 1, \\ 0, & \text{if } x = y = 1. \end{cases}$$

Then, it is easy to verify that $(X, p_b, 4)$ is a complete partial b -metric space. Moreover, we note that

$$\begin{aligned}
 p_b(1, 1) &= 0, \quad p_b(2, 2) = 2, \quad p_b(3, 3) = 3, \quad p_b(4, 4) = 4, \\
 p_b(1, 2) &= p_b(2, 1) = 5, \quad p_b(3, 1) = p_b(1, 3) = 13, \quad p_b(1, 4) = p_b(4, 1) = 25, \\
 p_b(2, 3) &= p_b(3, 2) = 10, \quad p_b(2, 4) = p_b(4, 2) = 20, \quad p_b(3, 4) = p_b(4, 3) = 17.
 \end{aligned}$$

Define the mapping $T : X \rightarrow X$ by

$$T(1) = 1, \quad T(2) = 1, \quad T(3) = 2, \quad T(4) = 2.$$

Following table shows that T satisfies the contractive condition (3.1) for $\frac{1}{6} \leq k < 1$.

$x \neq y$	$p_b(T(x), T(y))$	$\left(\frac{p_b(x, T(y)) + p_b(y, T(x))}{s(1 + p_b(x, T(x)) + p_b(y, T(y)))} + k \right) \mathcal{M}(x, y)$
(1, 2)	0	$5k + 25/24$
(1, 3)	5	$13k + 117/22$
(1, 4)	5	$25k + 125/14$
(2, 3)	5	$13k + 195/64$
(2, 4)	5	$20k + 540/104$
(3, 4)	2	$20k + 300/62$

$x = y \neq 1$	$p_b(T(x), T(y))$	$\left(\frac{p_b(x, T(y)) + p_b(y, T(x))}{s(1 + p_b(x, T(x)) + p_b(y, T(y)))} + k \right) \mathcal{M}(x, y)$
(2, 2)	0	$5k + 25/11$
(3, 3)	2	$10k + 50/21$
(4, 4)	2	$20k + 200/41$

Similarly (3.1) holds for $x = y = 1$. Now define

$$\alpha_s(x, y) = \begin{cases} 18, & \text{if } x \neq y, \\ 17, & \text{if } x = y \neq 1, \\ 16, & \text{if } x = y = 1. \end{cases}$$

There exists $x_0 = 1$ such that $\alpha_s(x_0, T(x_0)) = 16$. Clearly, T is both α_4 -non-expansive and α_4 -orbital admissible mapping and satisfies conditions (a) and (b). Hence, the conditions of Theorem 3.7 are satisfied. Note that $x = 1$ is a fixed point of T with $p_b(1, 1) = 0$. Since $p_b(2, 2) = 2 \neq 0$, it follows that p_b is not a b -metric. Also p_b is not a partial metric. Indeed, $p_b(4, 1) = 25 > 23 = p_b(4, 2) + p_b(2, 1) - p_b(2, 2)$. It is remarked that the results in [5], [16] and [27] are not applicable while Theorem 3.7 is applicable.

Definition 3.10. Let (X, p_b) be an ordered partial b -metric space and \prec_1 be a binary relation on X . We say the mapping $T : X \rightarrow X$ is an order preserving if for each $x \in X$ such that $x \prec_1 T(x)$, we have $T(x) \prec_1 T^2(x)$.

We state the following.

Corollary 3.11. Let (X, p_b, s, \prec_1) be an ordered complete partial b -metric space and $T : X \rightarrow X$ be a non-expansive and order preserving mapping satisfying the following condition:

$$p_b(T(x), T(y)) \leq \left(\frac{p_b(x, T(y)) + p_b(y, T(x))}{s(p_b(x, T(x)) + p_b(y, T(y)) + 1)} + k \right) \mathcal{M}(x, y),$$

for all $x, y \in X$ such that $x \prec_1 y$ and $k \in [0, 1)$, where

$$\mathcal{M}(x, y) = \max \left\{ p_b(x, y), p_b(x, Tx), p_b(y, Ty), \frac{p_b(x, Ty) + p_b(y, Tx)}{2s} \right\}.$$

Assume that

(a) there exists $x_0 \in X$ such that $x_0 \prec_1 T(x_0)$ and

$$\frac{p_b(x_0, T(x_0)) + p_b(T(x_0), T^2(x_0))}{1 + p_b(x_0, T(x_0)) + p_b(T(x_0), T^2(x_0))} + k < \frac{1}{s};$$

(b) (X, p_b, s, \prec_1) is regular.

Then

- (i) T has at least one fixed point;
- (ii) the Picard iterative sequence converges to a fixed point of T .
- (iii) If x^* and y^* are two distinct fixed points of T such that $x^* \prec_1 y^*$, then

$$p_b(x^*, y^*) \geq \frac{s(1-k)}{2}.$$

Proof. It suffices to consider

$$\alpha_s(x, y) = \begin{cases} s^2, & \text{if } x \prec_1 y, \\ 0, & \text{otherwise} \end{cases}$$

in Theorem 3.7. ■

Now we state our second main result.

Theorem 3.12. Let (X, p_b, s) be a complete partial b -metric space and $T : X \rightarrow X$ be an α_s -non-expansive and an α_s -orbital admissible mapping satisfying the following contractive condition:

$$p_b(T(x), T(y)) \leq \left(\frac{p_b(x, T(y)) + p_b(y, T(x))}{s(p_b(x, T(x)) + p_b(y, T(y)) + 1)} + k \right) \mathcal{M}_1(x, y), \quad (3.7)$$

for all $x, y \in X$ such that $\alpha_s(x, y) \geq s^2$ and $k \in [0, 1)$, where

$$\mathcal{M}_1(x, y) = p_b(x, y) + |p_b(x, Tx) - p_b(y, Ty)|,$$

Assume that

(a) there exists $x_0 \in X$ such that $\alpha_s(x_0, T(x_0)) \geq s^2$ and

$$\frac{p_b(x_0, T(x_0)) + p_b(T(x_0), T^2(x_0))}{1 + p_b(x_0, T(x_0)) + p_b(T(x_0), T^2(x_0))} + k < \frac{1}{s};$$

(b) X is α_s -regular.

Then

- (i) T has at least one fixed point;
- (ii) the Picard iterative sequence converges to a fixed point of T .

(iii) If x^* and y^* are two distinct fixed points of T such that $\alpha_s(x^*, y^*) \geq s^2$, then

$$p_b(x^*, y^*) \geq \frac{s(1-k)}{2}.$$

Proof. Proceeding as in the proof of Theorem 3.7, we have

$$\alpha_s(x_n, x_{n+1}) = \alpha_s(T^n(x_0), T^{n+1}(x_0)) \geq s^2, \text{ for all } n \in \mathbb{N}.$$

Since $\alpha_s(x_n, x_{n+1}) \geq s^2$ for all $n \geq 0$ and the mapping T is an α_s -non-expansive, we get

$$p_b(x_n, x_{n+1}) \leq p_b(x_{n-1}, x_n), \text{ for all } n \in \mathbb{N}. \quad (3.8)$$

By contractive condition (3.7), we have

$$p_b(x_n, x_{n+1}) \leq \left(\frac{p_b(x_{n-1}, x_{n+1}) + p_b(x_n, x_n)}{s(p_b(x_{n-1}, x_n) + p_b(x_n, x_{n+1}) + 1)} + k \right) \mathcal{M}_1(x_{n-1}, x_n). \quad (3.9)$$

By (3.8), we have

$$\begin{aligned} \mathcal{M}_1(x_{n-1}, x_n) &= p_b(x_{n-1}, x_n) + |p_b(x_{n-1}, x_n) - p_b(x_n, x_{n+1})| \\ &= 2p_b(x_{n-1}, x_n) - p_b(x_n, x_{n+1}). \end{aligned}$$

By Lemma 3.5, we know that the sequence

$$\left\{ \frac{p_b(x_{n-1}, x_n) + p_b(x_n, x_{n+1})}{p_b(x_{n-1}, x_n) + p_b(x_n, x_{n+1}) + 1} \right\}_{n \in \mathbb{N}}$$

is non-increasing. Consequently,

$$\frac{p_b(x_{n-1}, x_n) + p_b(x_n, x_{n+1})}{p_b(x_{n-1}, x_n) + p_b(x_n, x_{n+1}) + 1} \leq \frac{p_b(x_0, x_1) + p_b(x_1, x_2)}{p_b(x_0, x_1) + p_b(x_1, x_2) + 1}.$$

Let $\delta = \frac{p_b(x_0, x_1) + p_b(x_1, x_2)}{p_b(x_0, x_1) + p_b(x_1, x_2) + 1} + k$. By (3.9), we obtain

$$\begin{aligned} p_b(x_n, x_{n+1}) &\leq \left(\frac{p_b(x_{n-1}, x_{n+1}) + p_b(x_n, x_n)}{s(1 + p_b(x_{n-1}, x_n) + p_b(x_n, x_{n+1}))} + k \right) \mathcal{M}_1(x_{n-1}, x_n) \\ &\leq \left(\frac{p_b(x_0, x_1) + p_b(x_1, x_2)}{1 + p_b(x_0, x_1) + p_b(x_1, x_2)} + k \right) \mathcal{M}_1(x_{n-1}, x_n) \\ &= \delta \mathcal{M}_1(x_{n-1}, x_n) = \delta(2p_b(x_{n-1}, x_n) - p_b(x_n, x_{n+1})), \\ (1 + \delta)p_b(x_n, x_{n+1}) &\leq 2\delta p_b(x_{n-1}, x_n), \\ p_b(x_n, x_{n+1}) &\leq \frac{2\delta}{(1 + \delta)} p_b(x_{n-1}, x_n). \end{aligned}$$

By induction we obtain

$$p_b(x_n, x_{n+1}) \leq \lambda^n p_b(x_0, x_1), \text{ for all } n \in \mathbb{N} \text{ where } \lambda = \frac{2\delta}{(1 + \delta)} < \frac{2}{1 + s} < 1.$$

The remaining part of this proof can easily be followed by the proof of Theorem 3.7. ■

Corollary 3.13. *Let (X, p_b, s) be a complete partial b -metric space and $T : X \rightarrow X$ be an α_s -non-expansive mapping satisfying the following condition:*

$$p_b(T(x), T(y)) \leq \left(\frac{p_b(x, T(y)) + p_b(y, T(x))}{s(1 + p_b(x, T(x)) + p_b(y, T(y)))} + k \right) \mathcal{M}_2(x, y),$$

for all $x, y \in X$ $\alpha_s(x, y) \geq s^2$ and $k \in [0, 1)$, where

$$\mathcal{M}_2(x, y) = \max\{\mathcal{M}(x, y), \mathcal{M}_1(x, y)\}.$$

Assume that the conditions (a) and (b) assumed in the statement of Theorem 3.7 hold. Then

- (i) T has at least one fixed point;
- (ii) the Picard iterative sequence converges to a fixed point of T .
- (iii) If x^* and y^* are two distinct fixed points of T such that $\alpha_s(x^*, y^*) \geq s^2$, then

$$p_b(x^*, y^*) \geq \frac{s(1-k)}{2}.$$

Corollary 3.14. Let (X, p_b, s) be a complete partial b -metric space and $T : X \rightarrow X$ be an α_s -non-expansive mapping satisfying the following condition:

$$p_b(T(x), T(y)) \leq \left(\frac{p_b(x, T(y)) + p_b(y, T(x))}{s(1 + p_b(x, T(x)) + p_b(y, T(y)))} + k \right) \mathcal{M}_3(x, y),$$

where

$$\mathcal{M}_3(x, y) = \frac{1}{2} \{\mathcal{M}(x, y) + \mathcal{M}_1(x, y)\}.$$

Assume that the conditions (a) and (b) assumed in the statement of Theorem 3.7 hold. Then

- (i) T has at least one fixed point;
- (ii) the Picard iterative sequence converges to a fixed point of T .
- (iii) If x^* and y^* are two distinct fixed points of T such that $\alpha_s(x^*, y^*) \geq s^2$, then

$$p_b(x^*, y^*) \geq \frac{s(1-k)}{2}.$$

Theorem 3.15. Let (X, p_b, s) be a complete partial b -metric space and $T : X \rightarrow X$ be a non-expansive mapping satisfying the following conditions:

$$p_b(T(x), T(y)) \leq \left(\frac{p_b(x, T(y)) + p_b(y, T(x))}{s(p_b(x, T(x)) + p_b(y, T(y)) + 1)} + k \right) \mathcal{M}(x, y),$$

for all $x, y \in X$, where

$$\mathcal{M}(x, y) = \max \left\{ p_b(x, y), p_b(x, Tx), p_b(y, Ty), \frac{p_b(x, Ty) + p_b(y, Tx)}{2s} \right\}.$$

Assume that

$$\frac{p_b(x_0, T(x_0)) + p_b(T(x_0), T^2(x_0))}{1 + p_b(x_0, T(x_0)) + p_b(T(x_0), T^2(x_0))} + k < \frac{1}{s}.$$

Then

- (i) T has at least one fixed point;
- (ii) the Picard iterative sequence converges to a fixed point of T .
- (iii) If x^* and y^* are two distinct fixed points of T , then

$$p_b(x^*, y^*) \geq \frac{s(1-k)}{2}.$$

Proof. The proof follows as the same lines in the proof of Theorem 3.7. ■

Remark 3.16. If we replace $\mathcal{M}(x, y)$ with $\mathcal{M}_1(x, y)$ in the Theorem 3.15, Corollary 3.11 stated above, we get new results. For $s = 1$, we get all the stated results in partial metric spaces (new results). For $p_b(x, x) = 0$ for all $x \in X$, we get all the stated results in b -metric spaces (new results). For $s = 1$ with $p_b(x, x) = 0$ for all $x \in X$, we get all the stated results in metric spaces.

4. APPLICATION

In this section, we present an application of Theorem 3.15 to show the existence of the solution of the boundary valued problem given by

$$\begin{cases} -\frac{d^2x}{dt^2} = f(t, x(t)), & t \in [0, 1] \\ x(0) = x(1) = 0, \end{cases} \tag{4.1}$$

where $f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous mapping. The Green function associated to the boundary valued problem (4.1) is defined by

$$G(t, \tau) = \begin{cases} t(1 - \tau), & 0 \leq t \leq \tau \leq 1 \\ \tau(1 - t), & 0 \leq \tau \leq t \leq 1. \end{cases}$$

Let $C[0, 1]$ be the space of all continuous mappings defined on $[0, 1]$. Let $X = (C[0, 1], \mathbb{R})$. Define the mapping $p_b : X \times X \rightarrow \mathbb{R}_0^+$ by

$$p_b(x, y) = \left\| (x - y)^2 \right\|_\infty + \eta = \sup_{t \in [0, 1]} |x(t) - y(t)|^2 + \eta; \eta > 0$$

It is known that (X, p_b, s) is a complete partial b -metric space with constant $s = 2$. Define the mapping $T : X \rightarrow X$ by

$$Tx(t) = \int_0^1 G(t, \tau) f(\tau, x(\tau)) d\tau,$$

for all $t \in [0, 1]$. Note that problem (4.1) has a solution if and only if the operator T has a fixed point.

Theorem 4.1. Let $X=C([0, 1], \mathbb{R})$. Define the mappings $T : X \rightarrow X$ by

$$Tx(t) = \int_0^1 G(t, \tau) f(\tau, x(\tau)) d\tau, \tag{4.2}$$

where $f : [0, 1] \times X \rightarrow \mathbb{R}$ is a continuous mapping. Assume that T satisfies following conditions:

(a) there exists $x_0 = x_0(t) \in X$ such that

$$\frac{p_b(x_0, T(x_0)) + p_b(T(x_0), T^2(x_0))}{1 + p_b(x_0, T(x_0)) + p_b(T(x_0), T^2(x_0))} + k < \frac{1}{s};$$

(b) the mapping $f : [0, 1] \times X \rightarrow \mathbb{R}$ satisfies

$$|f(t, x) - f(t, y)|^2 \leq 64 \ln \left(\frac{e^{kp_b(x, y)}}{\rho} \right),$$

for all $t \in [0, 1]$, $x, y \in X$ and $\rho > 1$.

Then boundary valued problem (4.1) has a solution.

Proof. It is remarked that $x^*(t) \in (C^2[0, 1], \mathbb{R})$ (say) is a solution of (4.1) if and only if $x^*(t) \in X$ is a solution of the integral equation (4.2). The solution of (4.2) is given by the fixed point of T i.e $x^*(t) = T(x^*(t))$.

Let $x, y \in X$ and $t \in [0, 1]$, by assumption (b), we get

$$\begin{aligned} |Tx(t) - Ty(t)|^2 &= \left[\int_0^1 G(t, \tau) [f(\tau, x(\tau)) - f(\tau, y(\tau))] d\tau \right]^2 \\ &\leq \left[\int_0^1 G(t, \tau) |f(\tau, x(\tau)) - f(\tau, y(\tau))| d\tau \right]^2 \\ &\leq \left[8 \int_0^1 G(t, \tau) \sqrt{\ln \left(\frac{e^{kp_b(x, y)}}{\rho} \right)} d\tau \right]^2 \\ &\leq \left[8 \int_0^1 G(t, \tau) \sqrt{\ln \left(\frac{e^{kp_b(x, y)}}{\rho} \right)} d\tau \right]^2 \\ &= 8^2 \ln \left(\frac{e^{kp_b(x, y)}}{\rho} \right) \left(\sup_{t \in [0, 1]} \left[\int_0^1 G(t, \tau) d\tau \right]^2 \right). \end{aligned}$$

Since $\int_0^1 G(t, \tau) d\tau = -\frac{t^2}{2} + \frac{t}{2}$ for all $t \in [0, 1]$, then we have $\left(\sup_{t \in [0, 1]} \left[\int_0^1 G(t, \tau) d\tau \right]^2 \right) = \frac{1}{8^2}$,

which implies that

$$\begin{aligned} |Tx(t) - Ty(t)|^2 + \eta &\leq kp_b(x, y), \text{ where } \eta = \ln(\rho) \\ &\leq \left(\frac{p_b(x, T(y)) + p_b(y, T(x))}{s(p_b(x, T(x)) + p_b(y, T(y)) + 1)} + k \right) \mathcal{M}(x, y). \end{aligned}$$

We note that

$$p_b(Tx(t), Ty(t)) \leq p_b(x(t), y(t))$$

which shows that T is non-expansive. Let $\mathcal{M}(x, y)$ be defined as in Theorem 3.15. Then it can easily be proved that

$$\mathcal{M}(x, y) = \sup_{t \in [0, 1]} \mathcal{M}(x(t), y(t)).$$

Thus,

$$p_b(Tx(t), Ty(t)) \leq \left(\frac{p_b(x, T(y)) + p_b(y, T(x))}{s(p_b(x, T(x)) + p_b(y, T(y)) + 1)} + k \right) \mathcal{M}(x, y)$$

holds true for all $x, y \in X$ and $t \in [0, 1]$. Hence, application of Theorem 3.15 ensures that T has at least one fixed point $x^*(t) \in X$, that is, $T(x^*(t)) = x^*(t)$ which is a solution of (4.2). ■

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