Thai Journal of Mathematics
Vol. 18, No. 1 (2020),
Pages 1 - 13

ISSN 1686-0209

COMMON FIXED POINTS OF MODIFIED PICARD-S ITERATION PROCESS INVOLVING TWO G-NONEXPANSIVE MAPPINGS IN CAT(0) SPACE WITH DIRECTED GRAPH

Sabia Khatoon[1], **Izhar Uddin**[1], **Nuttapol Pakkaranang**[2], **Nopparat Wairojjana**[3,*]

[1] *Department of Mathematics, Jamia Millia Islamia, New Delhi-110025, India*
e-mail : sabiakhatoon522@gmail.com (S. Khatoon), e-mail : izharuddin1@jmi.ac.in (I. Uddin)
[2] *Department of Mathematics, Faculty of Science, King Mongkut's University of Technology Thonburi(KMUTT), 126 Pracha-Uthit Road Bang Mod, Thung Khru, Bangkok 10140 Thailand*
e-mail : nuttapol.pak@mail.kmutt.ac.th (N. Pakkaranang)
[3] *Applied Mathematics Program, Faculty of Science and Technology, Valaya Alongkorn Rajabhat University under the Royal Patronage (VRU), 1 Moo 20 Phaholyothin Road, Klong Neung, Klong Luang, Pathumthani, 13180, Thailand*
e-mail : nopparat@vru.ac.th (N. Wairojjana)

Abstract The aim of this paper is to bring and study the convergence behaviour of modified Picard-S iteration involving two G-nonexpansive mappings in CAT(0) space with directed graph. We prove Δ and strong convergence theorems for modified Picard-S iteration process in CAT(0) space with a directed graph. We also construct a numerical example to validate our results and to ensure the better rate of convergence of the proposed method with modified Ishikawa iteration, modified S-iteration and Thianwan's new iteration.

MSC: 47H10; 47H09; 47E10

Keywords: G-nonexpansive mapping; $\Delta-$convergence and strong convergence; common fixed point; CAT(0) space; directed graph

Submission date: 01.10.2019 / Acceptance date: 21.11.2019

1. Introduction

Fixed point theory is an active area of research due to its applications in multiple fields. It addresses the results which state that, under certain conditions, a self map on a set admits a fixed point. Banach contraction principle [6] is one of the most important theorem due to its simplicity and ease of application in major areas of mathematics. In 2008, Jachymski [16] generalized the Banach contraction principle in a complete metric space endowed with a directed graph. Many authors have investigated fixed point theorems for

*Corresponding author.

nonexpansive mappings on both Hilbert spaces and Banach spaces. The initial existence theorems for nonexpansive mapping have been obtained by Browder [8], Göhde [12] and Kirk [17] independently.
Construction of approximating fixed points of nonexpansive mappings is an important subject in the theory of nonexpansive mappings and its applications in a number of applied areas. Thus three iteration methods often prevail to approximate a fixed point of a nonexpansive mapping are Halpern [14], Mann [20] and Ishikawa [15] iteration process. Some of well known iterative process are Agarwal [2], Noor [21], Abbas [1], SP[22], CR[9], Picard-S[13] iteration process.
In 2012, Aleomraninejad et al. [5] presented some iterative scheme for G-contraction and G-nonexpansive mappings in a Banach space with a graph. Tiammee et al. [31] proved Browder's convergence theorem for G-nonexpansive mappings in a Hilbert space with a directed graph. In 2016, Tripak [32] prove the weak and strong convergence of a sequence generated by the Ishikawa iteration to some common fixed points of two G-nonexpansive mappings defined on a Banach space endowed with a graph. In 2017, Suparatulatorn et al. [28] introduced and studied the modified S-iteration for two G-nonexpansive mappings in a uniformly convex Banach space endowed with a graph. Recently, Thianwan and Yambangwai [30], introduced a new two-step iteration process for two G-nonexpansive mappings and studied the strong and weak convergence theorems for such mappings in a uniformly convex Banach space endowed with a graph.
They use a uniformly convex Banach space as a base space and prove strong and weak convergence theorems. On the other hand, we know that every Banach space is a CAT(0) space. Motivated by the recent works, we introduce modified Picard-S iteration process including two G-nonexpansive mappings, where the sequence $\{x_n\}$ is generated iteratively as follows:
Let C be a nonempty convex subset of a CAT(0) space X, for any arbitrary $x_0 \in C$,

$$\begin{aligned} z_n &= (1-\beta_n)x_n \oplus \beta_n S_2 x_n, \\ y_n &= (1-\alpha_n)S_1 x_n \oplus \alpha_n S_2 z_n, \\ x_{n+1} &= S_1 y_n. \end{aligned} \tag{1.1}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are appropriate real sequences in $(0,1)$.

The purpose of this paper is to approximate common fixed points of two G-nonexpansive mappings for modified Picard-S iteration and to study the convergence analysis of such mappings in CAT(0) space endowed with a graph. We also perform the numerical experiments for supporting our main results and comparing rate of convergence of the proposed method (1.1) with the Ishikawa iteration process [32], the modified S-iteration process [28], Thianwan's new iteration process [30].

2. Preliminaries

In this section, we collect some well-known concepts and relevant results which will be used frequently in our subsequent results.
A metric space (X, d) is a CAT(0) space if it is geodesically connected and every geodesic triangle in X is at least as thin as its comparison triangle in the Euclidean plane. Let X be complete CAT(0) space and $\{x_n\}$ be a bounded sequence in X. For $x \in X$ set:

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n).$$

The asymptotic radius $r(\{x_n\})$ is given by

$$r(\{x_n\}) = \inf\{r(x, x_n) : x \in X\},$$

and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is defined as:

$$A(\{x_n\}) = \{x \in X : r(x, x_n) = r(\{x_n\})\}.$$

It is known that in a CAT(0) space (see in, [33–41]), $A(\{x_n\})$ consists of exactly one point.

In 2008, Kirk and Panyanak [18] gave a concept of convergence in CAT(0) spaces which is analogue of weak convergence in Banach spaces and restriction of Lim's concepts of convergence [19] to CAT(0) spaces.

Definition 2.1. A sequence $\{x_n\}$ in X is said to Δ-converges to $x \in X$ if x is the unique asymptotic center of u_n for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case we write $\Delta - \lim_n x_n = x$ and read as x is the Δ-limit of $\{x_n\}$.

Notice that given $\{x_n\} \subset X$ such that $\{x_n\}$ Δ-converges to x and given $y \in X$ with $y \neq x$, by uniqueness of asymptotic center we have,

$$\limsup_{n\to\infty} d(x_n, x) < \limsup_{n\to\infty} d(x_n, y).$$

Thus every CAT(0) space satisfies the Opial property. Now we collect some basic fact about CAT(0) spaces which will be used throughout the text frequently.

Lemma 2.2. *([18]). Every bounded sequence in a complete CAT(0) space admits a Δ-convergent subsequence.*

Lemma 2.3. *([10]). If C is closed convex subset of a complete CAT(0) space and if $\{x_n\}$ is a bounded sequence in C, then the asymptotic center of $\{x_n\}$ is in C.*

Lemma 2.4. *([11]). Let $(X,\ d)$ be a CAT(0) space. For $x,\ y \in X$ and $t \in [0, 1]$, there exists a unique $z \in [x,\ y]$ such that*

$$d(x, z) = td(x, y) \text{ and } d(y, z) = (1 - t)d(x, y).$$

We use the notation $(1 - t)x \oplus ty$ for the unique point z of the above lemma.

Lemma 2.5. *For $x, y, z \in X$ and $t \in [0, 1]$ we have*

$$d((1 - t)x \oplus ty, z) \leq (1 - t)d(x, z) + td(y, z).$$

Let C be a nonempty subset of a complete CAT(0) X. Let $\triangle$ denote the diagonal of the cartesian product $C \times C$, i.e., $\triangle = \{(x, x) : x \in C\}$.

Consider a directed graph G such that the set $V(G)$ of its vertices coincides with C, and the set $E(G)$ of its edges contains all loops, i.e., $E(G) \supseteq \triangle$. We assume G has no parallel edge. So we can identify the graph G with the pair $(V(G), E(G))$. By G^{-1} we denote the conversion of a graph G i.e., the graph obtained from G by reversing the direction of edges. Thus we have

$$E(G^{-1}) = \{(x, y) \in X \times X : (y, x) \in E(G)\}.$$

A set B dominates x_0 if for each $x \in B$, $(x, x_0) \in E(G)$ and is dominated by x_0 if for each $x \in B$, $(x_0, x) \in E(G)$. Let C be a subset of a CAT(0) space X. A mapping $S : C \to C$ is semicompact [27] if for a sequence $\{x_n\}$ in C with $\lim_{n\to\infty} d(x_n, Sx_n) = 0$, there exists a

subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \to p \in C$.
If x and y are vertices in a graph G, then a path in G from x to y of length $N (N \in \mathbb{N} \cup 0)$ is a sequence $\{x_i\}_{i=0}^{N}$ of $N+1$ vertices such that $x_0 = x, x_N = y$ and $(x_i, x_{i+1}) \in E(G)$ for $i = 0, 1, ...N-1$. A graph G is connected if there is path between any two vertices. A directed $G = (V(G), E(G))$ is said to be transitive if, for any $x, y, z \in V(G)$ such that (x, y) and (y, z) are in $E(G)$, implies $(x, z) \in E(G)$.
Let $S : C \to C$ be a self map. An edge preserving mapping i.e.($(x, y) \in E(G) \Rightarrow (Sx, Sy) \in E(G)$) is said to be:

(1) G-contraction if

$$d(Sx, Sy) \leq \alpha d(x, y), \qquad \forall (x, y) \in E(G).$$

where $\alpha \in (0, 1)$.

(1) G-nonexpansive if

$$d(Sx, Sy) \leq d(x, y), \qquad \forall (x, y) \in E(G).$$

Let C be a subset of a complete CAT(0) space X and let $G = (V(G), E(G))$ be a directed graph such that $V(G) = C$. Then, C is said to have Property $DG(SG)$ if for each sequence $\{x_n\}$ in C converging Δ(strongly) to $x \in C$ and $(x_n, x_{n+1}) \in E(G)$, there is a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $(x_{n_j}, x) \in E(G)$ for all $n \in \mathbb{N}$.
The following lemmas are useful in our main results.
Lemma 2.6 Let X be a complete $CAT(0)$ space and let $x \in X$. Suppose $\{t_n\}$ is a sequence in $[b, c]$ for some $b, c \in (0, 1)$ and $\{x_n\}$, $\{y_n\}$ are sequences in X such that $\limsup_{n\to\infty} d(x_n, x) \leq r$, $\limsup_{n\to\infty} d(y_n, x) \leq r$, and $\lim_{n\to\infty} d((1-t_n)x_n \oplus t_n y_n, x) = r$ for some $r \geq 0$. Then

$$\lim_{n\to\infty} d(x_n, y_n) = 0.$$

3. Main Results

We start with proving the following proposition.

Proposition 3.1. *Let S_1 and S_2 be two G-nonexpansive from C to C with $F = F(S_1) \cap F(S_2)$ nonempty, where C is a nonempty closed convex subset of a complete CAT(0) space X endowed with the directed graph. Let $V(G) = C$, $E(G)$ is convex and the graph G is transitive. For an arbitrary $x_0 \in C$, defined the sequence $\{x_n\}$ by (1.1). Let $p_0 \in F$ be such that (x_0, p_0), (p_0, x_0) are in $E(G)$. Then (x_n, p_0), (y_n, p_0), (z_n, p_0), (p_0, x_n), (p_0, y_n), (p_0, z_n), (x_n, y_n), (x_n, z_n) and (x_n, x_{n+1}) are in $E(G)$.*

Proof. We proceed by induction. Since S_2 is edge-preserving and $(x_0, p_0) \in E(G)$, we have $(S_2x_0, p_0) \in E(G)$ and so $(z_0, p_0) \in E(G)$, by convexity of $E(G)$. By edge preservingness of S_1 and $(x_0, p_0) \in E(G)$, we get $(S_1x_0, p_0) \in E(G)$ and as S_2 is edge preserving and $(z_0, p_0) \in E(G)$, we have $(S_2z_0, p_0) \in E(G)$. By convexity of $E(G)$ and $(S_1x_0, p_0), (S_2z_0, p_0) \in E(G)$, we get $(y_0, p_0) \in E(G)$. Thus, by edge-preserving of S_1, $(S_1y_0, p_0) \in E(G)$, we get $(x_1, p_0) \in E(G)$.
Again, by edge-preserving of S_2, $(S_2x_1, p_0) \in E(G)$. By convexity of $E(G)$ and $(x_1, p_0), (S_2x_1, p_0) \in E(G)$, we get $(z_1, p_0) \in E(G)$. Thus by edge-preserving of S_2, $(S_2z_1, p_0) \in E(G)$. Again by convexity of $E(G)$ and $(S_1x_1, p_0), (S_2z_1, p_0) \in E(G)$, we have $(y_1, p_0) \in E(G)$. By edge-preserving of S_1, $(S_1y_1, p_0) \in E(G)$, we get $(x_2, p_0) \in E(G)$. Next we assume that $(x_k, p_0) \in E(G)$. By edge-preserving of S_2 and convexity of $E(G)$, we get $(S_2x_k, p_0) \in$

$E(G)$ and $(z_k, p_0) \in E(G)$. By applying edge-preserving of S_2 on $(z_k, p_0) \in E(G)$, we get $(S_2 z_k, p_0) \in E(G)$. By using convexity of $E(G)$ and $(S_1 x_k, p_0), (S_2 z_k, p_0) \in E(G)$, we have $(y_k, p_0) \in E(G)$. As S_1 is edge-preserving and $(y_k, p_0) \in E(G)$ implies $(S_1 y_k, p_0) \in E(G)$ which implies $(x_{k+1}, p_0) \in E(G)$. Owing to edge-preserving of S_2, we obtain $(S_2 x_{k+1}, p_0) \in E(G)$ and so $(z_{k+1}, p_0) \in E(G)$, since $E(G)$ is convex. By edge-preserving of S_2, we get $(S_2 z_{k+1}, p_0) \in E(G)$ so $(y_{k+1}, p_0) \in E(G)$, since $E(G)$ is convex. Therefore, $(x_n, p_0), (y_n, p_0), (z_n, p_0) \in E(G)$ for all $n \geq 1$. Using a similar arguement, we can show that $(p_0, x_n), (p_0, y_n), (p_0, z_n) \in E(G)$ under the assumption that $(p_0, z_0) \in E(G)$. By using transitivity of G, we get $(x_n, y_n), (x_n, z_n), (y_n, z_n), (x_n, x_{n+1}) \in E(G)$. This completes the proof. ■

Lemma 3.2. *Let X, C, S_1, S_2, F and $\{x_n\}$ be same as in Proposition* (3.1). *Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequence in $[\delta, 1-\delta]$ for some $\delta \in (0,1)$ and $(x_0, p_0), (p_0, x_0) \in E(G)$ for arbitrary $x_0 \in C$ and $p_0 \in F$. Then*
(i) $\lim_{n\to\infty} d(x_n, p_0)$ exists;
(ii) $\lim_{n\to\infty} d(x_n, S_1 x_n) = 0 = \lim_{n\to\infty} d(x_n, S_2 x_n)$.

Proof. (i) Let $p_0 \in F$. By Proposition (3.1), we have $(x_n, p_0), (y_n, p_0), (z_n, p_0) \in E(G)$. Then, by G-nonexpansiveness of S_1 and S_2 and using (1.1), we have

$$\begin{aligned} d(z_n, p_0) &= d((1-\beta_n)x_n \oplus \beta_n S_2 x_n, p_0) \\ &\leq (1-\beta_n)d(x_n, p_0) + \beta_n d(S_2 x_n, p_0) \\ &\leq (1-\beta_n)d(x_n, p_0) + \beta_n d(x_n, p_0) \\ &\leq d(x_n, p_0), \end{aligned} \tag{3.1}$$

$$\begin{aligned} d(y_n, p_0) &= d((1-\alpha_n)S_1 x_n \oplus \alpha_n S_2 z_n, p_0) \\ &\leq (1-\alpha_n)d(S_1 x_n, p_0) + \alpha_n d(S_2 z_n, p_0) \\ &\leq (1-\alpha_n)d(x_n, p_0) + \alpha_n d(z_n, p_0) \\ &\leq (1-\alpha_n)d(x_n, p_0) + \alpha_n d(x_n, p_0) \\ &\leq d(x_n, p_0), \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} d(x_{n+1}, p_0) &= d(S_1 y_n, p_0) \\ &\leq d(y_n, p_0) \\ &\leq d(x_n, p_0). \end{aligned} \tag{3.3}$$

This implies that sequence $\{d(x_n, p_0)\}$ is decreasing and bounded below for all $p_0 \in F(S)$. Hence $\lim_{n\to\infty} d(x_n, p_0)$ exists.
(ii) Assume that $\lim_{n\to\infty} d(x_n, p_0) = c$. If c=0, then by G-nonexpansiveness of S_1 and S_2 , we get

$$\begin{aligned} d(x_n, S_i x_n) &\leq d(x_n, p_0) + d(p_0, S_i x_n) \\ &\leq d(x_n, p_0) + d(p_0, x_n). \end{aligned}$$

Therefore, the result follows.
Suppose that $c > 0$.
Taking the lim sup on both sides in the inequality (3.1) and (3.2), we obtain

$$\limsup_{n\to\infty} d(z_n, p_0) \leq \limsup\nolimits_{n\to\infty} d(x_n, p_0) = c. \tag{3.4}$$

$$\limsup_{n\to\infty} d(y_n, p_0) \leq \limsup\nolimits_{n\to\infty} d(x_n, p_0) = c. \tag{3.5}$$

In addition, by G-nonexpansiveness of $S_i (i = 1, 2)$, we have $d(S_i y_n, p_0) \leq d(y_n, p_0)$ and $d(S_i z_n, p_0) \leq d(z_n, p_0)$.
We have,

$$\limsup_{n\to\infty} d(S_i y_n, p_0) \leq c. \tag{3.6}$$

and

$$\limsup_{n\to\infty} d(S_i z_n, p_0) \leq c. \tag{3.7}$$

Since $\lim_{n\to\infty} d(x_{n+1}, p_0) = c$. Letting $n \to \infty$ in the inequality (3.3), we have,

$$\lim_{n\to\infty} d(S_1 y_n, p_0) = c. \tag{3.8}$$

we have,

$$d(S_1 y_n, p_0) \leq d(y_n, p_0).$$

By taking the lim inf on both sides;

$$c \leq \liminf_{n\to\infty} d(y_n, p_0). \tag{3.9}$$

By using (3.5) and (3.9) we have

$$\lim_{n\to\infty} d(y_n, p_0) = c. \tag{3.10}$$

From (3.2) and (3.10), we have

$$\lim_{n\to\infty} d((1-\alpha_n)S_1 x_n \oplus \alpha_n S_2 z_n, p_0) = c. \tag{3.11}$$

In addition, $\limsup_{n\to\infty} d(S_1 x_n, p_0) \leq c$ and $\limsup_{n\to\infty} d(S_2 z_n, p_0)) \leq c$, using (3.11) and Lemma (2.6), we have

$$\lim_{n\to\infty} d(S_1 x_n, S_2 z_n) = 0. \tag{3.12}$$

We have,

$$\begin{aligned} d(x_{n+1}, p_0) &= d(S_1 y_n, p_0) \\ &\leq d(y_n, p_0) \\ &\leq d((1-\alpha_n)S_1 x_n \oplus \alpha_n S_2 z_n, p_0) \\ &\leq (1-\alpha_n) d(S_1 x_n, p_0) + \alpha_n d(S_2 z_n, p_0) \\ &\leq (1-\alpha_n) d(x_n, p_0) + \alpha_n d(z_n, p_0) \\ &\leq d(x_n, p_0) - \alpha_n d(x_n, p_0) + \alpha_n d(z_n, p_0) \end{aligned}$$

This implies that,

$$\frac{d(x_{n+1}, p_0) - d(x_n, p_0)}{\alpha_n} \leq d(z_n, p_0) - d(x_n, p_0)$$

So

$$d(x_{n+1}, p_0) - d(x_n, p_0) \leq \frac{d(x_{n+1}, p_0) - d(x_n, p_0)}{\alpha_n} \leq d(z_n, p_0) - d(x_n, p_0).$$

This implies,

$$d(x_{n+1}, p_0) \leq d(z_n, p_0).$$

By taking lim inf both sides, we have

$$c \leq \liminf_{n\to\infty} d(z_n, p_0). \tag{3.13}$$

By using (3.4) and (3.13), we get

$$\lim_{n\to\infty} d(z_n, p_0) = 0. \tag{3.14}$$

From, (3.1) and (3.14), we have

$$\lim_{n\to\infty} d((1-\gamma_n)x_n \oplus \gamma_n S_2 x_n, p_0) \quad = \quad 0. \tag{3.15}$$

In addition, $\limsup_{n\to\infty} d(x_n, p_0) \leq c$ and $\limsup_{n\to\infty} d((S_2 x_n, p_0) \leq c$. By using Lemma (2.6) and (3.15), we have

$$\lim_{n\to\infty} d(S_2 x_n, x_n) \quad = \quad 0. \tag{3.16}$$

Thus, it follows from (3.16) that

$$\begin{aligned} d(z_n, x_n) &= d((1-\beta_n)x_n \oplus \beta_n S_2 x_n, x_n) \\ &\leq (1-\beta_n)d(x_n, x_n) + \beta_n d(S_2 x_n, x_n) \\ &\to 0(as\ n \to \infty). \end{aligned} \tag{3.17}$$

Now, By using (3.12), (3.16) and (3.17) we have,

$$\begin{aligned} d(S_1 x_n, x_n) &\leq d(S_1 x_n, S_2 z_n) + d(S_2 z_n, x_n) \\ &\leq d(S_1 x_n, S_2 z_n) + d(S_2 z_n, S_2 x_n) + d(S_2 x_n, x_n) \\ &\leq d(S_1 x_n, S_2 z_n) + d(z_n, x_n) + d(S_2 x_n, x_n) \\ &\to 0(as\ n \to \infty) \end{aligned}$$

Therefore, we conclude $\lim_{n\to\infty} d(x_n, S_1 x_n) = 0 = \lim_{n\to\infty} d(x_n, S_2 x_n)$. This completes the proof. ∎

We now prove the Δ convergence of the sequence (1.1) for two G-nonexpansive mappings in CAT(0) space.

Theorem 3.3. *Let X, C, S_1, S_2, F and $\{x_n\}$ be same as in Proposition* (3.1) *with C has property DG. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequence in $[\delta, 1-\delta]$ for some $\delta \in (0,1)$ and $(x_0, p_0), (p_0, x_0) \in E(G)$ for arbitrary $x_0 \in C$ and $p_0 \in F$ then sequence $\{x_n\}$, Δ converges to a common fixed point of S_1 and S_2.*

Proof. Let $p_0 \in F$ be such that $(x_0, p_0), (p_0, x_0) \in E(G)$. From Lemma (3.2)(i), we have $\lim_{n\to\infty} d(x_n, p_0)$ exists, so $\{x_n\}$ is bounded. It follows from Lemma (3.2)(ii) that $\lim_{n\to\infty} d(x_n, S_1 x_n) = 0 = \lim_{n\to\infty} d(x_n, S_2 x_n)$. Let $W_\omega(\{x_n\}) =: \cup A(\{u_n\})$, where union is taken over all subsequence $\{u_n\}$ over $\{x_n\}$. To show the Δ-convergence of $\{x_n\}$ to a common fixed point of S_1 and S_2, we show that $W_\omega(\{(x_n\}) \subset F(S)$ and $W_\omega(\{x_n\})$ is a singleton set. To show that $W_\omega(\{x_n\}) \subset F(S)$ let $r \in W_\omega(\{(x_n\})$. Then, there exists a subsequence $\{r_n\}$ of $\{x_n\}$ such that $A(\{r_n\}) = r$. By Lemmas (2.2), there exists a subsequence $\{s_n\}$ of $\{r_n\}$ such that $\Delta - \lim_n s_n = s$ and $s \in C$. Since $\lim_{n\to\infty} d(s_n, S_1 s_n) = 0 = \lim_{n\to\infty} d(s_n, S_2 s_n)$ By Opial property

$$\limsup_{n\to\infty} d(s_n, s) < \limsup_{n\to\infty} d(s_n, S_1 s).$$

and

$$\limsup_{n\to\infty} d(s_n, s) < \limsup_{n\to\infty} d(s_n, S_2 s).$$

Hence $S_1s = S_2s = s$, i.e. $s \in F(S)$. Now, we claim that $s = r$. If not, by Lemma (3.1), $\lim_n d(x_n, s)$ exists and owing to the uniqueness of asymptotic centers,

$$\begin{aligned}\limsup_{n\to\infty} d(s_n, s) &< \limsup_{n\to\infty} d(s_n, r)\\ &\leq \limsup_{n\to\infty} d(r_n, r)\\ &< \limsup_{n\to\infty} d(r_n, s)\\ &= \limsup_{n\to\infty} d(x_n, s)\\ &= \limsup_{n\to\infty} d(s_n, s),\end{aligned}$$

which is a contradiction. Hence $r = s$. To assert that $W_\omega(\{(x_n\})$ is a singleton let $\{r_n\}$ be a subsequence of $\{x_n\}$. In view of Lemmas (2.2) and (2.3), there exists a subsequence $\{s_n\}$ of $\{r_n\}$ such that $\Delta - \lim_n s_n = s$. Let $A(\{r_n\}) = r$ and $A(\{x_n\}) = x$. Earlier, we have shown that $r = s$. Therefore it is enough to show $s = x.$. If $s \neq x$, then in view of Lemma (3.1) $\{d(x_n, s)\}$ is convergent. By uniqueness of asymptotic centers

$$\begin{aligned}\limsup_{n\to\infty} d(s_n, s) &< \limsup_{n\to\infty} d(s_n, x)\\ &\leq \limsup_{n\to\infty} d(x_n, x)\\ &< \limsup_{n\to\infty} d(x_n, s)\\ &= \limsup_{n\to\infty} d(s_n, s),\end{aligned}$$

which is a contradiction so that conclusion follows. ■

Theorem 3.4. *Let C be a nonempty compact convex subset subset of a complete CAT(0) space, S_1, S_2, F and $\{x_n\}$ be same as in Proposition (3.1) with C has property SG, $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequence in $[\delta, 1-\delta]$ for some $\delta \in (0, 1)$. If $(x_0, p_0), (p_0, x_0) \in E(G)$ for arbitrary $x_0 \in C$ and $p_0 \in F$, then $\{x_n\}$ converges strongly to a common fixed point of S_1 and S_2.*

Proof. We have $F(S) \neq \emptyset$ and so by Lemma (3.2) we have $\lim_{n\to\infty} d(x_n, S_1x_n) = 0 = \lim_{n\to\infty} d(x_n, S_2x_n)$. Since C is compact, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges strongly to p for some $p \in C$. By using the property SG and Lemma (3.2), we have

$$\lim_{n\to\infty} d(x_{n_k}, S_1x_{n_k}) = 0 = \lim_{n\to\infty} d(x_{n_k}, S_2x_{n_k})$$

Then

$$d(x_{n_k}, S_ip) \leq d(x_{n_k}, S_ix_{n_k}) + d(S_ix_{n_k}, S_ip)$$

$$d(x_{n_k}, S_ip) \leq d(x_{n_k}, S_ix_{n_k}) + d(x_{n_k}, p)$$

for all $n \geq 1$. Letting $k \to \infty$, we get $Tp = p$, i.e., $p \in F(S)$. By Lemma (3.2), $\lim_{n\to\infty} d(x_n, p)$ exists for every $p \in F(S)$ and so the sequence $\{x_n\}$ converge strongly to p. ■

Let C be a nonempty closed convex subset of a uniformly convex Banach space X. Recall that the mappings S_1 and S_2 on C are said to satisfy condition (B) [27] if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r > 0$ such that for all $x \in C$,

$$\max\{\|x - S_1x\|, \|x - S_2x\|\} \geq f(d(x, F)),$$

where $F = F(S_1) \cap F(S_2)$, $F(S_1)$ and $F(S_2)$ are the sets of fixed points of S_1 and S_2 and $d(x, F) = \inf\{\|x - q\| : q \in F\}$.

Theorem 3.5. *Let X, C, S_1, S_2 F and $\{x_n\}$ be same as in Proposition (3.1). Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequence in $[\delta, 1-\delta]$ for some $\delta \in (0, 1)$, $S_i(i = 1, 2)$ satisfies condition(B) and F is dominated by x_0 and F dominates x_0. Then $\{x_n\}$ converges strongly to a common fixed point of S_1 and S_2.*

Proof. From Lemma (3.2)(i), we have $\lim_{n\to\infty} d(x_n, q)$ exists and so $\lim_{n\to\infty} d(x_n, F)$ exists for any $q \in F$. Also from Lemma (3.2)(ii), $\lim_{n\to\infty} d(x_n, S_1x_n) = 0 = \lim_{n\to\infty} d(x_n, S_2x_n)$. Owing to condition (B),

$$f(d(x, F)) \leq \max\{d(x, S_1x), d(x, S_2x)\},$$

we have $\lim_{n\to\infty} f(d(x_n, F)) = 0$. As $f : [0, \infty) \to [0, \infty)$ is a nondecreasing function satisfying $f(0) = 0, f(r) > 0$ for all $r \in [0, \infty)$, we obtain that $\lim_{n\to\infty} d(x_n, F) = 0$.
Hence, we can find a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and a sequence $\{u_j\} \subset F$ such that $d(x_{n_j}, u_j) \leq \frac{1}{2^j}$. Put $n_{j+1} = n_j + h$ for some $h \geq 1$. Then

$$d(x_{n_{j+1}}, u_j) \leq d(x_{n_j+h-1}, u_j) \leq d(x_{n_j}, u_j) \leq \tfrac{1}{2^j}.$$

$$d(u_{j+1}, u_j) \leq d(u_{j+1}, x_{j+1}) + d(x_{j+1}, u_j) \leq \tfrac{1}{2^{j+1}} + \tfrac{1}{2^j} = \tfrac{1}{2^{j-1}}.$$

So $\{u_j\}$ is a Cauchy sequence. We assume that $u_j \to q_0 \in C$ as $j \to \infty$. Since F is closed, we get $q_0 \in F$. So we have $x_{n_j} \to q_0$ as $j \to \infty$. Since $\lim_{n\to\infty} d(x_n, q_0)$ exists, we get $x_n \to q_0$. This completes the proof. ∎

Theorem 3.6. *Let C be a nonempty closed convex subset of a complete CAT(0) space with property SG, $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequence in $[\delta, 1-\delta]$ for some $\delta \in (0, 1)$, F is dominated by x_0 and F dominates x_0. If one of $S_i(i = 1, 2)$ is semicompact then $\{x_n\}$ converges strongly to a common fixed point of S_1 and S_2.*

Proof. It follows from Lemma (3.2) $\{x_n\}$ is bounded and $\lim_{n\to\infty} d(x_n, S_1x_n) = 0 = \lim_{n\to\infty} d(x_n, S_2x_n)$. Since one of S_1 and S_2 is semicompact, then there exists a subsequences $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \to q \in C$ as $j \to \infty$. Since C has Property SG and transitivity of graph G, we obtain $(x_{n_j}, q) \in E(G)$. Notice that, for each $i \in 0, 1$, $\lim_{j\to\infty} d(x_{n_j}, S_ix_{n_j}) = 0$.
Then

$$\begin{aligned} d(q, S_iq) &\leq d(q, x_{n_j}) + d(x_{n_j}, S_ix_{n_j}) + d(S_ix_{n_j}, S_iq) \\ &\leq d(q, x_{n_j}) + d(x_{n_j}, S_ix_{n_j}) + d(x_{n_j}, q) \\ &\to 0(as\ j \to \infty) \end{aligned}$$

Hence $q \in F$. Thus $\lim_{n\to\infty} d(x_n, F)$ exists by Theorem 3.5. We note that $d(x_{n_j}, F) \leq d(x_{n_j}, q) \to 0$ as $j \to \infty$. Hence $\lim_{n\to\infty} d(x_n, F) = 0$. It follows, as in the proof of Theorem (3.5), that $\{x_n\}$ converges strongly to a common fixed point of S_1 and S_2. This completes the proof. ∎

4. Numerical Example

Now we will discuss a numerical experiments which support our main theorem.

Example 4.1. Let $X = \mathbb{R}$ and $C = [0, 2]$. Let $G = (V(G), E(G))$ be a directed graph defined by $V(G) = C$ and $(x, y) \in E(G)$ if and only if $0.65 \leq x, y \leq 1.65$. Define a mapping $S_1, S_2 : C \to C$ by

$$S_1 x = x^{2\log x},$$
$$S_2 x = x^{\frac{1}{3}},$$

for any $x \in C$

It is easy to show that S_1, S_2 are G-nonexpansive but S_1, S_2 are not nonexpansive because

$$|S_1 x - S_1 y| > |x - y|,$$

and

$$|S_2 u - S_2 v| > |u - v|,$$

when $x = 0.1$, $y = 1.90$, $u = 0.5$ and $v = 0.03$.

iteration no.	modified Ishikawa	modified S-iteration	Thianwan's new	modified Picard S-iteration
1	1.450000000000000	1.450000000000000	1.450000000000000	1.450000000000000
2	1.446937747316145	1.135151919359837	1.131080728764182	1.043236108230096
3	1.439978424804714	1.042576363439713	1.038677525097028	1.004565686484902
4	1.428350360114987	1.013325277028436	1.010915892564614	1.000476511629518
5	1.411433812946091	1.004089484416440	1.002905477559686	1.000048707721444
6	1.388873575644693	1.001227490994499	1.000726883239842	1.000004870758310
7	1.360698339367269	1.000360147984224	1.000170618155917	1.000000476251854
8	1.327418711668000	1.000103251370589	1.000037500910039	1.000000045508510
9	1.290068993515692	1.000028911223261	1.000007700340779	1.000000004247461
10	1.250161362039185	1.000007902472312	1.000001473338871	1.000000000386991
11	1.209538616935949	1.000002107331501	1.000000261927179	1.000000000034399
12	1.170140985639988	1.000000547906589	1.000000043130684	1.000000000002981
13	1.133734370966920	1.000000138803029	1.000000006555864	1.000000000000252
14	1.101667785773435	1.000000034238082	1.000000000916364	1.000000000000021
15	1.074723914188363	1.000000008217140	1.000000000117295	1.000000000000002
16	1.053096214938510	1.000000001917333	1.000000000013684	1.000000000000000
17	1.036481654435387	1.000000000434595	1.000000000001448	1.000000000000000
18	1.024242648677044	1.000000000095611	1.000000000000138	1.000000000000000

Table 1. Comparative Sequences

Let $\alpha_n = \frac{n}{50}$, $\beta_n = \frac{n}{20}$. Choose $x_0 = y_0 = z_0 = 1.45$ and $x = 1$ is a common fixed point of S_1 and S_2. Let $\{x_n\}$ be the sequence generated by (1.1) and $\{y_n\}$, $\{z_n\}$ and $\{t_n\}$ be sequences generated by Thianwan's new iteration, modified S-iteration and modified Ishikawa iteration, respectively. We get the following numerical experiments for common fixed point of S_1 and S_2 and rate of convergence of $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{t_n\}$.
Table 1 shows the numerical experiment for supporting our main results and comparing rate of convergence of modified Picard-S iteration with Thianwan's new iteration, modified S-iteration and modified Ishikawa iteration.

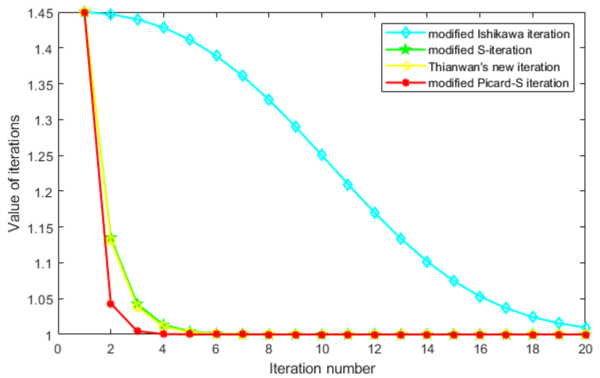

FIGURE 1. Numerical experiment of Example 4.1 using Ishikawa iteration, modified S-iteration, Thianwan's new iteration and modified Abbas iteration.

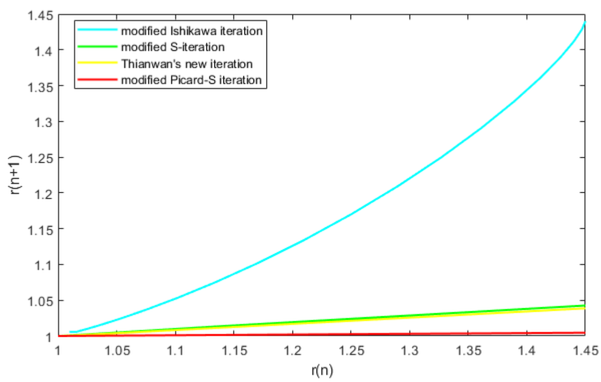

FIGURE 2. Convergence comparision of sequence generated by Ishikawa iteration, modified S-iteration, Thianwan's new iteration and modified Picard-S iteration for example 4.1.

Figure 1 shows the convergence of Ishikawa iteration, modified S-iteration, Thianwan's new iteration and modified Abbas iteration to the common fixed point of S_1 and S_2 which is 1 in this numerical experiment and proposed iteration process converges faster than other iteration process.

ACKNOWLEDGEMENTS

I would like to thank the referee(s) for his comments and suggestions on the manuscript. Nuttapol Pakkaranang would like to thank the Thailand Research Fund (TRF) and King Mongkut's University of Technology Thonburi (KMUTT) for their joint support through the Royal Golden Jubilee Ph.D. (RGJ-PHD) Program (Grant No. PHD/0205/2561).

References

[1] M. Abbas, T. Nazir, A new faster iteration process applied to constrained minimization and feasibility problems, Mat. Vesnik 66(2)(2014) 223–234.

[2] R.P. Agarwal, D. O'Regan, D.R. Sahu, Iterative construction of fixed points of nearly asymptotically nonexpansive mappings, J. Nonlinear Convex Anal. 8(1)(2007) 61–79.

[3] M.R. Alfuraidan, Fixed points of monotone nonexpansive mappings with a graph, Fixed Point Theory Appl. 49(2015).

[4] M.R. Alfuraidan, M.A. Khamsi, Fixed points of monotone nonexpansive mappings on a hyperbolic metric space with a graph, Fixed Point Theory Appl. 44(2015).

[5] S.M.A. Aleomraninejad, Sh. Rezapour, N. Shahzad, Some fixed point result on a metric space with a graph, Topology Appl. 159(3)(2012) 659-663.

[6] S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales, Fund. Math. 3(1922) 133–181.

[7] V. Berinde, Iterative Approximation of Fixed Points, Editura Efemeride, Baia Mare, 2002.

[8] F.E. Browder, Convergence of approximants to fixed points of nonexpansive nonlinear mappings in Banach spaces, Arch. Rational Mech. Anal. 24(1967) 82–90.

[9] R. Chugh, V. Kumar, S. Kumar, Strong convergence of a new three step iterative scheme in Banach spaces, Amer. J. Comput. Math. 2(4)(2012) 345–357.

[10] S. Dhompongsa, W.A. Kirk, B. Panyanak, Nonexpansive set-valued mappings in metric and Banach spaces, J. Nonlinear Convex Anal. 8(1)(2007) 35–45.

[11] S. Dhompongsa, B. Panyanak, On Δ-convergence theorems in CAT(0) space, Comput. Math. Appl. 56(10)(2008) 2572–2579.

[12] D. Göhde, Zum Prinzip der kontraktiven Abbildung, Math. Nachr. 30(1965) 251–258.

[13] F. Gursoy, V. Karakaya, A Picard-S hybrid type iteration method for solving a differential equation with retarded argument, arXiv:1403.2546 (2014).

[14] B. Halpern, Fixed points of nonexpanding maps, Bull. Amer. Math. Soc. 73(1967) 957–961 .

[15] S. Ishikawa, Fixed points by a new iteration method, Proc. Amer. Math. Soc. 44(1974) 147–150 .

[16] J. Jachymski, The contraction principle for mappings on a metric space with a graph, Proc. Am. Math. Soc. 136(4)(2008) 1359–1373.

[17] W.A. Kirk, A Fixed point theorem for mappings which do not increase distances. Amer. Math. Monthly 72(1965) 1004–1006.

[18] W.A. Kirk, B. Panyanak, A concept of convergence in geodesic spaces, Nonlinear Anal. 68(2008) 3689-3696.

[19] T.C. Lim, Remarks on some fixed point theorems, Proc. Amer. Math. Soc. 60(1976) 179–182.

[20] W.R. Mann, Mean value methods in iteration, Proc. Am. Math. Soc. 4(1953) 506–510.

[21] M.A. Noor, New approximation schemes for general variational inequalities, J. Math. Anal. Appl. 251(1)(2000) 217–219.

[22] W. Phuengrattana, S. Suantai, On the rate of convergence of Mann, Ishikawa, Noor and SP-iterations for continuous functions on an arbitrary interval, J. Comput. Appl. Math. 235(2011) 3006–3014.

[23] B.E. Rhoades, Comments on two fixed point iteration method, J. Math. Anal. Appl. 56(3)(1976) 741–750.

[24] M.G. Sangago, Convergence of iterative schemes for nonexpansive mappings, Asian-Eur. J. Math. 4(4)(2011) 671–682.

[25] J. Schu,Weak and strong convergence to fixed points of asymptotically nonexpansive mappings, Bull. Austral. Math. Soc. 43(1)(1991) 153–159.

[26] S. Suantai, Weak and strong convergence criteria of Noor iterations for asymptotically nonexpansive mappings, J. Math. Anal. Appl. 331(2)(2005) 506–517.

[27] N. Shahzad, R. Al-Dubiban, Approximating common fixed points of nonexpansive mappings in Banach spaces, Georgian Math. J. 13(3)(2006) 529–537.

[28] R. Suparatulatorn, W. Cholamjiak, S. Suantai, A modified S-iteration process for G-nonexpansive mappings in Banach spaces with graphs, Numer. Algorithms 77(4)(2017) 479–490.

[29] K.K. Tan, H.K. Xu, Approximating fixed points of nonexpansive mapping by the Ishikawa iteration process, J. Math. Anal. Appl. 178(2)(1993) 301–308.

[30] T. Thianwan, D. Yambangwai, Convergence analysis for a new two-step iteration process for G-nonexpansive mappings with directed graphs, J. Fixed Point Theory Appl. 21(2)(2019) 1–16.

[31] J. Tiammee, A. Kaewkhao, S. Suantai, On Browder's convergence theorem and Halpern iteration process for G-nonexpansive mappings in Hilbert spaces endowed with graphs, Fixed Point Theory Appl. 2015(187)(2015) 1–12.

[32] O. Tripak, Common fixed points of G-nonexpansive mappings on Banach spaces with a graph, Fixed Point Theory Appl. 87(2016) 1–8.

[33] P. Cholamjiak, SP-iteration process for nonexpansive mappings in CAT(κ) spaces, J. Nonlinear Convex Anal. 16(2015) 109–118.

[34] R. Suparatulatorn, P. Cholamjiak, The modified S-iteration process for nonexpansive mappings in CAT(κ) spaces, Fixed Point Theory Appl. 25(2016) 1–12.

[35] P. Cholamjiak, A.A. Abdou, Y.J. Cho, Proximal point algorithms involving fixed points of nonexpansive mappings in CAT(0) spaces, Fixed Point Theory Appl. 227(2015) 1–13.

[36] R. Suparatulatorn, P. Cholamjiak, S. Suantai, On solving the minimization problem and the fixed point problem for nonexpansive mappings in CAT(0) spaces, Optim. Methods Softw. 32(2017) 182–192.

[37] N. Pakkaranang, P. Kumam, P. Cholamjiak, N. Pholasa, Convergence analysis of modified Picard-S hybrid with errors for total asymptotically nonexpansive mappings in CAT(0) spaces, Linear Nonlinear Anal. 4(3)(2018) 377–391.

[38] W. Kumam, N. Pakkaranang, P. Kumam, Modified viscosity type iteration for total asymptotically nonexpansive mappings in CAT(0) spaces and its application to optimization problems, J. Nonlinear Sci. Appl. 11(2018) 288–302.

[39] N. Pakkaranang, P. Sa Ngiamsunthorn, P. Kumam, Y.J. Cho, Convergence theorems of the modified S-type iterative method for (α, β)-generalized hybrid mappings in CAT(0) spaces, J. Math. Anal. 8(1)(2017) 103–112.

[40] P. Chaipunya, P. Kumam, On the proximal point method in Hadamard spaces, Optimization 66(2017) 1647–1665.

[41] P. Kumam, D. Kitkuan, A. Padcharoen, P. Kumam, Proximal point algorithm for nonlinear multivalued type mappings in Hadamard spaces, Math. Meth. Appl. Sci. 42(17)(2019) 5758–5768.