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Solving the Poisson Process in Conformable Fractional Calculus Sense by Homotopy Perturbation Method

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Abstract : In this paper, we focus on solving a non-local problem for a differential equation, which contains the fractional Poisson process involving conformable fractional calculus. We apply an improved homotopy perturbation method which is efficient and powerful in solving widely fractional order equations to solve a solution of fractional Poisson process.

Keywords : fractional Poisson processes; fractional derivatives; homotopy perturbation method.
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1 Introduction

The fractional Poisson probability distribution captures the long-memory effect which results in the non-exponential waiting time probability distribution function empirically observed in complex classical and quantum systems. Fractional Poisson process and fractional Poisson probability distribution function can

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be considered as a natural generalization of the famous Poisson process and the Poisson probability distribution. The idea behind the fractional Poisson process was to design the counting process with non-exponential waiting time probability distribution. Mathematically the idea was realized by substitution of the first-order time derivative in the Kolmogorov-Feller equation for the Poisson probability distribution function with the time fractional order derivative [1]. The fractional Poisson process (FPP) was introduced and studied by Repin and Saichev [2], Jumarie [3], Laskin [4], Mainardi et al. [5, 6], Uchaikin et al. [7] and Beghin and Orsingher [8, 9]. The probability distribution function of the fractional Poisson process has been found for the first time in 2003 by Nick Laskin [4]. Mainardi et al. [5] provided a generalization of the pure and compound Poisson processes via fractional calculus, by resorting to a renewal process-based approach involving waiting time distributions expressed in term of the Mittag-Leffler function. Beghin and Orsingher [9] studied the properties of Poisson-type fractional processes, governed by fractional recursive differential equations, obtained substituting regular derivatives with fractional derivatives. Meerschaert et al. [10] showed that a Poisson process, with the time variable replaced by an independent inverse stable subordinator, is also a fractional Poisson process.

In 2017, Kataria and Vellaisamy [11] definded the Saigo space-time fractional Poisson process and obtained the state probability of various fractional version of the classical homogeneous Poisson process using Adomian decomposition method (ADM) for parameters $0 < \alpha \leq 1$, $0 < v \leq 1$, $\beta < 0$ and $\gamma \in \mathbb{R}$ that satisfies

$$\partial_t^{\alpha,\beta,\gamma} p_v^{\alpha,\beta,\gamma}(n,t) = -\lambda^v (1-B)^v p_v^{\alpha,\beta,\gamma}(n,t), \ \ n \geq 0, \tag{1.1}$$

with $p_v^{\alpha,\beta,\gamma}(-1,t) = 0$ and subject to the initial conditions $p_v^{\alpha,\beta,\gamma}(0,0) = 1$ and $p_v^{\alpha,\beta,\gamma}(n,0) = 0, n \geq 1$. Also,(1.1) can be rewritten as

$$\partial_t^{\alpha,\beta,\gamma} p_v^{\alpha,\beta,\gamma}(n,t) = -\lambda^v \sum_{r=0}^{n} (-1)^r \frac{(v)_r}{r!} p_v^{\alpha,\beta,\gamma}(n-r,t), \ \ n \geq 0,$$

where $(1-B)^v = \sum_{r=0}^{\infty} \frac{(v)_r}{r!}(-1)^r B^r$ is the fractional difference operator and $(v)_r = v(v-1)...(v-r+1)$ denotes the falling factorial.

In recent years, the homotopy perturbation method (HPM), first proposed by Ji Huan He [12, 13], has successfully been applied to solve many types of linear and nonlinear functional equations. This method, which is a combination of homotopy in topology and classic perturbation techniques, provides us with a convenient way to obtain analytical or approximate solutions for a wide variety of problems arising in different fields. The HPM is applied to Volterra's integro-differential equation [14], nonlinear oscillators [15], bifurcation of nonlinear problems [16], bifurcation of delay-differential equations [17], nonlinear wave equations [18], boundary value problems [19] and other fields.

The fractional Poisson process(FPP) with parameter $\lambda > 0$ and $0 < \alpha \leq 1$ is defined by

$$D_t^{\alpha} p_{\alpha}(n,t) = -\alpha\lambda \, (1-B) \, p_{\alpha}(n,t), \qquad n \geq 0 \tag{1.2}$$

when $p_\alpha(-1,t) = 0$, $t \geq 0$ and the initial conditions $p_\alpha(0,0) = 1$ and $p_\alpha(n,0) = 0$, $n \geq 1$. Here λ is a time rate for the events to happen, B is the backward shift operator acting on the state space, i.e., $B(p_\alpha(n,t)) = p_\alpha(n-1,t)$ and D_t^α denote the fractional derivative in the conformable sense. Also, (1.2) can be rewritten as

$$D_t^\alpha p_\alpha(n,t) = \left(\alpha n t^{-\alpha} - \alpha\lambda\right) p_\alpha(n,t), \qquad n \geq 0. \tag{1.3}$$

For $\alpha = 1$, (1.2)-(1.3) can be reversed to the homogeneous Poisson process.

In this paper, we develop the fractional Poisson process in the conformable sense and solve the fractional Poisson process (1.3) using the HPM. The main aim is to investigate the efficiency and accuracy of the method by comparison of the exact solutions and the solutions obtained by the HPM. Finally, the numerical solutions are illustrated and compared with the exact solutions using advantages of the Maple software.

2 Conformable Fractional Calculus

The definitions and theorems of conformable fractional calculus which will be used in this work are given below.

Definition 2.1 ([20]). Given a function $f : [0,\infty) \to \mathbb{R}$. Then the conformable fractional derivative of the function f of order α is defined by

$$D_t^\alpha f(t) = \lim_{\epsilon \to \infty} \frac{f(t + \epsilon t^{1-\alpha}) - f(t)}{\epsilon}$$

for all $t > 0$ and $\alpha \in (0,1]$.

Theorem 2.2 ([20]). *Let $\alpha \in (0,1]$ and f, g be α-differentiable at a point $t > 0$. Then*

(1) $D_t^\alpha(af + bg) = aD_t^\alpha(f) + bD_t^\alpha(g)$, for all $a, b \in \mathbb{R}$.

(2) $D_t^\alpha(t^p) = pt^{p-\alpha}$, for all $p \in \mathbb{R}$.

(3) $D_t^\alpha(\lambda) = 0$, for all constants λ.

(4) $D_t^\alpha(fg) = fD_t^\alpha(g) + gD_t^\alpha(f)$.

(5) $D_t^\alpha(\frac{f}{g}) = \dfrac{gD_t^\alpha(f) - fD_t^\alpha(g)}{g^2}$.

(6) If, in addition, f is differentiable, then $D_t^\alpha f(t) = t^{1-\alpha}\dfrac{d}{dt}f(t)$.

Definition 2.3 ([20]). The conformable fractional integrals of a function $f : [0, \infty) \to \mathbb{R}$ of order $0 \leq \alpha < 1$ is defined by

$$I_t^\alpha f(t) = \int_0^t x^{\alpha-1} f(x)dx, \quad \alpha \in (0, 1].$$

Lemma 2.4 ([21]). *Let $f : [0, \infty) \to \mathbb{R}$ be a differentiable function and $0 < \alpha \leq 1$. Then, for all $t > 0$ we have*

$$I_t^\alpha D_t^\alpha f(t) = f(t) - f(0).$$

3 Homotopy Perturbation Method

In recent years, fractional differential equations have successfully modelled to many physical and engineering phenomena [22, 23]. One of the most important classes of fractional differential equations is the fractional initial-value problems (FIVPs), which can be written as

$$D_t^\alpha y(t) = f(t, y(t)), \quad y^{(k)}(0) = y_0^{(k)}, k = 0, 1, 2, ..., n-1, \tag{3.1}$$

where f is an arbitrary function, D_t^α denotes the fractional derivative of order α with respect to t in the sense of conformable, $y^{(k)}(t)$ is the k^{th} derivative of y, and $y_0^{(k)}$ are the specified initial conditions.

The FIVPs (3.1) can be written in the operator form as

$$D_t^\alpha y(t) + Ly(t) + Ny(t) = g(t), \tag{3.2}$$
$$y^{(k)}(0) = c_k, \;\; k = 0, 1, ..., n-1, \tag{3.3}$$

where (3.3) are the initial conditions, L is the linear operator and N is the nonlinear operator.

In view of HPM, we construct the following homotopy that represents (3.2):

$$(1-h)D_t^\alpha y + h[D_t^\alpha y + Ly(t) + Ny(t) - g(t)] = 0, \tag{3.4}$$

or

$$D_t^\alpha y + h[Ly(t) + Ny(t) - g(t)] = 0, \tag{3.5}$$

where $h \in [0, 1]$ is an embedding parameter. If $h = 0$, (3.5) becomes

$$D_t^\alpha y = 0, \tag{3.6}$$

and when $h = 1$, both (3.4) and (3.5) turn out to be (3.2).

Using the parameter h, we expand the solution of (3.5) in the following form:

$$y(t) = y_0(t) + hy_1(t) + h^2 y_2(t) + h^3 y_3(t) + \cdots . \tag{3.7}$$

Letting $Ny(t) = H(y)$ and substituting (3.7) into (3.5), and then collecting the terms with the same powers of h, we obtain that

$$\begin{aligned} h^0 &: \quad D_t^\alpha y_0 = 0, \\ h^1 &: \quad D_t^\alpha y_1 = -Ly_0(t) - H_1(y_0) + g(t), \\ h^2 &: \quad D_t^\alpha y_2 = -Ly_1(t) - H_2(y_0, y_1), \\ h^3 &: \quad D_t^\alpha y_3 = -Ly_2(t) - H_3(y_0, y_1, y_2), \\ &\ \vdots \end{aligned} \tag{3.8}$$

where the functions $H_1, H_2, H_3, ...$ satisfy the following condition as

$$H(y_0 + hy_1 + h^2 y_2 + \cdots) = H_1(y_0) + hH_2(y_0, y_1) + h^2 H_3(y_0, y_1, y_2) + \cdots .$$

Applying the operator I_t^α on both sides of each linear equation (3.8), substituting them into (3.7) and setting $h = 1$, then the solution of (3.2) is

$$y(t) = y_0(t) + y_1(t) + y_2(t) + y_3(t) + \cdots . \tag{3.9}$$

4 Applications of HPM to FPP

Now, we consider the fractional conformable Poisson process

$$D_t^\alpha p_\alpha(n,t) = \left(\alpha n t^{-\alpha} - \alpha\lambda\right) p_\alpha(n,t), \qquad n \geq 0.$$

We can construct the following homotopy in the form of

$$D_t^\alpha p_\alpha(n,t) + h\left(\left(\alpha\lambda - \alpha n t^{-\alpha}\right) p_\alpha(n,t)\right) = 0, \quad 0 \leq \alpha < 1, \tag{4.1}$$

where $h \in [0,1]$ with the initial conditions $p_\alpha(0,0) = 1$ and $p_\alpha(n,0) = 0$, $n \geq 1$ and the exact solution is

$$p_\alpha(n,t) = \frac{e^{-\lambda t^\alpha}(\lambda t^\alpha)^n}{n!}. \tag{4.2}$$

Using the parameter h, the solution is expressed in the following form:

$$p_\alpha(n,t) = \sum_{i=0}^{\infty} h^i p_{\alpha,i}(n,t). \tag{4.3}$$

Substituting (4.3) into (4.1) and equating the terms with the identical power of h, we obtain that

$$D_t^\alpha\left(\sum_{i=0}^{\infty} h^i p_{\alpha,i}(n,t)\right) + \left(\alpha\lambda - \alpha n t^{-\alpha}\right)\sum_{i=0}^{\infty} h^{i+1} p_{\alpha,i}(n,t) = 0,$$

or

$$D_t^{\alpha} p_{\alpha,0}(n,t) + \sum_{i=1}^{\infty} h^i \left(D_t^{\alpha} p_{\alpha,i}(n,t)\right) + \left(\alpha\lambda - \alpha n t^{-\alpha}\right) \sum_{i=1}^{\infty} h^i p_{\alpha,i-1}(n,t) = 0.$$

Simplifying above equation, then we have

$$D_t^{\alpha} p_{\alpha,0}(n,t) + \sum_{i=1}^{\infty} h^i \left(D_t^{\alpha} p_{\alpha,i}(n,t) + \left(\alpha\lambda - \alpha n t^{-\alpha}\right) p_{\alpha,i-1}(n,t)\right) = 0.$$

Consider coefficient of h^i, then we obtain

$$\begin{array}{lcl} h^0 & : & D_t^{\alpha} p_{\alpha,0}(n,t) = 0, \\ h^1 & : & D_t^{\alpha} p_{\alpha,1}(n,t) = \left(\alpha n t^{-\alpha} - \alpha\lambda\right) p_{\alpha,0}(n,t), \\ h^2 & : & D_t^{\alpha} p_{\alpha,2}(n,t) = \left(\alpha n t^{-\alpha} - \alpha\lambda\right) p_{\alpha,1}(n,t), \\ h^3 & : & D_t^{\alpha} p_{\alpha,3}(n,t) = \left(\alpha n t^{-\alpha} - \alpha\lambda\right) p_{\alpha,2}(n,t), \\ & \vdots & \end{array}$$

So, the recursive formula of HPM is

$$D_t^{\alpha} p_{\alpha,i+1}(n,t) = \left(\alpha n t^{-\alpha} - \alpha\lambda\right) p_{\alpha,i}(n,t), \qquad i = 0, 1, 2, 3, \cdots. \tag{4.4}$$

When the initial approximation $p_{\alpha,0}(n,t)$ can be chosen, as

$$p_{\alpha,0}(n,t) = \frac{\lambda^n}{e^{\lambda} n!}.$$

Substituting $i = 0$ into (4.4), we obtain

$$D_t^{\alpha} p_{\alpha,1}(n,t) = \left(\alpha n t^{-\alpha} - \alpha\lambda\right) p_{\alpha,0}(n,t).$$

Using Theorem 2.2 and applying integral with respect to t on both sides, we get

$$\frac{d}{dt} p_{\alpha,1}(n,t) = \left(\frac{\alpha n t^{-\alpha} - \alpha\lambda}{t^{\alpha-1}}\right) \frac{\lambda^n}{e^{\lambda} n!},$$

which gives the solution as

$$p_{\alpha,1}(n,t) = \left((\alpha n) lnt - \lambda t^{\alpha}\right) \frac{\lambda^n}{e^{\lambda} n!}.$$

Substituting $i = 1$ into (4.4), we obtain

$$D_t^{\alpha} p_{\alpha,2}(n,t) = \left(\alpha n t^{-\alpha} - \alpha\lambda\right) p_{\alpha,1}(n,t).$$

Using Theorem 2.2 and applying integral with respect to t on both sides, we get

$$\frac{d}{dt} p_{\alpha,2}(n,t) = \left(\frac{\alpha n t^{-\alpha} - \alpha\lambda}{t^{\alpha-1}}\right) \left((\alpha n) lnt - \lambda t^{\alpha}\right) \frac{\lambda^n}{e^{\lambda} n!}.$$

Hence, the solution is

$$p_{\alpha,2}(n,t) = \left(\frac{(\alpha^2 n)(lnt)^2}{2} - n\lambda t^\alpha - (\alpha^2 n\lambda t^\alpha)lnt + \alpha n\lambda t^\alpha + \frac{\lambda^2 t^{2\alpha}}{2} \right) \frac{\lambda^n}{e^\lambda n!}.$$

Substituting $i = 2$ into (4.4), we obtain

$$D_t^\alpha p_{\alpha,3}(n,t) = \left(\alpha n t^{-\alpha} - \alpha\lambda\right) p_{\alpha,2}(n,t).$$

Using Theorem 2.2 and applying integral with respect to t on both sides, we get

$$\begin{aligned} \frac{d}{dt} p_{\alpha,3}(n,t) &= \left(\frac{\alpha n t^{-\alpha} - \alpha\lambda}{t^{\alpha-1}} \right) \left(\frac{(\alpha^2 n)(lnt)^2}{2} - n\lambda t^\alpha - (\alpha^2 n\lambda t^\alpha)lnt \right. \\ & \left. + \alpha n\lambda t^\alpha + \frac{\lambda^2 t^{2\alpha}}{2} \right) \frac{\lambda^n}{e^\lambda n!}. \end{aligned}$$

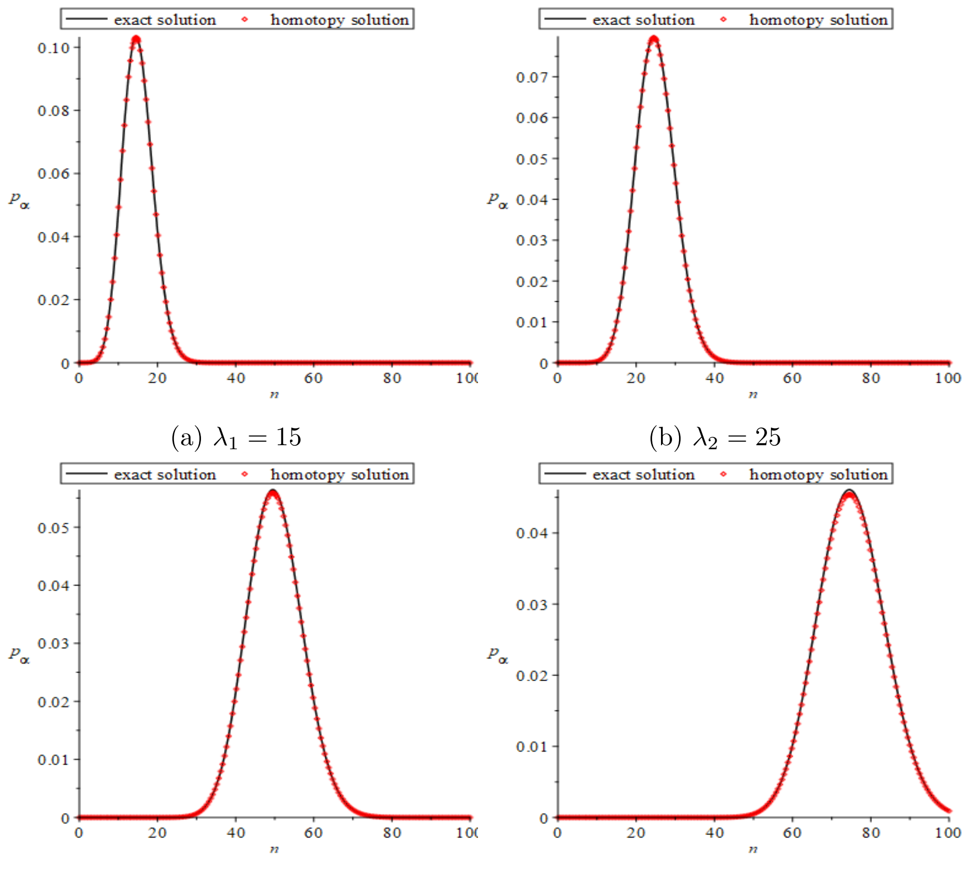
Then it obtains the solution as

$$\begin{aligned} p_{\alpha,3}(n,t) &= \left(\frac{(\alpha^3 n^2)(lnt)^3}{6} - n^2\lambda t^\alpha - (\alpha^2 n^2 \lambda t^\alpha)lnt + 2\alpha n^2 \lambda t^\alpha \right. \\ & + \frac{3n\lambda^2 t^{2\alpha}}{4} - \frac{(\alpha^2 n\lambda t^\alpha)(lnt)^2}{2} + (\alpha n\lambda t^\alpha)lnt - n\lambda t^\alpha \\ & \left. + \frac{(\alpha^2 n\lambda^2 t^{2\alpha})lnt}{2} - \frac{3\alpha n\lambda t^{2\alpha}}{4} - \frac{\lambda^3 t^{3\alpha}}{6} \right) \frac{\lambda^n}{e^\lambda n!} \\ & \vdots \end{aligned}$$

Therefore, the homotopy solution of (4.1) is provided by

$$\begin{aligned} p_\alpha(n,t) &= p_{\alpha,0}(n,t) + p_{\alpha,1}(n,t) + p_{\alpha,2}(n,t) + p_{\alpha,3}(n,t) + ... \\ &= \left(1 + (\alpha n)lnt - \lambda t^\alpha + \frac{(\alpha^2 n)(lnt)^2}{2} - n\lambda t^\alpha - (\alpha^2 n\lambda t^\alpha)lnt \right. \\ & + \alpha n\lambda t^\alpha + \frac{\lambda^2 t^{2\alpha}}{2} + \frac{(\alpha^3 n^2)(lnt)^3}{6} - n^2\lambda t^\alpha - (\alpha^2 n^2\lambda t^\alpha)lnt \\ & + 2\alpha n^2\lambda t^\alpha + \frac{3n\lambda^2 t^{2\alpha}}{4} - \frac{(\alpha^2 n\lambda t^\alpha)(lnt)^2}{2} + (\alpha n\lambda t^\alpha)lnt \\ & \left. - n\lambda t^\alpha + \frac{(\alpha^2 n\lambda^2 t^{2\alpha})lnt}{2} - \frac{3\alpha n\lambda t^{2\alpha}}{4} - \frac{\lambda^3 t^{3\alpha}}{6} \right) \frac{\lambda^n}{e^\lambda n!} + \cdots . \end{aligned}$$

Next, the homotopy solutions of fractional Poisson process with the first four terms can be calculated by using Maple program. We also compare with the exact solutions and the homotopy solutions by absolute errors that shown in Tables 1-4 . Some graphical solutions with several parameters of $\lambda = 15, 25, 50, 75$ were plotted and compared between the exact and the homotopy solutions in Figure 1, by setting $t = 1$, $\alpha = 1$. The 3D graphical homotopy and exact solutions of the fractional Poisson process are depicted Figure 2, by selecting $\lambda = 0.1$ with difference parameters $\alpha = 0.5, 0.75, 1$ as the following.



(a) $\lambda_1 = 15$

(b) $\lambda_2 = 25$

(c) $\lambda_3 = 50$

(d) $\lambda_4 = 75$

Figure 1: Comparison between exact solution (4.2) and homotopy solution with $t = 1$ and $\alpha = 1$.

Table 1: Comparison between exact solution (4.2) and homotopy solution with $t = 1$, $\alpha = 1$ and $\lambda_1 = 15$.

n	exact solution	homotopy solution	Absolute error
10	4.861×10^{-2}	4.846×10^{-2}	1.554×10^{-4}
20	4.181×10^{-2}	4.168×10^{-2}	1.337×10^{-4}
30	2.211×10^{-4}	2.204×10^{-4}	7.071×10^{-7}
40	4.146×10^{-8}	4.132×10^{-8}	1.326×10^{-10}
50	6.413×10^{-13}	6.393×10^{-13}	2.051×10^{-15}
80	5.225×10^{-32}	5.209×10^{-32}	1.600×10^{-34}
100	1.332×10^{-47}	1.328×10^{-47}	4.261×10^{-50}

Table 2: Comparison between exact solution (4.2) and homotopy solution with $t = 1$, $\alpha = 1$ and $\lambda_2 = 25$.

n	exact solution	homotopy solution	Absolute error
10	3.650×10^{-4}	3.631×10^{-4}	1.910×10^{-6}
20	5.192×10^{-2}	5.164×10^{-2}	2.800×10^{-4}
30	4.542×10^{-2}	4.518×10^{-2}	2.400×10^{-4}
40	1.408×10^{-3}	1.402×10^{-3}	6.100×10^{-6}
50	6.413×10^{-13}	6.393×10^{-13}	1.900×10^{-8}
80	1.328×10^{-18}	1.321×10^{-18}	7.000×10^{-21}
100	9.262×10^{-30}	9.215×10^{-30}	4.710×10^{-32}

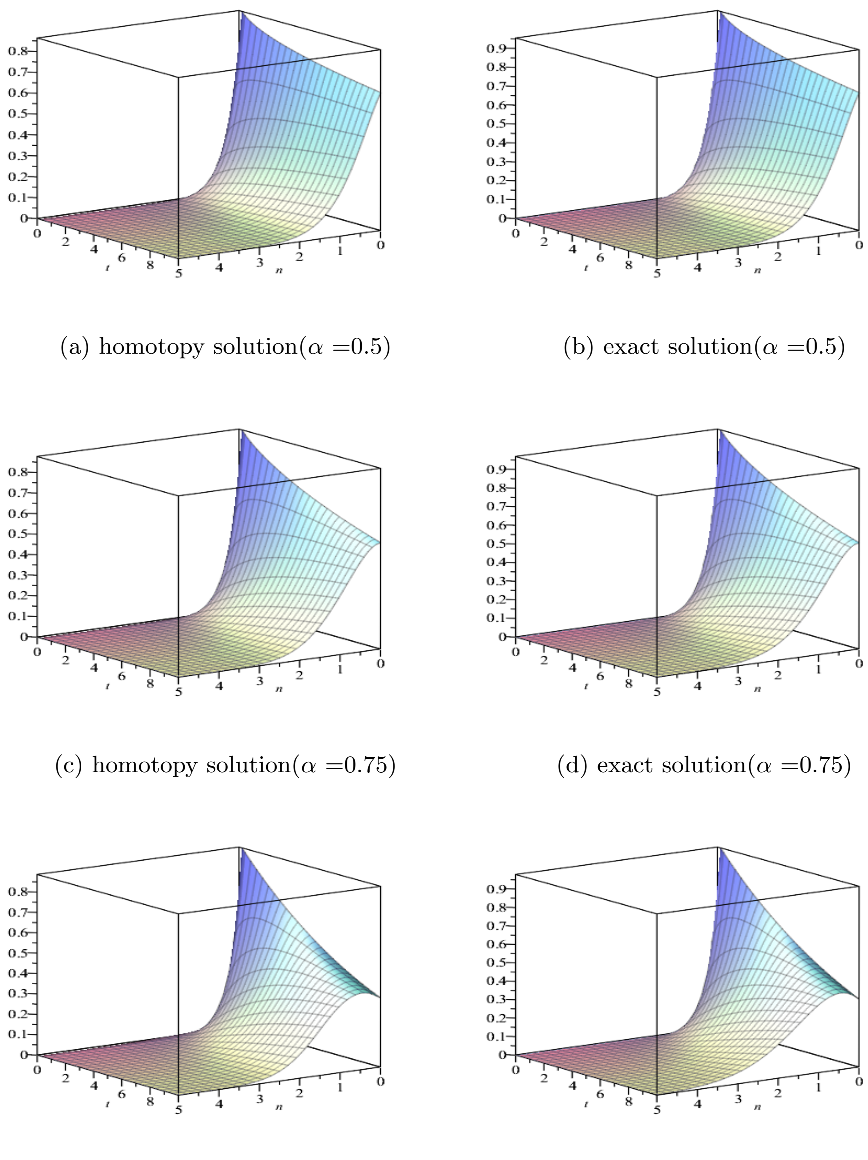
Table 3: Comparison between exact solution (4.2) and homotopy solution with $t = 1$, $\alpha = 1$ and $\lambda_3 = 50$.

n	exact solution	homotopy solution	Absolute error
10	5.191×10^{-12}	5.137×10^{-12}	5.410×10^{-14}
20	7.562×10^{-7}	7.485×10^{-7}	7.710×10^{-9}
30	6.773×10^{-4}	6.703×10^{-4}	7.010×10^{-6}
40	2.151×10^{-2}	2.130×10^{-2}	2.100×10^{-4}
50	5.633×10^{-2}	5.575×10^{-2}	5.800×10^{-4}
80	2.230×10^{-5}	2.207×10^{-5}	2.300×10^{-7}
100	1.631×10^{-10}	1.614×10^{-10}	1.710×10^{-12}

Table 4: Comparison between exact solution (4.2) and homotopy solution with $t = 1$, $\alpha = 1$ and $\lambda_4 = 75$.

n	exact solution	homotopy solution	Absolute error
10	4.158×10^{-21}	4.096×10^{-21}	6.200×10^{-23}
20	3.491×10^{-14}	3.437×10^{-14}	5.400×10^{-16}
30	1.804×10^{-9}	1.776×10^{-9}	2.800×10^{-11}
40	3.303×10^{-6}	3.252×10^{-6}	5.100×10^{-8}
50	4.988×10^{-4}	4.913×10^{-4}	7.500×10^{-6}
80	3.785×10^{-2}	3.729×10^{-2}	5.600×10^{-4}
100	9.208×10^{-4}	9.066×10^{-4}	1.420×10^{-5}

Figure 1(a) with the parameter $\lambda_1 = 15$ shows that value of probability, $p_\alpha(n,t)$, is increasing when the value n is increasing and closed to $\lambda_1 = 15$. The highest value of probability is taking place at $n = 15$ and after that it tends to zero when n getting sufficiently large. Similarly, from Figures 1(b), 1(c) and 1(d), we can see that the highest value of $p_\alpha(n,t)$ occurs when n approaches to $\lambda_2 = 25$, $\lambda_3 = 50$ and $\lambda_4 = 75$, respectively. As the results from Figure 1 and Tables 1-4, it can be concluded that the highest value of $p_\alpha(n,t)$ occurs when the value of n approaches λ, and it tends to zero where n is far enough from λ.



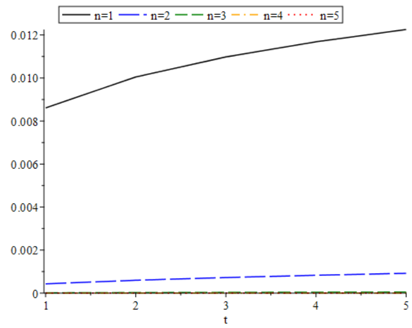
(a) homotopy solution(α =0.5) (b) exact solution(α =0.5)

(c) homotopy solution(α =0.75) (d) exact solution(α =0.75)

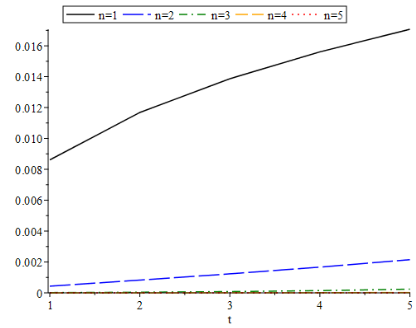
(e) homotopy solution(α =1) (f) exact solution(α =1)

Figure 2: Comparison between exact solutions in (4.2) and homotopy solutions with $\lambda = 0.1$.

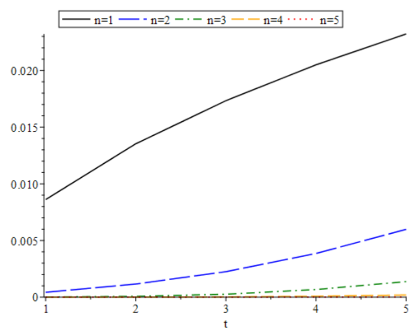
Figure 2 illustrates the comparison of the exact solutions for (4.2) and the homotopy solutions when $\alpha = 0.5, 0.75$ and 1. The results show that the homotopy solutions have similar pattern as the exact solutions when using different values of α.



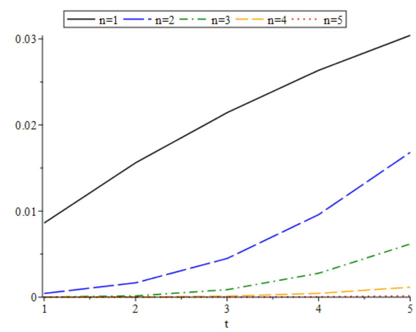
(a) $\alpha = 0.25$



(b) $\alpha = 0.5$



(c) $\alpha = 0.75$



(d) $\alpha = 1$

Figure 3: The absolute errors between exact solutions (4.2) and homotopy solutions with $\lambda = 0.1$.

Figure 3 shows the comparison between the exact solutions (4.2) and the homotopy solutions when using some n terms of perturbation. Here, the absolute errors, $|p_{\alpha,exact}(n,t) - p_\alpha(n,t)|$, are calculated when $n = 1, 2, .., 5$ and $\alpha = 0.25, 0.5, 0.75$ and 1. We also calculate the maximum absolute errors of the same case shown in Table 5.

Table 5: Maximum absolute errors between exact solutions in (4.2) and homotopy solutions with $\lambda = 0.1$.

α	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$
0.25	1.225×10^{-2}	9.161×10^{-4}	4.566×10^{-5}	1.707×10^{-6}	5.105×10^{-8}
0.50	1.701×10^{-2}	1.902×10^{-3}	1.418×10^{-4}	7.948×10^{-6}	3.609×10^{-7}
0.75	2.277×10^{-2}	3.808×10^{-3}	4.268×10^{-4}	3.770×10^{-5}	3.082×10^{-6}
1	2.885×10^{-2}	7.223×10^{-3}	1.260×10^{-3}	2.076×10^{-4}	3.530×10^{-5}

From Figure 3 and Table 5, we can conclude that the absolute errors and maximum absolute errors are decreasing when the number of terms n is increasing for all cases of α.

5 Conclusion

In this paper, the homotopy perturbation method (HPM) has been successfully applied to the fractional Poisson process (1.3) in the comformable sense. The results of HPM are compared with the exact solutions to determine the absolute error. The Maple program is used to illustrate the graphs in Figures 1 to 3 and calculate the errors in Table 1-5. We can see that the exact solutions and the homotopy solutions are closed to each other and have similar pattern when using several values of λ, α or n. Moreover, the HPM gives more accuracy when adding terms of perturbation but it will take more time to calculate and uses higher computational cost. The numerical results obtained in this work show that the homotopy perturbation method is efficiency and accuracy because the errors are small.

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