Thai Journal of Mathematics
Special Issue (Annual Meeting in Mathematics, 2007) : 11–23
www.math.science.cmu.ac.th/thaijournal

On Strongly θ - Semi - Continuous Functions

N. Puturong

Abstract : A function $f : X \to Y$ from a topological space X into a topological space Y to be strongly θ-semi-continuous if and only if for each $x \in X$ and each open set V containing $f(x)$, there exists a semi- open set U containing x such that $f(scl\ U) \subset V$. In this paper gives some characterizations of strongly θ-semi-continuous functions , including to apply strongly θ-semi-continuous to the retraction and strongly θ-semi-continuous fixed point property.

Keywords : semi-open set , semiclosure , semi θ-closure , semi θ-interior , sθ-converges, strongly θ-semi-continuous , strongly θ-semi-continuous retraction , strongly θ-semi-continuous fixed point property.

1 Introduction

N. Levine [3] has defined a function $f : X \to Y$ from a topological space X into a topological space Y to be semi-continuous (denoted by "s.c) if $f^{-1}(U)$ is semi-open set in X for every open set U in Y. Also, T. Noiri [6] has independently defined a function $f : X \to Y$ from a topological space X into a topological space Y to be strongly θ-continuous (denoted by $st.\ \theta c$) if for each $x \in X$ and each open set V containing $f(x)$, there exists an open set U containing x such that $f\left(\overline{U}\right) \subset V$. Seong Hoon Cho [4] has the notion of a type of converges for nets that we called sθ-converges.

F. Cammaroto and T. Noiri [1] defined and investigated the δ-continuous retraction and the δ-continuous fixed point property.

In the present paper, author has define and study the strongly θ-semi-continuous functions. In section 2, preliminaries. Section 3 gives some characterizations of strongly θ-semi-continuous functions. Section 4 deals with the retraction of a topological space by strongly θ-semi-continuous functions and the fixed point property in relation to strongly θ-semi-continuous functions.

2 Preliminaries

Definition 2.1. *[3] Let A be subset of a topological space X. A is said to be semi-open set in X if there exists an open set O of X such that $O \subset A \subset \overline{O}$. We will be denoted the class of all semi-open sets in X by S.O.(X).*

Remark 2.2. *[3] If O is an open set in X then $O \in S.O.(X)$.*

Definition 2.3. *[5] Let A be subset of a topological space X.*
(1) A is said to be semi-closed set in X if $A^c \in S.O.(X)$.
(2) The semi-closure of A , denoted by scl A ,
$scl\ A = \cap\{F/F \text{ is semi} - \text{closed set in } X \text{ such that } A \subset F\}$
(3) The semi-interior of A, denoted by sInt A ,
$sInt\ A = \cup\{O/O \in S.O.(X) \text{such that } O \subset A\}$.

Theorem 2.4. *[5] Let A and B be subsets of a topological space X. Then :*
(1) If A is closed set, then A is semi-closed set.
(2) $A \subset scl\ A \subset \overline{A}$.
(3) If $A \subset B$, then $scl\ A \subset scl\ B$.
(4) A is semi-closed set if and only if $scl\ A = A$.
(5) $A \in S.O.(X)$ if and only if $sInt\ A = A$.
(6) $sInt\ A = X - scl\ (X - A)$.
(7) $sInt\ A \subset A$.

Definition 2.5. *[2] Let A be subset of a topological space X and $x \in X$. A point x is called a semi θ-adherent point of A if $scl\ U \cap A \neq \emptyset$ for every semi-open set U containing x. The set of all semi θ-adherent points of A is called the semi θ-closure of A and is denoted by $scl_\theta A$.*

Definition 2.6. *Let D be a directed set and (x_d) is a net in a topological space X. A net (x_d) is said to $s\theta$-converges to $x_0 \in X$ if for each semi-open set U containing x_0, there exists $d_0 \in D$ such that $x_d \in scl\ U$ for all $d \geqslant d_0$.*

3 Characterizations

Definition 3.1. *Let $X = (X, \tau_X)$ and $Y = (Y, \tau_Y)$ be topological spaces.*
A mapping $f : X \to Y$ is said to be strongly θ-semi-continuous at a point $x_0 \in X$ if for each open set V containing $f(x_0)$, there exists a semi-open set U containing x_0 such that $f(scl\ U) \subset V$. f is said to be strongly θ-semi-continuous on X if it is strongly θ-semi-continuous at every point of X, we shall denote by f is st. θsc on X .

Example 3.2. *Let $X = \{a, b, c, d\}, Y = \{x, y, z\}$,*
$\tau_X = \{\emptyset, \{b\}, \{c\}, \{b, c\}, \{a, b, c\}, X\}$ be topology on X,
$\tau_y = \{\emptyset, \{y\}, \{z\}, \{y, z\}, Y\}$ be topology on Y
and $f : X \to Y$ such that $f(a) = x, f(b) = y and f(c) = f(d) = z$.
Show that f is st. θsc on X.

Solution For $(X, \tau_x\}$,
Closed sets in X ; $\emptyset, \{d\}, \{a, d\}, \{a, b, d\}, \{a, c, d\}$ and X.
Semi-open sets in $X(S.O.(X)); \emptyset, \{b\}, \{c\}, \{b, c\}, \{a, b\}, \{b, d\}, \{a, c\}, \{c, d\}, \{a, c, d\}$, $\{b, c, d\}, \{a, b, d\}$, $\{a, b, c\}$ and X
Semi-closed sets in X ; $\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, d\}, \{c, d\}, \{a, b, d\}$, $\{a, c, d\}$ and X.

Consider at a point a :
Let $V \in \tau_Y$ such that $f(a) \in V$ then V is to be Y. There exists $\{a, c\} \in S.O.(X)$ such that $a \in \{a, c\}$ and $f(scl\{a, c\}) = f(\{a, c\}) = \{x, z\} \subset V$. Hence f is $st.\ \theta sc$ at a point a.

Consider at a point b :
Let $V \in \tau_Y$ such that $f(b) \in V$ then V is to be $\{y\}, \{y, z\}$ and Y. There exists $\{b\} \in S.O.(X)$ such that $b \in \{b\}$ and $f(scl\{b\}) = f(\{b\}) = \{y\} \subset V$. Hence f is $st.\theta sc$ at a point b.

Consider at a point c :
Let $V \in \tau_Y$ such that $f(c) \in V$ then V is to be $\{z\}, \{y, z\}$ and Y. There exists $\{c\} \in S.O.(X)$ such that $c \in \{c\}$ and $f(scl\{c\}) = f(\{c\}) = \{z\} \subset V$. Hence f is $st.\ \theta sc$ at a point c.

Consider at a point d :
Let $V \in \tau_Y$ such that $f(d) \in V$ then V is to be $\{z\}, \{y, z\}$ and Y. There exists $\{c, d\} \in S.O.(X)$ such that $d \in \{c, d\}$ and $f(scl\ \{c, d\}) = f(\{c, d\}) = \{z\} \subset V$. Hence f is $st.\ \theta sc$ at a point d. Therefore, f is $st.\ \theta sc$ on X.

Example 3.3. *Let $X \neq \emptyset$ and $X = Y$. (X, τ_X) and (Y, τ_Y) be topological spaces such that τ_Y be trivial topology on Y, $f : X \to Y$ such that $f(x) = x$ for all $x \in X$. Show that f is st. θsc on X.*

Solution Since $f(x) = x$ for all $x \in X$ and τ_Y be trivial topology on Y, then an open set V containing $f(x)$ is to be Y. There exists an open set X such that $x \in X$, thus there exists a semi-open set X such that $x \in X$ while X be a closed set. By Theorem 2.4(1), we have X be a semi-closed set and by Theorem 2.4(4), we have $scl\ X = X$. Hence $f(scl\ X) = f(X) \subset Y$, thus f is $st.\ \theta sc$ at a point x. Therefore, f is $st.\ \theta sc$ on X.

Remark 3.4. *Let X and Y be topological spaces and $f : X \to Y$. Then :*
(1) If f is st. θc on X, then f is st. θsc on X.
(2) If f is st. θsc on X, then f is s.c on X.

Proof. (1) Let f is to be $st.\ \theta c$ on X. Then for each $x \in X$ and $V \in \tau_Y$ such that $f(x) \in V$, there exists an open set U such that $x \in U$ and $f(\overline{U}) \subset V$. Since if $U \in \tau_x$ then $U \in S.O.(X)$ and since $scl\ U \subset \overline{U}$, There exists a semi-open set U such that $x \in U$ and $f(scl\ U) \in f(\overline{U})$. Since $f(\overline{U}) \subset V$, hence we have $f(scl\ U) \subset V$. Therefore, f is $st.\ \theta sc$ on X.

(2) Let f is to be $st.\ \theta sc$ on X. Then for each $x \in X$ and $V \in \tau_Y$ such that $f(x) \in V$, there exists a semi-open set U of X such that $x \in U$ and $f(scl\ U) \subset V$.

Since $U \subset sclU$, we have $f(U) \subset f(scl\ U)$. Hence $f(U) \subset V$.Thus there exists a semi-open set U of X such that $x \in U$ and $f(U) \subset V$. Therefore, f is $s.c$ on X. $\square$

The converse of Remark 3.4(1), (2) are false, as shown by Example 3.5(1), (2).

Example 3.5. *(1) From Example 3.2, show that f is not st. θc on X*

Solution Consider at a point b :
Let $V \in \tau_Y$ such that $f(b) \subset V$, hence V is to be $\{y\}, \overline{\{y,z\}}$ and Y. Open sets in X containing b ; $\{b\}, \{b,c\}, \{a,b,c\}$ and X. Since $f(\overline{\{b\}}) = f(\{a,b,d\}) = \{x,y,z\} \not\subset V, f(\overline{\{b,c\}}) = f(X) = \{x,y,z\} \not\subset V, f(\overline{\{a,b,c\}}) = f(X) = \{x,y,z\} \not\subset V$ *and* $f(\overline{X}) = f(X) = \{x,y,z\} \not\subset V$ *for some* $V \in \tau_Y$. Hence f is not $st.\ \theta c$ on X. Therefore, f is $st.\theta sc$ on X but it is not $st.\ \theta c$ on X.

(2) Let $X = \{a,b,c,d\}, Y = \{1,2,3,4\}, \tau_X = \{\emptyset, \{a\}, \{c\}, \{a,c\}, \{b,c\}, \{a,b,c\}, X\}$ *be topology on X,* $\tau_Y = \{\emptyset, \{1\}, \{3\}, \{1,3\}, \{1,2\}, \{1,2,3\}, Y\}$ *be topology on Y, and* $f : X \to Y$ *such that* $f(a) = 3, f(b) = 2, f(c) = 1$ *and* $f(d) = 4$. *Show that f is s.c on X but it is not st.* θsc *on X.*

Solution For (X, τ_X) , Closed sets in $X; X, \{b,c,d\}, \{a,b,d\}, \{b,d\}, \{a,d\}, \{d\}$ and $\emptyset$. Semi-open sets in X ; $\emptyset, \{a\}, \{c\}, \{a,c\}, \{b,c\}, \{a,d\}, \{c,d\}, \{a,b,c\}, \{b,c,d\}, \{a,c,d\}$ and X. Semi-closed sets in X ; $X, \{b,c,d\}, \{a,b,d\}, \{b,d\}, \{a,d\}, \{b,c\}, \{a,b\}, \{a\}, \{b\}, \{d\}$ and $\emptyset$.

(1) Show that f is $s.c$ on X.
Open sets in Y ; $\emptyset, \{1\}, \{3\}, \{1,3\}, \{1,2\}, \{1,2,3\}$ and Y. Since $f^{-1}(\emptyset) = \emptyset$, $f^{-1}(\{1\}) = \{c\}$, $f^{-1}(\{3\}) = \{a\}$, $f^{-1}(\{1,3\}) = \{a.c\}$, $f^{-1}(\{1,2\}) = \{b,c\}$, $f^{-1}(\{1,2,3\}) = \{a,b,c\}$ and $f^{-1}(Y) = X$. Hence for each open set V in Y, we have $f^{-1}(V) \in S.O.(X)$. Therefore, f is $s.c$ on X.

(2) Show that f is not $st.\theta sc$ on X.
Consider at a point c :
Let $V \in \tau_Y$ such that $f(c) \in V$, hence V is to be $\{1\}, \{1,3\}, \{1,2\}, \{1,2,3\}$ and Y. Semi - open sets in X containing c ; $\{c\}, \{a,c\}, \{b,c\}, \{c,d\}, \{a,b,c\}, \{b,c,d\}, \{a,c,d\}$ and X. Since $f(scl\{c\}) = f(X \cap \{b,c,d\} \cap \{b,c\}) = f(\{b,c\}) = \{1,2\} \not\subset V$, $f(scl\{a,c\}) = f(X) = \{1,2,3,4\} \not\subset V$, $f(scl\{b,c\}) = f(\{b,c\}) = \{1,2\} \not\subset V$, $f(scl\{c,d\}) = f(X \cap \{b,c,d\}) = f(\{b,c,d\}) = \{1,2,4\} \not\subset V$, $f(scl\{a,b,c\}) = f(X) = \{1,2,3,4\} \not\subset V$, $f(scl\{b,c,d\}) = f(\{b,c,d\}) = \{1,2,4\} \not\subset V$, $f(scl\{a,c,d\}) = f(X) = \{1,2,3,4\} \not\subset V$, and $f(sclX) = f(X) = \{1,2,3,4\} \not\subset V$ for some $V \in \tau_Y$.

Hence f is not $st.\theta sc$ at a point c. Therefore , f is not $st.\ \theta sc$ on X.

Remark 3.6. *Let X and Y be topological spaces. If* $f : X \to Y$ *is continuous on X and* (Y, τ_Y) *is regular space, then f is st.* θsc *on X.*

Proof. Let $x \in X$ and $V \in \tau_Y$ such that $f(x) \in V$. We have V^c is a closed set in Y and $f(x) \notin V^c$. Since (Y, τ_Y) is regular space. There exists $A, B \in \tau_Y$ such that $A \cap B = \emptyset$, $f(x) \in A$ and $V^c \subset B$, we have $A \subset B^c$. Hence $\overline{A} \subset \overline{B^c}$ and $B^c \subset V$. Since B^c is a closed set, hence $B^c = \overline{B^c}$. Thus $\overline{A} \subset V$. Since f is continuous on X and $A \in \tau_Y$, we have $f^{-1}(A) \in \tau_X$. Since $f(x) \in A$, we have $x \in f^{-1}(A)$. Hence $x \in f^{-1}(A) \subset scl\ f^{-1}(A) \subset \overline{f^{-1}(A)} \subset f^{-1}(\overline{A}) \subset f^{-1}(V)$. Thus $f(scl\ f^{-1}(A)) \subset V$. Since $f^{-1}(A)$ is an open set, we have $f^{-1}(A)$ is a semi-open set. There exists a semi- open set $U = f^{-1}(A)$ in X such that $x \in U$ and $f(scl\ U) \subset V$. Therefore, f is *st.* θsc on X. □

From Remark 3.6 if (Y, τ_Y) is not regular space then f is not necessary to be *st.* θsc on X, as show by Example 3.7.

Example 3.7. *From Example 3.5(2) , show that if f is continuous on X and (Y, τ_Y) is not regular space, then f is not st. θsc on X.*

Solution Open sets in Y ; $\emptyset, \{1\}, \{3\}, \{1, 3\}, \{1, 2\}, \{1, 2, 3\}$ and Y. Closed sets in Y ; $Y, \{2, 3, 4\}, \{1, 2, 4\}, \{2, 4\}, \{3, 4\}, \{4\}$ and $\emptyset$. Since $f^{-1}(\emptyset) = \emptyset$, $f^{-1}(\{1\}) = \{c\}$, $f^{-1}(\{3\}) = \{a\}$, $f^{-1}(\{1, 3\}) = \{a, c\}$, $f^{-1}(\{1, 2\}) = \{b, c\}$, $f^{-1}(\{1, 2, 3\}) = \{a, b, c\}$ and $f^{-1}(Y) = X$. Thus for each open sets V in Y , hence $f^{-1}(V)$ is open sets in X. Therefore , f is continuous on X.

Next , we shall show that (Y, τ_Y) is not regular space. Consider at a point 2 , for $2 \in Y$ and a closed set $\{4\}$ such that $2 \notin \{4\}$. Open sets in Y containing 2 ; $\{1, 2\}, \{1, 2, 3\}$ and Y. Since $\{4\} \subset Y, \{1, 2\} \cap Y \neq \emptyset, \{1, 2, 3\} \cap Y \neq \emptyset$ and $Y \cap Y \neq \emptyset$, hence (Y, τ_Y) is not regular space. By Example 3.5(2) , we have f is not *st.* θsc on X. Therefore , if f is continuous on X and (Y, τ_Y) is not regular space then f is not *st.* θsc on X.

The converse of Remark 3.6 is false, as shown by Example 3.8.

Example 3.8. *From Example 3.2 , show that f is not continuous on X.*

noindent **Solution** Since $\{z\} \in \tau_Y$. But $f^{-1}(\{z\}) = \{c, d\} \notin \tau_X$. Hence , f is not continuous on X. Therefore , f is *st.* θsc on X but it is not continuous on X.

Definition 3.9. *Let A be subset of a topological space X. The semi θ-interior of A , denoted by $sInt_\theta A$,*

$$sInt_\theta A = \cup \{O | O \in S.O.(X) \text{ such that } scl\ O \subset A\}.$$

Example 3.10. *Let $X = \{a, b, c, d\}$ and $\tau_X = \{\emptyset, \{a\}, \{b, c\}, \{a, d\}, \{a, b, c\}, X\}$ be topology on X. Find $sInt_\theta\{a\}$, $sInt_\theta\{b\}$, $sInt_\theta\{b, c\}$, $sInt_\theta\{b, d\}$, $sInt_\theta\{a, c, d\}$ and $sInt_\theta\ X$.*

Solution Open sets in X ; $\emptyset, \{a\}, \{b,c\}, \{a,d\}, \{a,b,c\}$ and X.
Closed sets in X ; $X, \{b,c,d\}, \{a,d\}, \{b,c\}, \{d\}$ and $\emptyset$.
Semi-open sets in X ; $\emptyset, \{a\}, \{b,c\}, \{a,d\}, \{a,b,c\}$ and X.
Semi-closed sets in X ; $X, \{b,c,d\}, \{a,d\}, \{b,c\}, \{d\}$ and $\emptyset$.

Since $scl\ A = \cup\{F|F\ is\ semi-closed\ set\ in\ X\ such\ that\ A \subset F\}$, hence $scl\ \emptyset = \emptyset$, $scl\ \{a\} = \{a,d\} \cap X = \{a,d\}$, $scl\ \{b,c\} = \{b,c\}$, $scl\ \{a,d\} = \{a,d\}$, $scl\ \{a,b,c\} = X$ and $scl\ X = X$.

Since $sInt_\theta A = \cup\{O|O \in S.O.(X)\ such\ that\ \ scl\ O \subset A\}$, hence $sInt_\theta\{a\} = \emptyset, sInt_\theta\{b\} = \emptyset, sInt_\theta\{b,c\} = \{b,c\}, sInt_\theta\{a,c,d\} = \{a\}\cup\{a,d\} = \{a,d\}, Int_\theta\{b,d\} = \emptyset$ and $sInt_\theta = X$.

Lemma 3.11. *Let A be subset of a topological space X. Then :*
(1) $sInt_\theta(X-A) = X - scl_\theta A$.
(2) $sInt_\theta A \subset sInt_\theta A$.
(3) $A \subset scl_\theta A$.

Proof. (1) $sInt_\theta(X-A) = X - scl_\theta A$.
($\Rightarrow$) Let $x \in sInt_\theta(X-A)$. Then there exists a semi-open set U containing x such that $x \in scl\ U \subset (X-A)$. Thus $scl\ U \cap A = \emptyset$ and $x \notin scl_\theta A$. Hence $x \in X - scl_\theta A$. Therefore , we obtain $sInt_\theta(X-A) \subset X - scl_\theta A$.
($\Leftarrow$) Let $x \in X - scl_\theta A$. Then $x \notin scl_\theta A$. There exists a semi - open set U containing x such that $scl\ U \cap A = \emptyset$. So, $x \in U \subset scl\ U \subset X - A$. Hence $x \in sInt_\theta(X-A)$. Therefore, we obtain $X - scl_\theta A \subset sInt_\theta(X-A)$.

(2) $sInt_\theta A \subset sInt_\theta A$.
Let $x \in X$ and $x \in sInt_\theta A$. Since $sInt_\theta A = \cup\{O|O \in S.O.(X) such\ thatscl\ O \subset A\}$, hence $x \in \cup\{O|O \in S.O.(X) such\ thatscl\ O \subset A\}$. By Theorem 2.4 (2) , we have $O \subset scl\ O$. Hence, we have $x \in \cup\{O|O \in S.O.(X) such\ thatO \subset A\}$. By Definition 2.3(3), hence $x \in sInt\ A$. Therefore, $sInt_\theta A \subset sInt\ A$.

(3)$A \subset scl_\theta A$.
Since $sInt_\theta(X-A) = X - scl_\theta A$ and $sInt\ A = X - scl(X-A)$. By (2), we have $sInt_\theta(X-A) \subset sInt(X-A)$. Thus $X - scl_\theta A \subset X - scl\ A$. Hence $scl\ A \subset scl_\theta A$. By Theorem 2.4(2), we have $A \subset scl\ A$. Therefore, $A \subset scl_\theta A$. □

Theorem 3.12. *Let X and Y be topological spaces. For a function $f : X \to Y$, the following statements are equivalent :*
(1) f is st. θsc on X.
(2) For each $x \in X$ and each $V \subset Y$ such that $f(x) \in IntV$, there exists $U \subset X$ such that $x \in sInt_\theta U$ and $f(U) \subset V$.
(3) $f^{-1}(IntB) \subset sInt_\theta f^{-1}(B)$ for each $B \subset Y$.
(4) $scl_\theta f^{-1}(B) \subset f^{-1}(\overline{B})$ for each $B \subset Y$.
(5) $f^{-1}(G) \subset sInt_\theta f^{-1}(G)$ for each open set G in Y.

Proof. (1) $\Rightarrow$ (2) Let $x \in X$ and $V \subset Y$ such that $f(x) \in IntV$.
There exists $G \in \tau Y$ such that $f(x) \in G$ and $G \subset V$. Since f is *st.* θsc on X,

hence there exists $A \in S.O.(X)$ such that $x \in A$ and $f(scl\ A) \subset G \subset V$. Since $sInt_\theta(sclA) = \cup\{A|A \in S.O.(X)$ such that $scl\ A \subset scl\ A\}$. Let $U = scl\ A$. Hence, there exists $U \subset X$ such that $x \in sInt_\theta U$ and $f(U) \subset V$.

(2) $\Rightarrow$ (3) Let $B \subset Y$ show that $f^{-1}(IntB) \subset sInt_\theta f^{-1}(B)$. Let $x \in f^{-1}(IntB)$then $f(x) \in IntB$. By (2), there exists $U \subset X$ such that $x \in sInt_\theta U$ and $f(U) \subset B$. Hence $f^{-1}(f(U)) \subset f^{-1}(B)$. By Lemma 3.11(2), $sInt_\theta U \subset sInt\ U$ and by Theorem 2.4(7) , $sInt\ U \subset U$. Hence $sInt_\theta U \subset U$ and since $U \subset f^{-1}(f(U))$. Hence $x \in sInt_\theta U \subset f^{-1}(B)$. Since $x \in sInt_\theta U = \cup\{O|O \in S.O.(X)\ such\ that\ sclO \subset U\}$ and $U \subset f^{-1}(B)$. Therefore, $x \in sInt_\theta f^{-1}(B) = \cup\{O|O \in S.O.(X)\ such\ that\ sclO \subset U \subset f^{-1}(B)\}$. Thus $f^{-1}(IntB) \subset f^{-1}(B)$ for each $B \subset Y$.

(3) $\Rightarrow$ (4) Let $B \subset Y$ show that $scl_\theta f^{-1}(B) \subset f^{-1}(\overline{B})$.

$$\begin{aligned} X - f^{-1}(\overline{B}) &= X - f^{-1}(Y - Int(Y - B)) \\ &= X - f^{-1}(Y) + f^{-1}(Int(Y - B)) = X - X + f^{-1}(Int(Y - B)) \end{aligned}$$

$$\begin{aligned} X - X + f^{-1}(Int(Y - B)) &= f^{-1}(Int(Y - B)) \\ &\subset sInt_\theta f^{-1}(Y - B) \qquad (By\ (3)) \\ &= sInt_\theta[f^{-1}(Y) - f^{-1}(B)] \\ &= sInt_\theta[X - f^{-1}(B)] \\ &= X - scl_\theta f^{-1}(B) \qquad (By\ Lemma\ 3.11(1)) \end{aligned}$$

Thus $X - f^{-1}(\overline{B}) \subset scl_\theta f^{-1}(B)$. Therefore, $scl_\theta f^{-1}(B) \subset f^{-1}(\overline{B})$ for each $B \subset Y$.

(4) $\Rightarrow$ (5) Let $G \in \tau_Y$ show that $f{-}1(G) \subset sInt_\theta f^{-1}(G)$.

$$\begin{aligned} X - sInt_\theta f^{-1}(G) &= scl_\theta(X - f^{-1}(G)) \qquad (By\ Lemma\ 3.11(1)) \\ &= scl_\theta(f^{-1}(Y) - f^{-1}(G)) \\ &= scl_\theta f^{-1}(YG) \\ &= \subset f^{-1}(\overline{Y - G}) \qquad (By\ (4)) \\ &= f^{-1}(Y - Int\ (Y - (Y - G))) \\ &= f^{-1}(Y - Int\ G) \\ &= f^{-1}(Y) - f^{-1}(Int\ G) = X - f^{-1}(G) \end{aligned}$$

Thus $X - sInt_\theta f^{-1}(G) \subset X - f^{-1}(G)$. Therefore, $f^{-1}(G) \subset sInt_\theta f^{-1}(G)$ for each $G \in \tau_Y$.

(5) $\Rightarrow$ (1) Let $x \in X$ and $V \in \tau_Y$ such that $f(x) \in V$. Hence $x \in f^{-1}(V)$. By (5) , we have $f^{-1}(V) \subset sInt_\theta f^{-1}(V)$. Hence $x \in sInt_\theta f^{-1}(V)$. Since $sInt_\theta f^{-1}(V) = \cup\{O|O \in S.O.(X)\ such\ that\ scl\ O \subset f^{-1}(V)\}$. Thus there exists $U \in S.O.(X)$ such that $x \in U$ and $f(scl\ U) \subset V$. Therefore, f is *st.* θsc on X. □

Theorem 3.13. *Let X and Y be topological spaces. For a function $f : X \to Y$, the following statements are equivalent :*

(1) f is st. θsc on X.

(2) For each $x_0 \in X$ and each net (x_d) in X. If $(x_d)s\theta$-converges to x_0, then the net $(f(x_d))$ converges to $f(x_0)$.

Proof. (1) $\Rightarrow$ (2) Let $x_0 \in X$ and (x_d) be a net in X such that (x_d) $s\theta$-converges to x_0. Let V be an open set containing $f(x_0)$. Since f is *st.* θsc on X , there exists a semi-open set U containing x_0 such that $f(scl\ U) \subset V$. Since $(x_d)s\theta$-converges to x_0, there exists d_0 such that $x_d \in scl\ U$ for all $d \geqslant d_0$. Hence $f(x_d) \in f(scl\ U)$ for all $d \geqslant d_0$. Since $f(scl\ U) \subset V$, hence $f(x_d) \subset V$ for all $d \geqslant d_0$. Thus $(f(x_d))$ converges to $f(x_0)$.

(2) $\Rightarrow$ (1) Suppose that f is not *st.* θsc on X. Then there exists $x_0 \in X$ and an open set V containing $f(x_0)$ such that $f(scl\ U) \not\subset V$ for all semi-open sets U containing x_0. Thus there exists $x_U \in scl\ U$ such that $f(x_U) \notin V$. Consider the net $\{x_U | U is\ semi-open\ set\ containing\ x_0\}$. Then $(x_U)s\theta$-converges to x_0 but $(f(x_U))$ does not converges to $f(x_0)$. Since this contradiction (2). Therefore, f is *st.* θsc on X. □

4 The strongly θ - semi - continuous retraction and fixed point property

Definition 4.1. *Let X be topological space and $A \subset X$. A is said to be strongly θ-semi-continuous retract of X if there exists $f : X \to A$ is st. θsc on X such that f is the identity on A. f is called a strongly θ-semi-continuous retraction.*

Example 4.2. *Let $X = \{a, b, c, d\}, A = \{a, b, c\}$,*
$\tau_X = \{\emptyset, \{b\}, \{c\}, \{b, c\}, \{a, b, c\}, X\}$ be topology on X,
$\tau_A = \{\emptyset, \{b\}, \{c\}, \{b, c\}, A\}$ be topology on A and $f : X \to A$ such that $f(a) = a, f(b) = b$ and $f(c) = f(d) = c$. Show that f is a strongly θ-semi-continuous retraction.

Solution Similar to Example 3.2, hence f is *st.* θsc on X. Since f is the identity on A. Hence f is a strongly θ-semi-continuous retraction.

Lemma 4.3. *Let X, Y and Z be topological spaces. If $f : X \to Y$ is st. θsc on X and $g : Y \to Z$ is st. θc on Y , then gof is st. θsc on X.*

Proof. Let $x \in X$ and $V \in \tau_Z$ such that $(gof)(x) \in V$. Hence $g(f(x)) \in V$ and $f(x) \in Y$. Since g is *st.* θc on Y, there exists an open set O containing $f(x)$ such that

$$g(\overline{O}) \subset V. \tag{4.1}$$

Since f is $st.\theta c$ on X , there exists a semi - open set U containing x such that $f(sclU) \subset O$. By Theorem 2.4 (2) , we have $O \subset \overline{O}$. Hence

$$g(O) \subset g(\overline{O}). \tag{4.2}$$

Since $f(sclU) \subset O$, hence

$$g(f(sclU)) \subset g(O). \tag{4.3}$$

By (4.1) , (4.2) and (4.3) , hence $g(f(scl\ U) \subset g(O) \subset g(\overline{O}) \subset V$. Thus $(gof)(scl\ U) \subset V$. Therefore, gof is $st.\ \theta sc$ on X. □

Theorem 4.4. *Let X be topological space and $A \subset X$. If A is a strongly θ-semi-continuous retract of X , then for every space Y , every $g : A \to Y$ is st. θc on A can be extended to $g : X \to Y$ is st. θsc on X.*

Proof. Let Y be topological space and $g : A \to Y$ is $st.\ \theta c$ on A. Since A is a strongly θ-semi-continuous retract of X, there exists $f : X \to A$ is $st.\ \theta sc$ on X and f is the identity on A. By Lemma 4.3, we have $gof : X \to Y$ is $st.\ \theta c$ on X. Since $gof(x) = g(f(x)) = g(x)$ for all $x \in A$. Therefore, gof is an extension of g. □

Theorem 4.5. *Let X is Hausdorff space. If A is a strongly θ-semi-continuous retract of X, then $scl_\theta A = A$.*

Proof. Suppose that $scl_\theta\ A \neq A$. By Lemma 3.11(3), we have $A \subset scl_\theta A$. Hence there exists $x \in (scl_\theta A - A)$, Thus $x \notin A$. Since A is a strongly θ-semi-continuous retract, hence $f(x) \neq x$ for some $x \in X$. Since X is Hausdorff space , there exists disjoint open sets U and V such that $x \in U$ and $f(x) \in V$. Thus $U \subset X - V$. By Theorem 2.4(3), we have $scl\ U \subset scl(X - V) = X - V$, thus $scl\ U \cap V = \emptyset$. Since $U \subset scl\ U$, hence $x \in scl\ U$ for open set U containing x.

Let W is an open set containing x. Since $W \subset scl\ W$, hence $x \in scl\ W$. Since $(U \cap W) \subset scl(U \cap W)$, hence $x \in scl(U \cap W)$ for open set $U \cap W$ containing x. Since $x \in scl_\theta A$ such that $scl_\theta A = \{x \in X | scl\ U \cap A \neq \emptyset\ for\ each\ semi - open\ set\ U\ containing\ x\}$, hence $scl(U \cap W) \cap A \neq \emptyset$. Since $scl(U \cap W) \subset scl\ U \cap scl\ W$, hence$(scl\ U \cap scl\ W) \cap A \neq \emptyset$.

Let $a \in (scl\ U \cap scl\ W) \cap A \neq \emptyset$. We have a $\in scl\ U, a \in scl\ W$ and $a \in A$. Since $a \in A$ hence $f(a) = a, a \in scl\ W$ hence $f(a) \in f(scl\ W)$ and $a \in scl\ U$ hence $a \notin V$. Thus $f(a) \notin V$, we have $f(scl\ W) \not\subset V$ for semi- open set w containing x. Thus this contradiction f is $st.\ \theta sc$ on X. Therefore, $scl_\theta A = A$. □

The converse of Theorem 4.2 is false , as shown by Example 4.6 , 4.7.

Example 4.6. *Let* $X = \{a, b, c, d\}, A = \{a, d\}$ *,*
$\tau_X = \{\emptyset, \{a\}, \{b, c\}, \{a, b, c\}, X\}$ *be topology on* X *,*
$\tau_A = \{\emptyset, A\}$ *be topology on* A
and $f : X \to A$ *such that* $f(a) = a, f(b) = f(c) = f(d) = d$*. Show that*
(1) (X, τ_X) *is not Hausdorff space.*
(2) A *is a strongly* θ *- semi - continuous retract of* X*.* *(3)* $scl_\theta A = A$*.*

Solution

(1) **Consider at** b, c **such that** $b \neq c$ **:**
Open set V containing b is to be $\{b, c\}, \{a, b, c\}$ and X.
Open set U containing c is to be $\{b, c\}, \{a, b, c\}$ and X.
Hence $V \cap U \neq \emptyset$, thus (X, τ_X) is not Hausdorff space.
(2) We must show , f is $st.\theta sc$ on X and f is the identity on A.
Open sets in $A; \emptyset, \{a\}$ and A.
Since $f^{-1}(\emptyset) = (\emptyset) \subset sInt_\theta f^{-1}(\emptyset), f^1(\{a\}) = \{a\} \subset sInt_\theta f^{-1}(\{a\}) = \{a\}$ and $f^{-1}(A) = A \subset AsInt_\theta f^{-1}(A) = A$.
By Theorem 3.12 (5) $\Rightarrow$(1) , hence f is $st.\theta sc$ on X. By defined of f , we have f is identity on A. Therefore , A is a strongly θ - semi - continuous retract of X.
(3) Closed sets in $X; X, \{b, c, d\}, \{a, d\}, \{d\}$ and$\emptyset$.
Semi - open sets in $X; \emptyset, \{a\}, \{b, c\}, \{a, d\}, \{a, b, c\}, \{b, c, d\}$ and X. Semi - closed sets in $X; X, \{b, c, d\}, \{a, d\}, \{b, c\}, \{d\}, \{a\}$ and $\emptyset$.
Consider at a point a **:**
Semi - open set V in X containing a is to be $\{a\}, \{a, d\}, \{a, b, c\}$ and X. Since $scl\{a\} = \{a\}, scl\{a, d\} = \{a, d\}, scl\{a, b, c\} = X$ and $scl\ X = X$, hence $scl\ V \cap A \neq \emptyset$ for each semi - open set V containing a. Thus $a \in scl_\theta A$.
Consider at a point b **:**
Semi - open set U in X containing b is to be $\{b, c\}, \{a, b, c\}, \{b, c, d\}$ and X. Since $scl\{b, c\} = \{b, c\}$, hence $scl\{b, c\} \cap A = \emptyset$. Thus $b \notin scl_\theta A$.
Consider at a point c **:**
Similar to a point b , hence $c \notin scl_\theta A$.
Consider at a point d **:**
Semi - open set K in X containing d is to be $\{a, d\}, \{b, c, d\}$ and X. Since $scl\{a, d\} = \{a, d\}, scl\{b, c, d\} = \{b, c, d\}$ and $scl\ X = X$, hence $scl\ K \cap A$Thus $d \in scl_\theta A$, hence $scl_\theta A = \{a, d\}$. Therefore , $scl_\theta A = A$.

Example 4.7. *Let* $X = \{x, y, z\}, A = \{x, y\}$*,*
$\tau_X = \{, \{x\}, \{y\}, \{z\}, \{x, y\}, \{x, z\}, \{y, z\}, X\}$ *be topology on* X *,*
$\tau_A = \{\emptyset, \{x\}, \{y\}, A\}$ *be topology on* A
and $f : X \to A$ *such that* $f(x) = f(y) = f(z) = x$*. Show that*
(1) (X, τ_X) *is Hausdorff space.*
(2) A *is not a strongly* θ *- semi - continuous retract of* X*.*
(3) $scl_\theta A = A$*.*

Solution

(1)Consider at x, y such that $x \neq y$, there exists disjoint open sets $\{x\}, \{y\}$

such that $x \in \{x\}, y \in \{y\}$. Consider at x, z such that $xneqz$, there exists disjoint open sets $\{x\}, \{z\}$ such that $x \in \{x\}, z \in \{z\}$. Consider at y, z such that $y \neq z$, there exists disjoint open sets $\{y\}, \{z\}$ such that $y \in \{y\}, z \in \{z\}$. Hence (X, τ_X) is Hausdorff space.

(2) We must show , f is not $st.\theta sc$ on X or f is not the identity on A. By defined of f , we have f is not the identity on A. Therefore , A is not a strongly θ - semi - continuous retract of X.

(3) Closed sets in $X; X, \{y, z\}, \{x, z\}, \{x, y\}, \{z\}, \{y\}, \{x\}$ and $\emptyset$.
Semi open sets in $X; \emptyset, \{x\}, \{y\}, \{z\}, \{x, y\}, \{x, z\}, \{y, z\}$ and X.
Semi- closed sets in $X; X, \{y, z\}, \{x, z\}, \{x, y\}, \{z\}, \{y\}, \{x\}$ and $\emptyset$.

By Lemma 3.11 (3) , we have $A \subset scl_\theta A$, thus $x \in scl_\theta A$ and $y \in scl_\theta A$. Next , we shall show that $z \notin scl_\theta A$. Semi open set S in X containing z is to be $\{z\}, \{x, z\}, \{y, z\}$ and X. Since $scl\{z\} = \{z\}$, hence $scl\{z\} \cap A = \emptyset$. Thus $z \notin scl_\theta A$. Therefore , $scl_\theta A = A$.

Definition 4.8. *Let X be topological space. A space X is said to has the strongly θ - semi - continuous fixed point property if for every $f : X \to X$ is $st.\theta sc$ on X , there exists an $x \in X$ such that $f(x) = x$. We shall denote by X has the $st.\theta scFPP$.*

Example 4.9. *Let $X = \{a, b, c\}$ and $\tau_X = \{\emptyset, \{a\}, \{b, c\}, X\}$ be topology on X. Consider the existence of $st.\theta scFPP$ for X.*

Solution

Claim that X has not the $st.\theta scFPP$. We must show , there exists $f : X \to X$ is $st.\theta sc$ on X such that $f(x) \neq x$ for all $x \in X$. Let $f : X \to X$ such that $f(a) = f(c) = b$ and $f(b) = c$. Next , we shall show that f is $st.\theta sc$ on X. Open sets in $X; \emptyset, \{a\}, \{b, c\}$ and X. Since $f-1(\emptyset) = \emptyset \subset sInt_\theta f^{-1}(\emptyset) = \emptyset, f^{-1}(\{a\}) = \emptyset \subset sInt_\theta f^{-1}(\{a\}) = \emptyset, f^{-1}(\{b, c\}) = X \subset sInt_\theta f^{-1}(\{b, c\}) = sInt_\theta X = X$, and $f^{-1}(X) = X \subset f^{-1}(X) = sInt_\theta X = X$. By Theorem 3.12 (5) $\Rightarrow$ (1) , hence f is $st.\theta sc$ on X. Since $f(x) \neq x$ for all $x \in X$. Therefore , X has not the $st.\theta scFPP$.

Lemma 4.10. *Let X, Y and Z be topological space. If $gof : X \to Z$ is continuous on X and $g : Y \to Z$ is an open bijection , then f is continuous on X.*

Proof. Let V is any open set in Y. Since g is an open mapping , hence $g(V)$ is an open set in Z. Since gof is continuous on X , hence $(gof)^{-1}(g(V))$ is an open set in X. Since $(gof)^{-1}(g(V)) = f^{-1}(g^{-1}(g(V))) = f^{-1}(V)$, hence $f^{-1}(V)$ is an open set in X. Therefore , f is continuous on X. □

Theorem 4.11. *Let (X, τ) is regular space with the $st.\theta scFPP$. If σ is a topology for X stronger than τ and $scl\ G^{(\tau)} = scl\ G^{(\sigma)}$ for every $G \in \sigma$, then (X, σ) has the fixed point property.*

Proof. Suppose that $f : (X, \sigma) \to (X, \sigma) is any continuous function.$ Let $g : (X, \sigma) \to (X, \tau)$ and $h : (X, \tau) \to (X, \tau)$ be the functions defined by $g(x) = h(x) = f(x)$ for all $x \in X$. Let $i : (X, \tau) \to (X, \sigma)$ be the identity function. Since $\tau \subset \sigma$, hence i is an open bijection. Since $f = iog$ is continuous , by Lemma 4.10 g is continuous . Next , we shall show that h is $st.\theta sc$ on X. Let $x \in X$ and $h(x) \in V$. For each open set V in (X, τ) , hence V^c is closed set in (X, τ) and $h(x) \notin V^c$. Since (X, τ) is regular space , there exists disjoint open sets A and B such that $h(x) \in A$ and $V^c \subset B$. We have $A \subset B^c$, then $\overline{A} \subset \overline{B^c}$ and $B^c \subset V$. Since B^c is closed set , we have $B^c = \overline{B^c}$. Hence $(\overline{A}^{(\tau)} \subset V$. Thus $h(x) \in A \subset (\overline{A}^{(\tau)} \subset V$. Since g is continuous , hence $g^{-1}(A) \in \sigma$. Since $h^{-1}(A) = f^{-1}(A) = g^{-1}(A)$, hence $h^{-1}(A) = f^{-1}(A) \in \sigma$. By Theorem 2.4(2) and $scl\ G^{(\tau)} = scl\ G^{(\sigma)}$ for every $G \in \sigma$, we obtain

$$x \in h^{-1}(A) \subset scl\ h^{-1}(A)^{(\tau)} = scl\ h^{-1}(A)^{(\sigma)} = scl\ f^{-1}(A)^{(\sigma)} \subset \overline{f^{-1}}(A)^(\sigma) \quad (4.4)$$

Since f is continuous , $\tau \subset \sigma$ and $(\overline{A}^{(\tau)} \subset V$, we obtain

$$\overline{f^{-1}}(A)^(\sigma) \subset f^{-1}(\overline{A})^{(\tau)} \subset f^{-1}(\overline{A})^{(\tau)} \subset f^{-1}(V). \quad (4.5)$$

By (4.4) and (4.5) , we have $scl\ h^{-1}(A) \subset f^{-1}(V)$. Since $h(x) = f(x)$ for all $x \in X$, hence $f^{-1}(V) = h^{-1}(V)$. Thus $scl\ h^{-1}(A) \subset h^{-1}(V)$. Now , we set $U = h^{-1}(A)$, then we have semi - open set U in (X, τ) with $x \in U$ such that $h(scl\ U) \subset V$. Hence h is $st.\theta sc$ on (X, τ). Since (X, τ) has the $st.\tau scFPP$, there exists $x \in X$ such that $x = h(x) = f(x)$. Therefore , (X, σ) has the fixed point property. □

Lemma 4.12. *Let X, Y and Z be topological spaces. If $gof : X \to Z$ is $st.\theta sc$ on X and $g : Y \to Z$ is an open bijection , then f is $st.\theta sc$ on X.*

Proof. Let V is any open set in Y. Since g is an open mapping , hence $g(v)$ is an open set in Z. Since gof is $st.\theta sc$ on X , hence $(gof)^{-1}(g(V)) \subset sInt_\theta(gof)^{-1}(g(V))$. Since g is bijection , hence $(gof)^{-1}(g(V)) = f^{-1}(g^{-1}(g(V)) = f^{-1}(V)$. Hence $f^{-1}(V) \subset sInt_\theta f^{-1}(V)$. By Theorem 3.12 (5) ⇒ (1), hence f is $st.\theta sc$ on X. □

Theorem 4.13. *Let (X, τ) is regular space with the fixed point property. If σ is a topology for X stronger than τ and $scl_\theta G^{(\tau)} = scl_\theta G^{(\sigma)}$ and $G \in \sigma$ for each semi - open set G in (X, σ) , then (X, σ) has the $st.\theta scFPP$.*

Proof. Suppose that $f : (X, \sigma) \to (X, \sigma)$ is any $st.\theta sc$ on X. Let $g : (X, \sigma) \to (X, \tau)$ and $h : (X, \tau) \to (X, \tau)$ be the functions defined by $g(x) = h(x) = f(x)$ for all $x \in X$. Let $i : (X, \tau) \to (X, \tau)$ be the identity function. Since $f = iog$ is $st.\theta sc$ on X and i is an open bijection. By Lemma 4.12 , g is $st.\theta sc$ on X. By Remark 3.4 (2) , hence g is $s.c$ on X. Next , we shall show that h is continuous. The same argument as in proof of Theorem 4.11 that , $h(x) \in A \subset \overline{A}^{(\tau)} \subset V$ for open set V in (X, τ) containing $h(x), A \in \tau$. Since g is $s.c$ on X , hence $g^{-1}(A)$ is semi -

open in (X, σ). By assumption $G \in \sigma$ for each semi - open set G in (X, σ) , hence $g^{-1}(A) \in \sigma$. Since $h^{-1}(A) = f^{-1}(A) = g^{-1}(A)$, hence $h^{-1}(A) = f^{-1}(A) \in \sigma$. By Lemma 3.11 (3) and $scl_\theta G^{(\tau)} = scl_\theta G^{(\sigma)}$ for every $G \in \sigma$, we obtain

$$x \in h^{-1}(A) \subset scl_\theta h^{-1}(A)^{(\tau)} = scl_\theta h^{-1}(A)^{(\sigma)} = scl_\theta f^{-1}(A)^{(\sigma)}. \tag{4.6}$$

Since f is $st.\theta sc$ on $X, \tau \subset \sigma$ and $\overline{A}^{(\tau)} \subset V$, we obtain

$$scl_\theta f^{-1}(A)^{(\sigma)} \subset f^{-1}\overline{A}^{(\sigma)} \subset f^{-1}\overline{A}^{(\tau)} \subset f^{-1}(V). \tag{4.7}$$

By (4.6) and (4.7) , we have $h^{-1}(A) \subset f^{-1}(V)$. Since $h(x) = f(x)$ for all $x \in X$, hence $f^{-1}(V) = h^{-1}(V)$. Thus $h^{-1}(A) \subset h^{-1}(V)$. Now , we set $U = h^{-1}(A)$, then we have open set U in (X, τ) with $x \in U$ such that $h(U) \subset V$. Hence h is continuous on (X, τ). Since (X, τ) has the fixed point property , there exists $x \in X$ such that $x = h(x) = f(x)$. Therefore , (X, σ) has the $st.\theta scFPP$. □

References

[1] F. Cammaroto and T. Noiri, On the *delta* - continuous fixed point property , *Internat. J. Math .& Math . Sci* . 13(1)(1990), 45 - 50.

[2] G. Di Maio and T. Noiri, Weak and strong forms of irresolute functions, *Third national Conference on Topology (Italian)(Trieste , 1986).Rend. Circ. Math. Palermo(2) suppl.*18(1988), 255-273.

[3] N. Levine, Semi open sets and semi continuity in topological spaces, *Amer . Math . Monthly* 70 (1963) , 36 41.

[4] Seong hoon cho, A Note on almost s continuous functions , *Kyungpook Math. J .* , 42 (2002) , 171 -175 27(4)(1981), 343-348.

[5] S.G. Crossley and S.K. Hildebrand, Semi closed sets and semi continuity in topological spaces , *Texas J. Sci .* , 22 (1971) , 123-126.

[6] T. Noiri, On the δ - continuous functions, *J. Korean Math . Soc.*, 16 (1980), 161-166.

(Received 30 May 2007)

N. Puturong
Mathematics Program
Faculty of Science
Udonthani Rajabhat University
Udonthani 41000, THAILAND.
e-mail: `naputurong@hotmail.com`