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Left and Right Magnifying Elements in Certain Linear Transformation Semigroups

Pongsan Prakitsri[1]

Faculty of Science at Sriracha, Kasetsart University Sriracha Campus,
Chonburi, Thailand, 20230
e-mail : pongsan.pr@ku.th

Abstract : An element a in a semigroup S is called a left (right) magnifying element in S if $aM = S$ ($Ma = S$) for some proper subset M of S. In this paper, we determine whether or not the linear transformation semigroups with infinite nullity and co-rank have left and right magnifying elements, and provide a characterization if such elements exist.

Keywords : left (right) magnifying element; linear transformation semigroups; nullity; co-rank.
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1 Introduction

In 1964, Ljapin [1] introduced the notion of left and right magnifying elements in a semigroup, that is, an element a in a semigroup S is said to be a *left* (*right*) *magnifying element* if there is a proper subset M of S such that $aM = S$ ($Ma = S$). Furthermore, a left (right) magnifying element a in S is said to be *strong* if there is a proper subsemigroup M such that $aM = S$ ($Ma = S$). These were introduced by Tolo [2] in 1969. It is obvious that strongly left (right) magnifying elements in S are left (right) magnifying elements.

Magnifying elements in semigroups have long been studied and many properties have also been investigated. In 1992, Catino and Migliorini [3] characterized when a bisimple semigroup contains left magnifying elements. Gutan [4] showed that semigroups which contain magnifying elements are factorizable. In 1994, Mag-

[1]Corresponding author.

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ill [5] provided necessary and sufficient conditions for elements in linear transformation semigroups to be left or right magnifying elements. Chinram and Buapradist [6] extended, in 2018, the results of Magill by considering linear transformations with invariant subspaces and gave characterization on these semigroups.

Let V be a vector space and $L(V)$ the semigroup, under composition, of all linear transfomations on V. For any $\alpha \in L(V)$, let $\ker\alpha$ and $im\,\alpha$ denote the kernel of α and the image of α, respectively. The dimension of a subspace U of V is denoted by $\dim U$. The quotient of V modulo by a subspace U is written as V/U.

We are interested in studying left or right magnifying elements in some subsemigroups of $L(V)$. For an infinite dimensional vector space V, let

$$OM(V) = \{\alpha \in L(V) \mid \dim(\ker\alpha) \text{ is infinite}\},$$
$$OE(V) = \{\alpha \in L(V) \mid \dim(V/im\,\alpha) \text{ is infinite}\}.$$

These are subsemigroups of $L(V)$, see [7] for more details. Note that $\dim(\ker\alpha)$ and $\dim(V/im\,\alpha)$ are called the *nullity* of α and the *co-rank* of α, respectively. Observe that the identity map is not an element in both $OM(V)$ and $OE(V)$. Moreover, $OM(V)$ does not contain any injective linear transformations on V, and similarly, surjective linear transformations on V are not contained in $OE(V)$.

In [5], the author characterized that $L(V)$ contains left (right) magnifying elements if and only if $\dim V$ is infinite. Moreover, in case $\dim V$ is infinite, he provided necessary and sufficient conditions when elements in $L(V)$ are left or right magnifying in $L(V)$, see below.

Theorem 1.1 ([5]). *Let $\alpha \in L(V)$ where $\dim V$ is infinite. The following statements hold.*
1. α is a left magnifying element if and only if α is surjective but not injective.
2. α is a right magnifying element if and only if α is injective but not surjective.

Below is a useful property that will be used in our results.

Proposition 1.2 ([8]). *Let $\alpha \in L(V)$ and let B_1 be a basis of $\ker\alpha$, B a basis of V containing B_1. Then*
(i) for each $v_1, v_2 \in B \setminus B_1$, $v_1 = v_2$ if and only if $\alpha(v_1) = \alpha(v_2)$;
(ii) $\alpha(B \setminus B_1)$ is a basis of $im\,\alpha$.

Let B be a basis of V and $u \in V$. A linear transformation on V can be defined on B. Now let $\{B_1, B_2\}$ be a partition of B. For $\alpha \in L(V)$ defined by $v\alpha = u$ and $w\alpha = v_w$ for all $v \in B_1$ and $w \in B_2$, we write

$$\alpha = \begin{pmatrix} B_1 & w \\ u & v_w \end{pmatrix}_{w \in B_2}.$$

We use this notation for an abbreviation of describing many linear transformations all along in this paper.

2 Left and right magnifiers in $OM(V)$ and $OE(V)$

Throughout this section, let V be an infinite dimensional vector space over a field. Our purpose is to provide a necessary and sufficient conditions for an element in $OM(V)$ and $OE(V)$ to be left or right magnifying elements. It has seen from Theorem 1.1 that a linear map in $L(V)$ that is surjective but not injective is a left magnifying element in $L(V)$. In $OM(V)$, every element is not injective. We first show a necessary and sufficient condition for element in $OM(V)$ to be a left magnifying element in $OM(V)$.

Theorem 2.1. *Let $\alpha \in OM(V)$. Then α is a left magnifying element in $OM(V)$ if and only if α is surjective.*

Proof. Assume that α is a left magnifying element in $OM(V)$. Then $\alpha M = OM(V)$ for some proper subset M of $OM(V)$. Let B be a basis of V and let $\{B_1, B_2\}$ be a partition of B such that $|B| = |B_1| = |B_2|$. Thus there is a bijection $\phi : B_2 \to B$. Define a linear transformation β in $L(V)$ by

$$\beta = \begin{pmatrix} B_1 & v \\ 0 & \phi(v) \end{pmatrix}_{v \in B_2}.$$

It can be seen that $\beta \in OM(V)$. Hence there exists $\gamma \in M$ such that $\alpha\gamma = \beta$. To show α is surjective, let $v \in B$. Hence $v = \phi(u_v) = \beta(u_v) = \alpha\gamma(u_v) = \alpha(\gamma(u_v))$ for some $u_v \in B_2$, so α is surjective.

Now suppose that α is surjective. Let

$$M = \{\gamma \in OM(V) \mid \gamma \text{ is not surjective}\}.$$

Next, let $\beta \in OM(V)$ and B_1 a basis of $\ker \beta$. Extend it to a basis B of V. Note that for each $v \in B \setminus B_1$, there is $u_v \in V$ such that $\alpha(u_v) = \beta(v)$ since α is surjective. Define $\gamma \in L(V)$ by

$$\gamma = \begin{pmatrix} B_1 & v \\ 0 & u_v \end{pmatrix}_{v \in B \setminus B_1}.$$

Thus $\gamma \in OM(V)$ since $\dim(\ker \gamma) = |B_1|$ is infinite. As $\alpha \in OM(V)$, we get γ is not surjective and hence $\gamma \in M$. Observe that for any $v \in B_1$, $\alpha\gamma(v) = 0 = \beta(v)$. Moreover, for any $v \in B \setminus B_1$, we have $\alpha\gamma(v) = \alpha(u_v) = \beta(v)$. Therefore, α is a left magnifying element in $OM(V)$. □

Remark 2.1. *The set M defined in the proof of the sufficiency of Theorem 2.1 is a subsemigroup of $OM(V)$. To show this, let $\gamma_1, \gamma_2 \in M$. Then they are not surjective. It follows that $\gamma_1\gamma_2$ is also not surjective.*

Hence we conclude a characterization for elements in $OM(V)$ to be strongly left magnifiers.

Corollary 2.2. *Let $\alpha \in OM(V)$. Then α is a strongly left magnifying elements if and only if α is surjective.*

Therefore the following fact is true.

Corollary 2.3. *Any left magnifying elements in $OM(V)$ are strong.*

We provide an example of a left magnifying element in $OM(V)$ as follows.

Example 2.4. Let B be a basis of V and let $\{B_1, B_2\}$ be a partition of B such that $|B| = |B_1| = |B_2|$ and B_0 a finite subset of B_1. Then there exists a bijection ϕ from B_2 to $B \setminus B_0$. Define $\alpha \in L(V)$ by

$$\alpha = \begin{pmatrix} B_1 \setminus B_0 & v & w \\ 0 & v & \phi(w) \end{pmatrix}_{v \in B_0, w \in B_2}.$$

This map is clearly in $OM(V)$ and surjective. By Theorem 2.1, α is a left magnifying element in $OM(V)$.

Note that any elements in $OM(V)$ are not injective. Next, we find that $OM(V)$ has no right magnifying elements in $OM(V)$.

Theorem 2.5. *$OM(V)$ has no right magnifying elements.*

Proof. Let $\alpha \in OM(V)$ and B_1 be a basis of $\ker \alpha$. Extend B_1 to a basis B of V. Since B_1 is infinite, there is a partition $\{B_1', B_1''\}$ of B_1 such that $|B_1| = |B_1'| = |B_1''|$. Define $\beta \in OM(V)$ by

$$\beta = \begin{pmatrix} B_1' & v \\ 0 & v \end{pmatrix}_{v \in B \setminus B_1'}.$$

Then, for any $\emptyset \neq M \subsetneq OM(V)$ and $\gamma \in M$, $\gamma\alpha(v) = 0$ but $\beta(v) = v \neq 0$ for all $v \in B_1'$. Hence α is not a right magnifying element. □

Corollary 2.6. *$OM(V)$ has no strongly right magnifying elements.*

We note that $OE(V)$ has no surjective elements. The following result is obtained.

Theorem 2.7. *$OE(V)$ has no left magnifying elements.*

Proof. Let $\alpha \in OE(V)$ and C_1 a basis of $im\,\alpha$. Extend C_1 to a basis C of V. Since $\alpha \in OE(V)$, we have $C \setminus C_1$ is infinite. Let $u \in C \setminus C_1$. Define $\beta \in L(V)$ by

$$\beta = \begin{pmatrix} C \setminus \{u\} & u \\ 0 & u \end{pmatrix}.$$

As $\dim(V/im\,\beta) = |C \setminus \{u\}| = |C|$ is infinite, we obtain $\beta \in OE(V)$. It is easy to see that, for any $\emptyset \neq M \subsetneq OE(V)$, $\alpha\gamma \neq \beta$ for all $\gamma \in M$ since $u \notin C_1$ and C_1 is a basis of $im\,\alpha$. Hence α is not a left magnifying element in $OE(V)$. □

Corollary 2.8. *$OE(V)$ has no strongly left magnifying elements.*

We next show that an injective linear transformation in $OE(V)$ is a right magnifying element and vice versa.

Theorem 2.9. *Let $\alpha \in OE(V)$. Then α is a right magnifying element if and only if α is injective.*

Proof. To show the necessity, suppose that α is a right magnifying element in $OE(V)$. Then there exists $M \subsetneq OE(V)$ such that $M\alpha = OE(V)$. Claim that α is injective. Let $u \in \ker\alpha$. Define $\beta \in L(V)$ by

$$\beta = \begin{pmatrix} w & u \\ 0 & u \end{pmatrix}_{w \notin \ker\alpha}.$$

Thus $\beta \in OE(V)$. It follows that there is $\gamma \in M$ such that $\gamma\alpha = \beta$. Hence $u = \beta(u) = \gamma\alpha(u) = 0$, so α is injective.

For the sufficiency, suppose that α is injective. Let B be a basis of V. Then $C_1 := \alpha(B)$ is a basis of $im\,\alpha$ and let C be a basis of V containing C_1. Now let

$$M = \{\gamma \in OE(V) \mid v \in \ker\gamma \text{ for all } v \in C \setminus C_1\}.$$

Then $M \subsetneq OE(V)$. Next, let $\beta \in OE(V)$. Note that if $v \in C_1$, there is $u_v \in B$ such that $\alpha(u_v) = v$. Define $\gamma \in L(V)$ by

$$\gamma = \begin{pmatrix} C \setminus C_1 & v \\ 0 & \beta(u_v) \end{pmatrix}_{v \in C_1}.$$

Then $\gamma \in M$ and $\dim(im\gamma) \leq \dim(im\beta)$. This implies that $\dim(V/im\gamma) \geq \dim(V/im\,\beta)$ and so $\gamma \in OE(V)$ since $\dim(V/im\,\beta)$ is infinite. Hence, for each $v \in B$, $\gamma\alpha(v) = \gamma(\alpha(v)) = \beta(v)$. Therefore, α is a right magnifying element in $OE(V)$. □

We give an example of $\gamma \in OE(V)$ that is not an element in the set M in the proof of the above theorem.

Example 2.10. We still use notations in the proof of the sufficiency of Theorem 2.9. Since $\alpha \in OE(V)$, we have $C \setminus C_1$ is infinite. Let $u \in C \setminus C_1$. Define $\gamma \in L(V)$ by

$$\gamma = \begin{pmatrix} C \setminus \{u\} & u \\ 0 & u \end{pmatrix}.$$

Then $\gamma \in OE(V)$ and $u \in C \setminus C_1$ but $u \notin \ker \gamma$. Hence $\gamma \notin M$. This guarantees that M is a proper subset of $OE(V)$.

Remark 2.2. *In the proof of Theorem 2.9, the set M is a subsemigroup of $OE(V)$. To show this, let $\gamma_1, \gamma_2 \in M$ and $v \in C \setminus C_1$. Then $v \in \ker \gamma_2$. It follows that $\gamma_1\gamma_2(v) = 0$ and thus $v \in \ker(\gamma_1\gamma_2)$. Hence $\gamma_1\gamma_2 \in M$.*

Therefore, a characterization for strongly magnifying elements in $OE(V)$ can be described by Theorem 2.9 and Remark 2.2.

Corollary 2.11. *Let $\alpha \in OE(V)$. Then α is a strongly right magnifying element if and only if α is injective.*

Corollary 2.12. *Any right magnifying elements in $OE(V)$ are strong.*

For the sake of completeness, we provide an example of $\alpha \in OE(V)$ which is injective.

Example 2.13. Let B be a basis of V. There is a partition $\{B_1, B_2\}$ of B such that $|B| = |B_1| = |B_2|$. Let $\phi : B \to B_1$ be a bijection. Then define $\alpha \in L(V)$ by

$$\alpha = \begin{pmatrix} v \\ \phi(v) \end{pmatrix}_{v \in B}.$$

Then $\dim(V/im\ \alpha) = |B \setminus B_1| = |B_2|$ is infinite. Hence $\alpha \in OE(V)$ and injective. Therefore, by Theorem 2.9, α is a right magnifying element in $OE(V)$.

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References

[1] E.S. Ljapin, Semigroups. Transl Math Monographs, 3, Rhode Island, 1963.

[2] K. Tolo, Factorizable semigroups, Pacific J. Math. 31 (1969) 523–535.

[3] F. Catino, F. Migliorini, Magnifying elements in semigroups, Semigroup Forum 44 (1992) 314–319.

[4] M. Gutan, Semigroups which contain magnifying elements are factorizable, Comm. Algebra 44 (1997) 314–319.

[5] K.D. Magill, Magnifying elements of transformation semigroups, Semigroup Forum 44 (1994) 314–319.

[6] R. Chinram, S. Buapradit, Magnifying elements in semigroups of linear transformations with invariant subspaces, J. Interdiscip. Math. 21 (6) (2018) 1457–1462.

[7] Y. Kemprasit, Algebraic Semigroup Theory. Pitak Press, Bangkok, 2002 (in Thai).

[8] S. Chaopraknoi, T. Phongpattanacharoen, P. Prakitsri, The natural partial order on linear transformation semigroups with nullity and corank bounded below. Bull. Aust. Math. Soc. 91 (2015) 104-115.

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