Thai Journal of Mathematics : 269-283
Special Issue : Annual Meeting in Mathematics 2019
http://thaijmath.in.cmu.ac.th
Online ISSN 1686-0209

A Non-standard Ternary Representation of Integers

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Abstract : In this paper, we introduce a family $\mathcal{A}_{\{0,1,5\}}$ of representations in base three with digit set $\{0,1,5\}$, i.e.,

$$\mathcal{A}_{\{0,1,5\}} = \left\{ \sum_{i=0}^{r} \epsilon_i 3^i : \epsilon_i \in \{0,1,5\},\ \text{for } 0 \le i \le r \text{ and } r \in \mathbb{N}_0 \right\}.$$

We discuss a structure and property of the increasing sequence of the elements in $\mathcal{A}_{\{0,1,5\}}$. Moreover, we show that a sequence associated with the maximal sets of consecutive integers called *max-sets* in $\mathcal{A}_{\{0,1,5\}}$ is related to Pell and Pell-Lucas numbers.

Keywords : non-standard representation; maximal set of consecutive integers; Pell number; Pell-Lucas number; combinatorial number theory.
2010 Mathematics Subject Classification : 11B85; 11B37; 05A19.

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1 Introduction

A non-negative integer can be uniquely expressed in the form of $\sum_{i=0}^{r} \epsilon_i d^i$, where $\epsilon_i \in \{0,1,2,\ldots,d-1\}$, $\epsilon_r \neq 0$ and $r \in \mathbb{N}_0$. In 1982, Matula [1] introduced a set $\mathcal{P}[D]$ called a *radix representation* system which is the set of polynomials with coefficients from finite integer-digit set D, where $0 \in D$. The set D is *basic* for integer base d if each integer n can be uniquely represented in the form of $\sum_{i=0}^{r} \epsilon_i d^i$, for some $r \geq 0$, where $\epsilon_i \in D$. In this work, we are interested in the

This research is financially supported by Mahasarakham University 2015.
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area of *non-standard representations*, which are the representations that the digit set is not basic.

The non-standard representation has been studied extensively in many aspects such as the number of representations of an integer n in a given digit set and a property of number in a representation. In 1989, Reznick [2] computed the number of representations in base d with a digit set $\{0, 1, 2, \ldots, n-1\}$ with a condition that $n > d$. In 2013, Anders, Dennison, Lansing and Reznick [3] created a digit set of a representation in base two such that the number of representations is periodic in modulo 2.

In this research, we are interested in the set of representations in base three with digit set $\{0, 1, 5\}$. We let

$$\mathcal{A}_{\{0,1,5\}} = \left\{ \sum_{i=0}^{r} \epsilon_i 3^i : \epsilon_i \in \{0, 1, 5\}, \text{ for } 0 \le i \le r \text{ and } r \in \mathbb{N}_0 \right\}.$$

We see that $2 \notin \mathcal{A}_{\{0,1,5\}}$; hence, the set $\{0, 1, 5\}$ is not basic. However, we are able to apply the method appearing in Matula's work to the increasing sequence of numbers in $\mathcal{A}_{\{0,1,5\}}$.

In this paper, we introduce the definition of $\mathcal{A}_{\{0,1,5\}}$. We use Matula's approach [1] to construct a directed tree of the elements in $\mathcal{A}_{\{0,1,5\}}$ and its complement. The main result in this research is on a property of a family of maximal sets of consecutive integers called *max-sets* in $\mathcal{A}_{\{0,1,5\}}$. By constructing a relation between max-sets, we are able to construct a family of isomorphic rooted trees of max-sets. The sequence of max-sets can be associated with the famous Pell and Pell-Lucas numbers.

2 Introduction to the ternary representation

A set D is a *residue* digit for d if, for each $i \in D$, there exists $j \in \{0, \ldots, d-1\}$ such that $i \equiv j \pmod{d}$. Matula [1] showed that if D is basic for d, then it is a residue digit for d. However, the converse is not true. Indeed, in this work, we are interested in a digit set $\{0, 1, 5\}$ for base 3. We can see that $\{0, 1, 5\}$ is a residue digit for 3, but it is not basic because $2 \notin \mathcal{A}_{\{0,1,5\}}$.

Let $\{x(n)\}_{n \ge 0}$ be the increasing sequence of the elements in $\mathcal{A}_{\{0,1,5\}}$ appearing in [4]:

$$0, 1, 3, 4, 5, 8, 9, 10, 12, 13, 14, 15, 16, 17, 20, 24, \ldots.$$

Theorem 2.1 is a part of a theorem given by Matula [1] on the necessary and sufficient condition for a basic set. Theorem 2.2 is the restriction of Matula's work to the set of non-negative integers.

Theorem 2.1 ([1]). *For any base d, if D is a residue digit for d, then each element in*

$$\left\{ \sum_{i=0}^{r} \epsilon_i d^i : \epsilon_i \in D, \text{ for } 0 \le i \le r \text{ and } r \in \mathbb{N}_0 \right\}$$

is uniquely represented.

Theorem 2.2 ([1]). *Let D be a residue digit set for base d such that $|D| = d$ and let $\epsilon_{min} = \min\{\epsilon : \epsilon \in D\}$, $\epsilon_{max} = \max\{\epsilon : \epsilon \in D\}$. Then D is basic for d if and only if there exist a representation for all integers i with $\frac{\epsilon_{min}}{d-1} \leq i \leq \frac{\epsilon_{max}}{d-1}$.*

By Theorem 2.1, each element in $\mathcal{A}_{\{0,1,5\}}$ is uniquely represented. To prove Theorem 2.1, Matula used a function $p(n)$ defined by $p(n) = \frac{n-\epsilon}{d}$, where $n \equiv \epsilon \pmod{d}$, for some $\epsilon \in D$ and $p^i(n) = p(p^{i-1}(n))$, for $i \geq 1$. In this paper, we use a term *predecessor* of n to denote $p(n)$. He also showed that the predecessor is uniquely determined. By Theorem 2.2, we see that the complement $\mathcal{A}^C_{\{0,1,5\}} = \mathbb{N}_0 \setminus \mathcal{A}_{\{0,1,5\}}$ is

$$\mathcal{A}^C_{\{0,1,5\}} = \left\{ 2 \cdot 3^r + \sum_{i=0}^{r-1} \epsilon_i 3^i : \epsilon_i \in \{0,1,5\}, \text{ for } 0 \leq i \leq r-1 \text{ and } r \in \mathbb{N}_0 \right\}.$$

Hence, $\mathcal{A}_{\{0,1,5\}}$ and $\mathcal{A}^C_{\{0,1,5\}}$ partition $\mathbb{N}_0$.

For a basic digit set D, Matula [1] used the predecessor function to construct a directed rooted tree where the vertex set is the set of integers and the edge relation is $(n, p(n))$. By the same construction, we are able to construct directed rooted trees G_0 and G_2 with the vertex sets $V(G_0) = \mathcal{A}_{\{0,1,5\}}$, $V(G_2) = \mathcal{A}^C_{\{0,1,5\}}$ and the edge set are

$$E(G_i) = \{(u, v) \in V(G_i) \times V(G_i) : v = 3u + \epsilon, \text{ for some } \epsilon \in \{0,1,5\}\},$$

for $i \in \{0, 2\}$. Since each $v \in \mathbb{N}_0$ has a unique predecessor, it follows that the constructed graphs are directed trees rooted at 0 and 2, respectively.

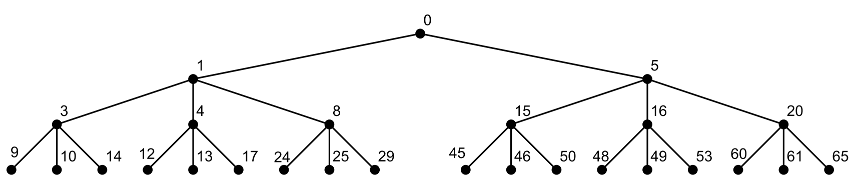

Figure 1: The underlying graph of G_0.

3 Maximal set of consecutive integers

In this section, we investigate a property of the maximal set of consecutive integers in $\mathcal{A}_{\{0,1,5\}}$. From now on, for any $s, t \in \mathbb{N}_0$ such that $s \leq t$, we use notation $[s, t]$ for the set $\{s, s+1, \ldots, t\}$. We give structures and properties of sequences related to two types of max-sets defined in Definition 3.2 and Definition

3.5. The sequences associated with such max-sets have a beautiful relation with Pell and Pell-Lucas numbers which is presented in Theorem 3.13 and Theorem 3.15, respectively. Then, later in this section, we construct a tree of blocks and a tree of dblocks by first creating the edge relations for the max-sets in an arbitrary subset of $\mathcal{A}_{\{0,1,5\}} \setminus \{0\}$ and $\mathcal{A}^C_{\{0,1,5\}}$. Then, we restrict the relations to the family of blocks and the family of dblocks.

Definition 3.1. For $\mathcal{A} \subsetneq \mathbb{N}_0$, a set $[s,t]$ is a *max-set* in $\mathcal{A}$ if $[s,t]$ is a maximal set of consecutive integers in $\mathcal{A}$; that is $s-1 \notin \mathcal{A}$ and $t+1 \notin \mathcal{A}$.

The consecutive 1's in the sequence of characteristic function of the elements in $\mathcal{A}$ represent the consecutive elements in $\mathcal{A}$ and the consecutive 0's represent the consecutive elements in $\mathcal{A}^C$. Hence, in order to study the max-set in $\mathcal{A}$, we can instead investigate the properties of the occurrence of 01^m0, for $m \geq 1$. Allouche and Shallit [5] showed that the number of occurrences of $P \in E = 1(0+1)^*$ in base d expansion of n with overlapping allowed is 2-regular. However, in order to count the number of max-sets, the number of each occurrence cannot be counted multiple times.

Definition 3.2. A *block* $[[s,t]]$ is a max-set $[s,t]$ in $\mathcal{A}_{\{0,1,5\}}$ and an *anti-block* $[[x,y]]$ is a max-set $[x,y]$ in $\mathcal{A}^C_{\{0,1,5\}}$.

In Example 3.3 and 3.4, we give some examples of the blocks and anti-blocks in $\mathcal{A}_{\{0,1,5\}}$.

Example 3.3. The following are examples of blocks in $\mathcal{A}_{\{0,1,5\}}$:
$[[0,1]]$, $[[3,5]]$, $[[8,10]]$, $[[12,17]]$, $[[20,20]]$, $[[24,25]]$, $[[27,32]]$, $[[35,37]]$, $[[39,53]]$, $[[56,56]]$, $[[60,61]]$, $[[65,65]]$, $[[72,73]]$, $[[75,77]]$.

Example 3.4. The following are examples of anti-blocks in $\mathcal{A}_{\{0,1,5\}}$:
$[[2,2]]$, $[[6,7]]$, $[[11,11]]$, $[[18,19]]$, $[[21,23]]$, $[[26,26]]$, $[[33,34]]$, $[[38,38]]$, $[[54,55]]$, $[[57,59]]$, $[[62,64]]$, $[[66,71]]$, $[[74,74]]$, $[[78,79]]$.

Next, we partition $\mathcal{A}_{\{0,1,5\}} \setminus \{0\}$ and $\mathcal{A}^C_{\{0,1,5\}}$ by the *degree* of n which is the maximum r such that $n = \sum_{i=0}^{r} \epsilon_i 3^i$, where $\epsilon_r \in \{1,2,5\}$ and $\epsilon_i \in \{0,1,5\}$ for all $i < r$. For $r \geq 1$, let $\mathcal{A}_{\{0,1,5\}}|_r$ and $\mathcal{A}^C_{\{0,1,5\}}|_r$ be the subset of $\mathcal{A}_{\{0,1,5\}}$ and $\mathcal{A}^C_{\{0,1,5\}}$ consisting of all the elements of degree r, respectively, i.e.,

$$\mathcal{A}_{\{0,1,5\}}|_r = \left\{ \sum_{i=0}^{r} \epsilon_i 3^i : \epsilon_i \in \{0,1,5\} \text{ and } \epsilon_r \neq 0 \right\}$$

and

$$\mathcal{A}^C_{\{0,1,5\}}|_r = \left\{ 2 \cdot 3^r + \sum_{i=0}^{r-1} \epsilon_i 3^i : \epsilon_i \in \{0,1,5\} \right\}.$$

By considering the max-sets in $\mathcal{A}_{\{0,1,5\}}|_r$ and $\mathcal{A}^C_{\{0,1,5\}}|_r$, we introduce another kind of max-sets associated with the degrees of the elements.

Definition 3.5. For $r \geq 1$, a *dblock* $\langle s,t\rangle$ in $\mathcal{A}_{\{0,1,5\}}$ is a max-set $[s,t]$ in $\mathcal{A}_{\{0,1,5\}}|_r$ with degree r. An *anti-dblock* $\langle s,t\rangle$ in $\mathcal{A}_{\{0,1,5\}}$ is a max-set in $\mathcal{A}^C_{\{0,1,5\}}|_r$ with degree r.

In example 3.6, the non-underlined numbers are in $\mathcal{A}_{\{0,1,5\}}$. They are divided into two parts consisting of elements with even and odd degrees. The non-underlined bold numbers represent the numbers with odd degree; otherwise, the degrees are even. In Table 1 and 2, we present the dblocks and the anti-dblocks in $\mathcal{A}_{\{0,1,5\}} \cap \{0,\ldots,80\}$.

Example 3.6. **1**, 2, 3, 4, **5**, 6, 7, 8, **9, 10**, 11, **12, 13, 14**, 15, 16, **17**, 18, 19, 20, 21, 22, 23, **24, 25**, 26, 27, 28, **29**, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, **45, 46**, 47, **48, 49, 50**, 51, 52, **53**, 54, 55, 56, 57, 58, 59, **60, 61**, 62, 64, **65**, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80.

Degree	Dblock
undefined	$\langle 0,0\rangle$
0	$\langle 1,1\rangle, \langle 5,5\rangle,$
1	$\langle 3,4\rangle, \langle 8,8\rangle, \langle 15,16\rangle, \langle 20,20\rangle$
2	$\langle 9,10\rangle, \langle 12,14\rangle, \langle 17,17\rangle, \langle 24,25\rangle, \langle 29,29\rangle, \langle 45,46\rangle,$ $\langle 48,50\rangle, \langle 53,53\rangle, \langle 60,61\rangle, \langle 65,65\rangle$
3	$\langle 27,28\rangle, \langle 30,32\rangle, \langle 35,37\rangle, \langle 39,44\rangle, \langle 47,47\rangle, \langle 51,52\rangle,$ $\langle 56,56\rangle, \langle 72,73\rangle, \langle 75,77\rangle, \langle 80,80\rangle$

Table 1: Example of dblocks in $\mathcal{A}_{\{0,1,5\}} \cap \{0,\ldots,80\}$

Degree	Anti-dblock
0	$\langle 2,2\rangle$
1	$\langle 6,7\rangle, \langle 11,11\rangle,$
2	$\langle 18,19\rangle, \langle 21,23\rangle, \langle 26,26\rangle, \langle 33,34\rangle, \langle 38,38\rangle,$
3	$\langle 54,55\rangle, \langle 57,59\rangle, \langle 62,64\rangle, \langle 66,71\rangle, \langle 74,74\rangle, \langle 78,79\rangle$

Table 2: Example of anti-dblocks in $\mathcal{A}_{\{0,1,5\}} \cap \{0,\ldots,80\}$

By comparing Table 1 and Example 3.3, we see that $[[0,1]]$ and $[[3,5]]$ represent two blocks in $\mathcal{A}_{\{0,1,5\}}$ but they are not dblocks in $\mathcal{A}_{\{0,1,5\}}$.

Let $\bar{\mathcal{A}} \in \{\mathcal{A}_{\{0,1,5\}}, \mathcal{A}^C_{\{0,1,5\}}\}$ and $\mathcal{A}_r$ be a subset of $\bar{\mathcal{A}}$ consisting of all elements of degree r.

Later in Theorem 3.13 and 3.15, we show that the number of blocks and the number of dblocks are related to Pell-numbers and Pell-Lucas numbers. In order to get the results in Theorem 3.13 and 3.15, we construct a morphism for the Pell numbers in Lemma 3.7, where a *morphism* [6] is a homomorphism function

h between two languages such that $h(xy) = h(x)h(y)$, for all x, y in the domain. Then, we construct an automaton M-DFAO to encode the characteristic sequence $\{\chi(n)\}_{n\geq 0}$, where χ is the characteristic function of $\mathcal{A} \subset \bar{\mathcal{A}}$.

Next, let us recall the defenitions and related properties of Pell and Pell-Lucas numbers. The *Pell numbers* P_n [7], [8, p. 45] is defined by

$$P_n = 2P_{n-1} + P_{n-2}, \tag{3.1}$$

for $n \geq 2$ and $P_0 = 0$ and $P_1 = 1$.

The *Pell-Lucas numbers* Q_n [9], [8, p. 23] is defined by

$$Q_n = 2Q_{n-1} + Q_{n-2},$$

for $n \geq 2$ and $Q_0 = 1, Q_1 = 1$.

The following identities [8, p. 193] are used in this research, for $n \geq 1$,

$$Q_n = P_n + P_{n-1}, \tag{3.2}$$

$$\sum_{i=0}^{n} P_i = \frac{Q_{n+1} - 1}{2}, \tag{3.3}$$

$$\sum_{i=0}^{n} Q_i = P_{n+1}. \tag{3.4}$$

The sequence of numbers in (3.3) also appears in [10].

Lemma 3.7. *Define a morphism* $\phi : \{a, b\}^* \to \{a, b\}^*$ *by*

$$\phi(a) = ab \text{ and } \phi(b) = aba.$$

Then, $|\phi^n(a)| = P_{n+1}$*, for all* $n \geq 0$.

Proof. We see that $|\phi^0(a)| = |a| = 1 = P_1$ and $|\phi(a)| = |ab| = 2 = P_2$. Let ϕ_i^n be the number of i's in $\phi^n(a)$ for $i \in \{a, b\}$. So

$$|\phi^n(a)| = \phi_a^n + \phi_b^n.$$

Since each a in $\phi^{n-1}(a)$ contributes to exactly one a and b in $\phi^n(a)$, and each b in $\phi^{n-1}(a)$ contributes to two a's and one b in $\phi^n(a)$, it follows that

$$\begin{aligned}\phi_a^n + \phi_b^n &= (\phi_a^{n-1} + 2\phi_b^{n-1}) + (\phi_a^{n-1} + \phi_b^{n-1}) \\ &= 2(\phi_a^{n-1} + \phi_b^{n-1}) + \phi_b^{n-1} \\ &= 2|\phi^{n-1}(a)| + \phi_b^{n-1} \\ &= 2|\phi^{n-1}(a)| + |\phi^{n-2}(a)|.\end{aligned}$$

Thus, $|\phi^n(a)| = P_{n+1}$ for all $n \geq 0$. $\square$

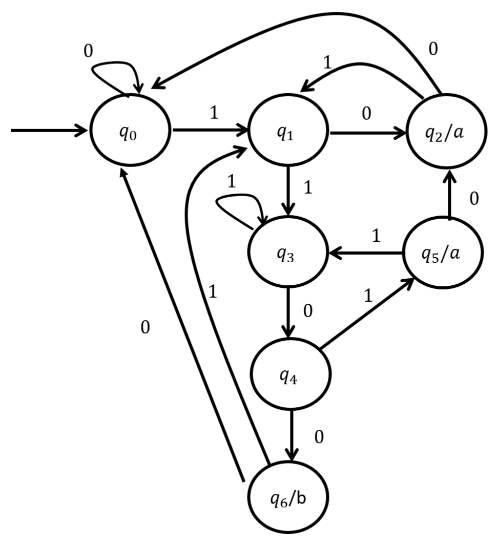

Figure 2: M-DFAO encoding characteristic sequence $\{\chi(n)\}_{n\geq 0}$

In Figure 2, we give an automaton

$$M = (Q, \{0,1\}^*, \delta, q_0, \{a,b\}^*, \tau)$$

encoding the characteristic sequence $\{\chi(n)\}_{n\geq 0}$. The following are the description of each state in M:

- q_0 is the starting state
- $q_1 = \delta(q_0, 0^*1)$ is the state that the first element of a max-set appears
- $q_2 = \delta(q_0, 0^*10)$ indicates that the max-set has only one element; in this state, we encode a
- $q_3 = \delta(q_0, 0^*111^*)$ indicates that the max-set has more than one element
- $q_4 = \delta(q_0, 0^*111^*0)$ indicates the end of the max-set from state q_3; the max-set in this state will be encoded in states q_5 or q_6
- $q_5 = \delta(q_0, 0^*111^*01)$ is the state that $\chi(t+2) = 1$ when there exist s such that $[s,t]$ is a max-set of size greater than one; the max-set is encoded by a
- $q_6 = \delta(q_0, 0^*111^*00)$ is the state that $\chi(t+2) = 0$ when there exist s such that $[s,t]$ is a max-set of size greater than one; the max-set is encoded by b.

In conclusion, the automaton M encodes a max-set as follows:

- a max-set with one element, i.e., 0^*10, by a

- a max-set $[s,t]$ with $t > s$ and $t+2 \in \mathcal{A}$, i.e., 0^*111^*01, by a
- a max-set $[s,t]$ with $t > s$ and $t+2 \notin \mathcal{A}$, i.e., 0^*111^*00, by b.

We note that the automaton M does not encode the max-set that is yet ended. This encoding is used to calculate the number of max-sets in $\mathcal{A}$. For convenience, we use notation $\chi([m,n]) = \chi(m)\dots\chi(n)$.

Theorem 3.8. *For a max-set $[s,t]$ in $\mathcal{A}$,*

$$\tau(\delta(q_0, \chi([3s, 3t+6]))) \in \{ab, aba, ba\}.$$

Especially, if $w = \chi([s-1, t+1])$ and $w' = \chi([3s, 3t+6])$, then

- *$\tau(\delta(q_0, w')) = ba$, where $w = 010$,*
- *$\tau(\delta(q_0, w')) = aba$, where $w = 01^{t-s+1}01$ and $t > s$,*
- *$\tau(\delta(q_0, w')) = ab$, where $w = 01^{t-s+1}00$ and $t > s$.*

Proof. Let $[s,t]$ be a max-set in $\mathcal{A}$. We note that $\chi(n) = \chi(p(n))$, where $p(n)$ is the predecessor of n. If $\chi([s-1, t+1]) = 010$, then, by using case analysis on $s+2$ modulo 3, we can show that $\chi(s+2) = 0$. Hence,

$$\chi([3s, 3s+6]) = \chi(s)\chi(s)\chi(s-1)\chi(s+1)\chi(s+1)\chi(s)\chi(s+2) = 1^2 0^3 10.$$

So

$$\tau(\delta(q_0, \chi([3s, 3t+6]))) = ba.$$

If $\chi([s-1, t+2]) = 01^{t-s+1}00$, where $t > s$, then

$$\chi([3s, 3t+6]) = 1^2 0 1^{3(t-s)} 0010.$$

So

$$\tau(\delta(q_0, \chi([3s, 3s+6]))) = aba.$$

Next, if $\chi([s-1, t+2]) = 01^{t-s+1}01$, where $t > s$, then

$$\chi([3s, 3t+6]) = 1^2 0 1^{3(t-s)} 0011.$$

So

$$\tau(\delta(q_0, \chi([3s, 3t+6]))) = ab.$$

□

In Theorem 3.8, we see that adding prefix 0^* to the image and pre-image of χ does not change the encoding of the max-set. We also note that, in the last case of Theorem 3.8, the automaton M does not encode the suffix 11.

Corollary 3.9. *A max-set encoded by a contributes to two max-sets consisting of a and b, and a max-set encoded by b contributes to three max-sets consisting of two a's and one b.*

Corollary 3.10. *If a max-set $[s,t]$ is encoded by a, then $\chi([3s,3t+6])$ is encoded by either ab or ba. If $[s,t]$ is encoded by b, then $\chi([3s,3t+6])$ is encoded by aba.*

For $\mathcal{A} \subset \bar{\mathcal{A}}$, let I be a family consisting of the max-sets in $\mathcal{A}$. We now define functions $\alpha_1, \alpha_2, \alpha_3$ which are the edge relations of the tree of blocks and the tree of dblocks. Define $\alpha_1, \alpha_2, \alpha_3 : \mathcal{A} \times I \to P(\mathbb{N}_0)$ by

$$\alpha_1(\mathcal{A}, [s,t]) = \begin{cases} [3s-1, 3s+1], & \text{if } s-2 \in \mathcal{A}, \\ [3s, 3s+1], & \text{otherwise}, \end{cases} \tag{3.5}$$

$$\alpha_2(\mathcal{A}, [s,t]) = \begin{cases} [3s+3, 3t+2], & \text{if } s \neq t, \\ \emptyset, & \text{otherwise}, \end{cases} \tag{3.6}$$

$$\alpha_3(\mathcal{A}, [s,t]) = \begin{cases} [3t+5, 3t+5], & \text{if } t+2 \notin \mathcal{A}, \\ \emptyset, & \text{otherwise}. \end{cases} \tag{3.7}$$

We note that the function $\alpha_1, \alpha_2, \alpha_3$ are not defined on $[0,1], [0,0]$. By the definition of $\alpha_1, \alpha_2, \alpha_3$, we see that the automaton M encodes the images of α_1 and α_3 by a if they are not empty, whereas, the image of α_2 is encoded by b. If $\mathcal{A}$ is clear from the context, then we omit $\mathcal{A}$ when applying α_i's.

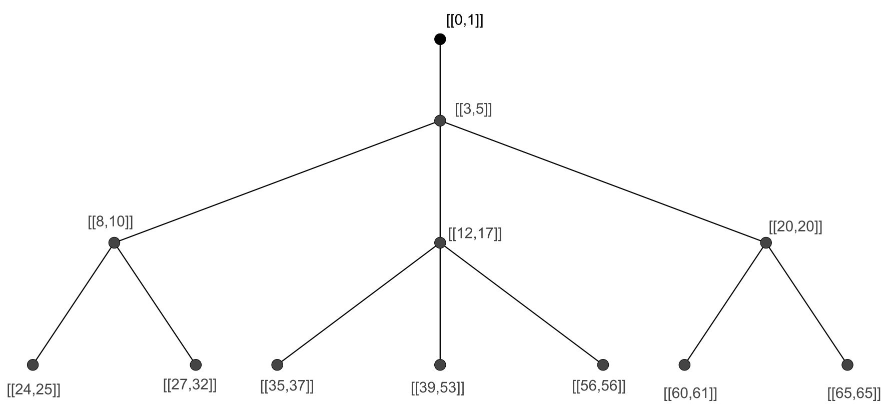

Figure 3: Tree of blocks for $\mathcal{A}_{\{0,1,5\}}$.

Let $[s',t']$ and $[s,t]$ be a disjoint pair of max-sets. If $s' < s$, then $x < y$, for all $x \in [s',t']$ and $y \in [s,t]$. We define an order on the max-sets by $[s',t'] \le [s,t]$ if $s' \le s$. By the definition of α_i's, if $\alpha_i([s,t]) \neq \emptyset$ and $\alpha_j([s,t]) \neq \emptyset$, for some $i, j \in \{1,2,3\}$, then

$$\alpha_i([s,t]) < \alpha_j([s,t]), \text{ for } i < j. \tag{3.8}$$

If $[s',t'] < [s,t]$ and $\alpha_i([s',t']), \alpha_i([s,t])$ are not empty, then

$$\alpha_i([s',t']) < \alpha_i([s,t]), \text{ for all } i = 1,2,3. \tag{3.9}$$

So, $\alpha_1, \alpha_2, \alpha_3$ preserve the order of the max-sets. For a family of max-sets I, we write

$$\alpha_i(I) = \{\alpha_i(x) : x \in I\}.$$

Next we define a function α from a family of max-sets to a set of non-negative integers by

$$\alpha(I) = \bigcup_{Y \in \alpha_1(I) \cup \alpha_2(I) \cup \alpha_3(I)} Y. \tag{3.10}$$

Lemma 3.11. *Let I be the family of max-sets in $\mathcal{A}$. For each max-set $z \in \alpha_1(I) \cup \alpha_2(I) \cup \alpha_3(I)$, there exists a unique max-set $x \in I$ and a unique $i \in \{1, 2, 3\}$ such that $z = \alpha_i(x)$.*

Proof. Let $[s, t], [s', t'] \in I$. Suppose $\alpha_i([s, t]) = \alpha_j([s', t'])$, for some $i, j \in \{1, 2, 3\}$. For the case $[s, t] \neq [s', t']$, without loss of generality, we suppose that $[s, t] < [s', t']$. It follows that $\alpha_i([s, t]) < \alpha_i([s', t'])$, and hence, $i \neq j$. If $i < j$, then $\alpha_i([s, t]) < \alpha_i([s', t']) < \alpha_j([s', t'])$. If $i > j$, we have $(i, j) \in \{(2, 1), (3, 1), (3, 2)\}$. Since, for any max-set $x \in \mathcal{A}$, $|\alpha_i(x)| = 1$ if and only if $i = 3$. It remains to consider $(i, j) = (2, 1)$. Suppose that $\alpha_2([s, t]) = \alpha_1([s', t'])$. It follows that either $[3s + 3, 3t + 2] = [3s' - 1, 3s' + 1]$ or $[3s + 3, 3t + 2] = [3s', 3s' + 1]$. We see that in either case, the last elements in the max-sets are incongruent modulo three. This completes the proof. $\square$

Lemma 3.11 allows us to compute the number of blocks and dblocks by counting the encoding obtained by the automaton M. It also allows us to construct a directed rooted trees of max-sets in Figure 3 and a tree of dblocks in Figure 5.

Lemma 3.12. *For $r \geq 3$, the minimum and maximum blocks in $\mathcal{A}_{\{0,1,5\}} \cap \{8 \cdot 3^{r-3}, \dots, 8 \cdot 3^{r-2} - 1\}$ are*

$$[[8 \cdot 3^{r-3}, 8 \cdot 3^{r-3} + 1]] \text{ and } \left[\left[\frac{5(3^{r-1} - 1)}{2}, \frac{5(3^{r-1} - 1)}{2}\right]\right],$$

respectively.

Proof. For $r = 3$, the minimum and the maximum blocks are $[[8, 10]]$ and $[[20, 20]]$, respectively. Let $[s, t], [s', t'] \in B$. By (3.8) and (3.9), the minimum block is $\alpha_1^{r-3}([[8, 10]]) = [[8 \cdot 3^{r-3}, 8 \cdot 3^{r-3} - 1]]$ and the maximum block is $\alpha_3^{r-3}([[20, 20]]) = \left[\left[\frac{5(3^{r-1}-1)}{2}, \frac{5(3^{r-1}-1)}{2}\right]\right]$.

$\square$

Theorem 3.13. *For each $r \in \mathbb{N}$, the number of blocks in $\mathcal{A}_{\{0,1,5\}} \cap \{0, \dots, 8 \cdot 3^r - 1\}$ is P_r.*

Proof. For each $x \leq 8 \cdot 3^r - 1$, we see that $p(x) \leq \left\lfloor \frac{8 \cdot 3^r - 1}{3} \right\rfloor$. Since $p(8 \cdot 3^r) \notin [0, 8 \cdot 3^{r-1} - 1]$, it follows that $[0, 8 \cdot 3^r - 1]$ is the maximal set that the predecessor of each element is contained in $[0, 8 \cdot 3^{r-1} - 1]$. We note that $\chi(8 \cdot 3^r - 1) =$

$0 = \chi(8 \cdot 3^r - 2)$. By inputing $\chi([0, 8 \cdot 3^r - 1])$ to M, it encodes all the max-sets in $\mathcal{A}_{\{0,1,5\}} \cap \{0, \ldots, 8 \cdot 3^r - 1\}$. So, we are able to find the number of blocks by considering the length of the output of M. Since $\tau(\delta(q_0, \chi([0, 0]))) = a$ and $\tau(\delta(q_0, \chi([0, 7]))) = ab$, it follows that

$$|\tau(\delta(q_0, \chi([0, 0])))| = 1 \text{ and } |\tau(\delta(q_0, \chi([0, 7])))| = 2.$$

By Corollary 3.9 and the induction on r, we have

$$\tau(\delta(q_0, \chi(\alpha([0, 8 \cdot 3^r - 1])))) = |\phi^r(a)| = P_{r+1}.$$

□

Corollary 3.14. *For $r \in \mathbb{N}$, let B_r be the set of blocks in $[8 \cdot 3^{r-1}, 8 \cdot 3^r - 1] \cap \mathcal{A}_{\{0,1,5\}}$. Then $|B_r| = Q_r$.*

Proof. By Theorem 3.13 and (3.4), $|B_r| = P_{r+1} - P_r = Q_r$. □

For each $m \in \mathbb{N}$, let χ_m be the characteristic function of

$$\left\{ n \in \mathbb{N} : n = m \cdot 3^r + \sum_{i=0}^{r-1} \epsilon_i 3^i, \text{ where } \epsilon_i \in \{0, 1, 5\}, r \in \mathbb{N}_0 \right\}.$$

Theorem 3.15. *For each positive integer r, let D_r be the set of dblocks of degree r in $\mathcal{A}_{\{0,1,5\}} \setminus \{0\}$. Then $|D_r| = 2P_r$ and the number of anti-dblocks with degree r is P_r.*

Proof. We note that $\mathcal{A}^C_{\{0,1,5\}}|_r \subset \{2 \cdot 3^r, \cdots, 2 \cdot 3^{r+1} - 1\}$. To ensure that the automaton M encodes every dblocks of such degree, for each $m \in \{1, 2, 5\}$, we construct χ'_m by adding suffix 00 to $\chi_m([m \cdot 3^r, m \cdot 3^{r+1} - 1])$, i.e.,

$$\chi'_m = \chi_m([m \cdot 3^r, m \cdot 3^{r+1} - 1])00, \text{ where } m \in \{1, 2, 5\}.$$

Since

$$\chi'_1([1, 2]) = 10^3, \chi'_2([2, 5]) = 10^5 \text{ and } \chi'_5([5, 14]) = 10^{12},$$

it follows that

$$\tau(\delta(q_0, \chi'_1([3, 8]))) = \tau(\delta(q_0, \chi'_2([5, 14]))) = \tau(\delta(q_0, \chi'_5([5, 14]))) = a.$$

Similar to Theorem 3.13, we can conclude that, for $m \in \{1, 2, 5\}$,

$$|\tau(\delta(q_0, \chi'_m([m \cdot 3^r, m \cdot 3^{r+1} - 1])))| = P_r.$$

Therefore, $|D_r| = 2P_r$ and the number of anti-blocks with degree r is P_r. □

4 Tree of max-sets

Next, we construct a rooted tree of max-set $[s,t]$, we use notation $G_{[s,t]}$ for a tree rooted at $[s,t]$ with α_i's as the relations of the edges, for $i = 1,2,3$. We say that a vertex $v \in G_{[s,t]}$ is in the *r-th row* if the distance from $[s,t]$ to v is $r-1$. In this section, we show that if a pair of max-sets is similar, then there is an isomorphism function preserving the similarity of the max-sets.

For a pair of sets $\bar{\mathcal{A}}, \bar{\mathcal{B}} \in \left\{\mathcal{A}_{\{0,1,5\}} \setminus \{0\}, \mathcal{A}^C_{\{0,1,5\}}\right\}$, let $\mathcal{A} \subset \bar{\mathcal{A}}$ and $\mathcal{B} \subset \bar{\mathcal{B}}$.

Definition 4.1. A max-set $[s,t]$ in $\mathcal{A}$ and a max-set $[s',t']$ in $\mathcal{B}$ with $t-s = t'-s'$ are said to be *similar*, denoted $[s,t] \sim [s',t']$, if one of the following is true;

- either $s-2 \in \mathcal{A}$ and $s'-2 \in \mathcal{B}$ or $s-2 \notin \mathcal{A}$ and $s'-2 \notin \mathcal{B}$,
- either $t+2 \in \mathcal{A}$ and $t'+2 \in \mathcal{B}$ or $t+2 \notin \mathcal{A}$ and $t'+2 \notin \mathcal{B}$.

We note that a pair of similar max-sets is encoded by the same alphabet in the automaton M; however, the converse is not true. We refer to the term *children* of a block and a dblock as the children of the corresponding vertex in the tree.

Definition 4.2. Let $[s,t]$ and $[s',t']$ be a pair of disjoint max-sets in $\mathcal{A}$ and $\mathcal{B}$, respectively. We say that $[s,t]$ and $[s',t']$ are *c-similar* if they have the same number of children. We write $[s,t] \mathring{\sim} [s',t']$ if $[s,t]$ and $[s',t']$ are c-similar.

Remark 4.3. *A pair of max-sets are c-similar if and only if they are encoded by the same alphabet in the automaton M.*

Remark 4.4. *Let $[s,t],[s',t']$ be a pair of max-sets in $\mathcal{A}$ and $\mathcal{B}$, respectively. If $[s,t] \sim [s',t']$, then, for $i = 1,2,3$*

1. $\alpha_i([s,t]) \sim \alpha_i([s',t'])$,
2. $\alpha_i([s,t]) \mathring{\sim} \alpha_i([s',t'])$.

Proof. Suppose that $[s,t] \sim [s',t']$. By the definitions of α_1, α_2 and α_3, we have $\alpha_i([s,t]) \sim \alpha_i([s',t'])$. By the fact that any pair of max-sets encoded by the same alphabet has the same number of children, we have $\alpha_i([s,t]) \mathring{\sim} \alpha_i([s',t'])$. □

The Remark 4.4 implies that the function α_i's preserve the similarity of the max-sets stated in Definition 4.1.

By Remark 4.4, automaton M encodes the max-sets with 2 children and 3 children by a and b, respectively. For a max-set $[s,t]$, let $I_r^{[s,t]}$ be the family of the max-sets in the r-th row in $G_{[s,t]}$.

Theorem 4.5. *Let $[s,t],[s',t']$ be a pair of max-sets in $\mathcal{A}$ and $\mathcal{B}$, respectively. If $[s,t] \sim [s',t']$, then there exists a graph isomorphism*

$$f : G_{[s,t]} \to G_{[s',t']} \tag{4.1}$$

preserving the similarity of the max-sets such that

$$f|_{I_r^{[s,t]}} : I_r^{[s,t]} \to I_r^{[s',t']}. \tag{4.2}$$

Proof. For convenience, we denote $f|_{I_r^{[s,t]}}$ by f_r. Firstly, we construct a bijective map $f_r : I_r^{[s,t]} \to I_r^{[s',t']}$, for $r \geq 1$, such that $[u,v] \sim f_r([u,v])$. For $r = 1$, the function is defined by

$$f_1([s,t]) = [s',t'].$$

Suppose there exists such bijective function $f_r : I_r^{[s,t]} \to I_r^{[s',t']}$ in the r-th row. We inductively construct the map $f_{r+1} : I_{r+1}^{[s,t]} \to I_{r+1}^{[s',t']}$ by defining

$$f_{r+1}(\alpha_i([u,v])) = \alpha_i([u',v']),$$

where $[u,v]$ and $[u',v']$ are max-sets in the r-th row such that $f_r([u,v]) = [u',v']$. Let $Dom(\alpha_i)$ and $Im(\alpha_i)$ be the domain and image of α_i respectively. By Lemma 3.11, the function $\alpha_i : Dom(\alpha_i) \to Im(\alpha_i)$ is bijective. So, the diagram in Figure 4

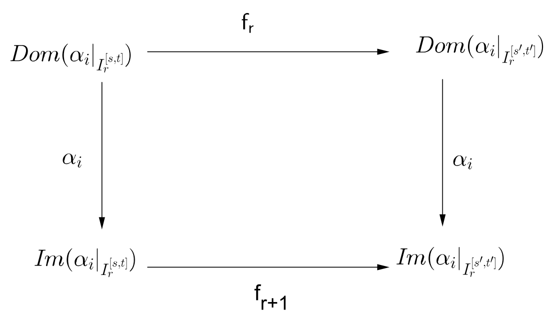

Figure 4: Commutative Diagram.

commutes. By Lemma 3.11, we can conclude that $f_{r+1}|_{Im(\alpha_i|_{I_r^{[s,t]}})}$ is bijective. By Remark 4.4 and the induction hypothesis, the function $f_{r+1}|_{Im(\alpha_i|_{I_r^{[s,t]}})}$ preserves the similarity of the max-sets, i.e., $\alpha_i([u,v]) \sim \alpha_i([u',v'])$, for $i = 1,2,3$. Since

$$Im(\alpha_1|_{I_r^{[s,t]}}) \cup (Im(\alpha_i|_{I_r^{[s,t]}})) \cup (Im(\alpha_i|_{I_r^{[s,t]}})) = I_{r+1}^{[s,t]}$$

and

$$Im(\alpha_i|_{I_r^{[s,t]}}) \cap Im(\alpha_j|_{I_r^{[s,t]}}) = \emptyset,$$

for $i \neq j$. It follows that f_{r+1} is bijective. Let $f = \bigcup_{r=1}^{\infty} f_r$. Then f is a bijection. By the definition of f_r, the function f preserves the edges and the similarity of the max-sets between $G_{[s,t]}$ and $G_{[s',t']}$. Thus, the function f satisfies the given condition. □

Corollary 4.6. *If* $[s,t] \sim [s',t']$, *then* $\left|I_r^{[s,t]}\right| = \left|I_r^{[s',t']}\right|$.

Corollary 4.7. *Let* $[s,t]$ *be a max-set in* $\mathcal{A}$ *and* $[s',t']$ *be a max-set in* $\mathcal{B}$ *such that* $[s,t] \mathring{\sim} [s',t']$. *For* $r \geq 1$, *the following statements are true:*

- *The number of the max-sets with 2 children in* $I_r^{[s,t]}$ *is equal to the number of the max-sets with 2 children in* $I_r^{[s',t']}$.
- *The number of the max-sets with 3 children in* $I_r^{[s,t]}$ *is equal to the number of the max-sets with 3 children in* $I_r^{[s',t']}$.
- $\left|I_r^{[s,t]}\right| = \left|I_r^{[s',t']}\right|$.

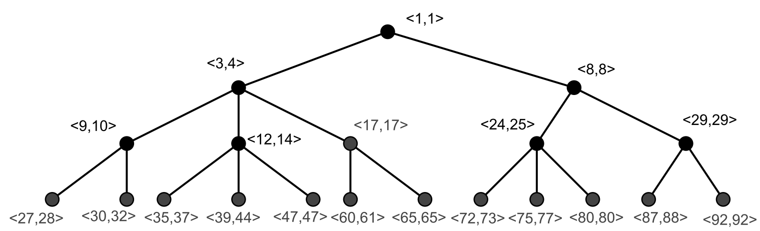

Figure 5: Tree of dblocks in $G_{\langle 1,1\rangle}$.

Acknowledgements : This research is financially supported by Mahasarakham University 2015. Some results in this paper appeared in the author's doctoral dissertation [11] at the University of Illinois at Urbana-Champaign. I would like to thank Professor Bruce Reznick for suggestion and comments on this paper, and Professor Shallit for the valuable suggestion on k-automatic sequence and its references. I would also like to thank the referees for the comments and suggestions.

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(Received 12 June 2019)
(Accepted 24 December 2019)