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Some Transformation Semigroups Admitting Nearing Structure

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Abstract : Denote by $T(X)$ and $P(X)$ the full transformation semigroup and the partial transformation semigroup on a nonempty set X, respectively. The semigroups $T(X)$ and $P(X)$ are known to admit a right nearring structure for any X and they admit a left nearring structure only the case that $|X| = 1$. We generalize these results to the semigroups $T(X,Y)$ and $P(X,Y)$ under composition where $\emptyset \neq Y \subseteq X$, $T(X,Y) = \{\alpha \in T(X)|\ \operatorname{ran}\alpha \subseteq Y\}$ and $P(X,Y) = \{\alpha \in P(X)|\ \operatorname{ran}\alpha \subseteq Y\}$. We obtain the analogous results that $T(X,Y)$ and $P(X,Y)$ admit a right nearring structure for any $\emptyset \neq Y \subseteq X$ and $|Y| = 1$ is necessary and sufficient for them to admit a left nearring structure.

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1 Introduction

The cardinality of a set X will be denoted by $|X|$.

If S is a semigroup which does not process a zero or $|S| = 1$, let S^0 denote the semigroup S with zero 0 ; otherwise $S^0 = S$.

By a *right nearring* we mean a triple $(N, +, \cdot)$ where $(N, +)$ is an abelian group, $(N, \cdot)$ is a semigroup and $(x+y)\cdot z = x\cdot z + y\cdot z$ for all $x, y, z \in N$. A *left nearring* is defined dually. Subnearrings of a right [left] nearring are defined naturally. A right [left] nearring $(N, +, \cdot)$ has the following basic properties :

$$0 \cdot x = 0\ [x \cdot 0 = 0] \text{ for all } x \in N \text{ where } 0 \text{ is the identity of } (N,+),$$
$$(-x)\cdot y = -(x\cdot y)\ [x\cdot(-y) = -(x\cdot y)] \text{ for all } x, y \in N$$

([2], page 19). Hence if $(N, +, \cdot)$ is a right [left] nearring, then $(N, \cdot)$ has a left [right] zero. A right [left] nearring is called *zero-symmetric* if $0 \cdot x = x \cdot 0 = 0$ for all $x \in N$.

Example 1.1. ([2], page 7 and page 19.) Let $(A, +)$ be an abelian group with identity 0, $M(A)$ the set of all mappings $f : A \to A$ and

$$M_0(A) = \{f \in M(A)| f(0) = 0\}.$$

Then $(M(A), +, \circ)$ is a right nearring and $(M_0(A), +, \circ)$ is a zero-symmetric right nearring where $+$ and $\circ$ are the usual addition and composition of functions.

Since the multiplicative structure of a ring is by definition a semigroup with zero, it is valid to ask whether a given semigroup S has S^0 isomorphic to the multiplicative structure of some ring $(R, +, \cdot)$. If ϕ is an isomorphism from the semigroup S^0 onto the semigroup $(R, \cdot)$ and define an operation $\oplus$ on S^0 by

$$x \oplus y = \phi^{-1}(\phi(x) + \phi(y)) \text{ for all } x, y \in S^0,$$

then $(S^0, \oplus, \circ)$ becomes a ring isomorphic to $(R, +, \cdot)$ through ϕ. If the semigroup S has S^0 isomorphic to the multiplicative structure of some ring, or equivalently, there is an operation $+$ on S^0 such that $(S^0, +, \cdot)$ is a ring where $\cdot$ is the operation on S^0, then S is said to *admit a ring structure*. Semigroups admitting ring structure have long been studied. A very nice brief survey was given by Peinado [7] in 1970. For further study, one can see, for example, in [1], [6], [9], [10] and [11].

Right [left] nearrings are a generization of rings. By definition and their properties, their multiplicative structures are semigroups with left [right] zero. The right nearrings in Example 1.1, the zero map θ, that is, $\theta(x) = 0$ for all $x \in A$, is a left zero of $(M(A), \circ)$ which is not a zero if $|A| > 1$ and θ is the zero of $(M_0(A), \circ)$. It is valid to ask that for a given semigroup S , whether S or S^0 is isomorphic to the multiplicative structure of some right [left] nearring. If it does, S shall be said to *admit a right* [*left*] *nearring structure*. By the same reason as above, S admits a right [left] nearring structure if and only if

(i) there is an operation $+$ on S such that $(S, +, \cdot)$ is a right [left] nearring where $\cdot$ is the operation on S or
(ii) there is an operation $+$ on S^0 such that $(S^0, +, \cdot)$ is a (zero-symmetric) right [left] nearring where $\cdot$ is the operation on S^0.

Notice that if S has no left [right] zero, then S cannot satisfies (i).

If X is a set, $(A, +)$ is an abelian group and $|X| = |A|$, then there is a bijective $\phi : X \to A$ and $(X, \oplus)$ becomes an abelian group isomorphic to $(A, +)$ by ϕ where

$$x \oplus y = \phi^{-1}(\phi(x) + \phi(y)) \quad \text{for all } x, y \in X.$$

In fact, the identity of $(X, \oplus)$ is $a \in X$ with $\phi(a) = 0$. Hence if $|X| = |A|$ and $a \in X$, then there is an operation $\oplus$ on X such that $(X, \oplus)$ is an abelian group with identity a. If X is a finite nonempty set, then $|X| = |\mathbb{Z}_n|$ for some positive integer n. Also, if X is infinite and $F(X)$ is the set of all finite subsets of X, then $|X| = |F(X)|$ ([8], page 154). Moreover, if we define an operation $+$ on $F(X)$ by

$$A + B = (A \setminus B) \cup (B \setminus A) \quad \text{for all } A, B \in F(X),$$

then $(F(X), +)$ is an abelian group having $\emptyset$ as its identity. Note that the inverse of $A \in F(X)$ in $(F(X), +)$ is A itself.

Therefore we have

Proposition 1.2. *For any nonempty set X, there is an operation $+$ on X such that $(X,+)$ is an abelian group.*

In addition, if $a \in X$ is given, then there is an operation $+$ on X such that $(X,+)$ is an abelian group with identity a.

The first part of this fact was also mentioned in [11].

For a nonempty set X, let $T(X)$ and $P(X)$ be respectively the full transformation semigroup on X (the semigroup, under composition, of all mappings $f : X \to X$) and the partial transformation semigroup on X (the semigroup, under composition, of all mappings from a subset of X into X). Then $T(X)$ is a subsemigroup of $P(X)$. Note that 1_X, the identity mapping on X, is the identity of $T(X)$ and $P(X)$ and 0, the empty transformation, is the zero of $P(X)$. The domain and the range of $f \in P(X)$ will be denoted by $\mathrm{dom}f$ and $\mathrm{ran}f$, respectively. For $\emptyset \neq A \subseteq X$ and $x \in X$, let A_x denote the constant mapping in $P(X)$ whose domain and range are A and $\{x\}$, respectively. For $x, y \in X$, $\{x\}_y$ may be written by $\begin{pmatrix} x \\ y \end{pmatrix}$.

The follwing basic fact of transformation semigroup is useful for our work.

Proposition 1.3. ([5], page 41). *Let X be a nonempty set θ a symbol not representing any element of X and*

$$Z(X \cup \{\theta\}) = \{g \in T(X \cup \{\theta\}) | g(\theta) = \theta\}.$$

For each $f \in P(X)$, define $f^* \in T(X \cup \{\theta\})$ by

$$f^*(x) = \begin{cases} f(x) & \text{if } x \in \mathrm{dom}f, \\ \theta & \text{if } x \in (X \cup \{\theta\}) \smallsetminus \mathrm{dom}f. \end{cases}$$

Then the following statements hold.

(i) *$Z(X \cup \{\theta\})$ is a subsemigroup of $T(X \cup \{\theta\})$ and*

$$Z(X \cup \{\theta\}) = \{f^* | f \in P(X)\}.$$

(ii) *The mapping $f \mapsto f^*$ is an isomorphism from $P(X)$ onto the subsemigroup $Z(X \cup \{\theta\})$ of $T(X \cup \{\theta\})$.*

The following theorems were provided in [3] and [4].

Theorem 1.4. ([3], [4]). *For a nonempty set X, the following statements hold.*

(i) *$T(X)$ admits a right nearring structure.*
(ii) *$T(X)$ admits a left nearring structure if and only if $|X| = 1$.*

Theorem 1.5. ([3], [4]). *For any nonempty set X, the following statements hold.*

(i) *$P(X)$ admits a right nearring structure.*
(ii) *$P(X)$ admits a left nearring structure if and only if $|X| = 1$.*

In this paper, Theorem 1.4 is generalized by considering the semigroup $T(X, Y)$, under composition, where $\emptyset \neq Y \subseteq X$ and

$$T(X, Y) = \{f \in T(X) | \operatorname{ran} f \subseteq Y\}.$$

In 1975, Symons [12] introduced the semigroup $T(X, Y)$ and described all the automorphisms of this semigroup. Moreover, he determined when the two semigroups of this type are isomorphic. To generalize Theorem 1.5, we introduce the semigroup $P(X, Y)$ analogously, that is, $\emptyset \neq Y \subseteq X$,

$$P(X, Y) = \{f \in P(X) | \operatorname{ran} f \subseteq Y\}$$

and the operation is the composition of functions. Notice that $T(X, Y)$ is a subsemigroup of $T(X)$ and $T(X, Y) = T(X)$ if and only if $Y = X$, and this is also true for $P(X, Y)$.

2 The Transformation Semigroups $T(X, Y)$ and $P(X, Y)$

The following theorem is our first main result.

Theorem 2.1. *Let X be a nonempty set and $\emptyset \neq Y \subseteq X$. Then the following statemants hold.*

(i) *The semigroup $T(X, Y)$ admits a right nearring structure.*
(ii) *The semigroup $T(X, Y)$ admits a left nearring structure if and only if $|Y| = 1$.*

Proof. (i) By Proposition 1.2, there is an operation $+$ on Y such that $(Y, +)$ is an abelian group. For $f, g \in T(X, Y)$, define $f + g \in T(X, Y)$ by

$$(f + g)(x) = f(x) + g(x) \quad \text{for all } x \in X.$$

Since $(Y, +)$ is an abelian group. We deduce that $(T(X, Y), +)$ is an abelian group. We also have that for all $f, g, h \in T(X, Y)$,

$$((f + g) \circ h)(x) = (f + g)(h(x)) = f(h(x)) + g(h(x)) = (f \circ h + g \circ h)(x) \quad \text{for all } x \in X.$$

Hence $(T(X, Y), +, \circ)$ is a right nearring. Thus $T(X, Y)$ admits a right nearring structure.

(ii) Assume that $|Y| > 1$. Let a, b be distinct elements of Y. Then $X_a, X_b \in T(X, Y)$. Since $X_a f = X_a$ and $X_b f = X_b$ for all $f \in T(X, Y)$, it follows that $T(X, Y)$ has no right zero. Suppose that $T(X, Y)$ admits a left nearring structure. Then there is an operation $+$ on $T^0(X, Y)$ such that $(T^0(X, Y), +, \circ)$ is a left nearring. Hence $X_a + X_b = f$ for some $f \in T^0(X, Y)$.

Case 1: $f = 0$. Then $X_a + X_b = 0$ and

$$X_a + X_a = X_a X_a + X_a X_b = X_a(X_a + X_b) = X_a 0 = 0$$

which implies that $X_b = X_a$. This is a contradiction since $a \neq b$.

Case 2: $f \neq 0$. Then

$$X_a + X_a = X_a(X_a + X_b) = X_a f = X_a$$

which implies that $X_a = 0$, a contradiction.

This proves that if $T(X, Y)$ admits a left nearring structure, then $|Y| = 1$.

The converse is obvious since $|T(X, Y)| = 1$ if $|Y| = 1$. □

Therefore Theorem 1.4 is a direct consequence of Theorem 2.1

Corollary 2.2. *Let X be a nonempty set. Then the following statements hold.*

(i) *$T(X)$ admits a right nearring structure.*
(ii) *$T(X)$ admits a left nearring structure if and only if $|X| = 1$.*

By the given definitions, if a semigroup S admits a ring structure, then S admits left [right] nearring structure. From this fact and $T^0(X, Y) \cong (\mathbb{Z}_2, \cdot)$ if $|Y| = 1$, by Theorem 2.1(ii), we have

Corollary 2.3. *Let X be a nonempty set and $\emptyset \neq Y \subseteq X$. Then $T(X, Y)$ admits a ring structure if and only if $|Y| = 1$.*

In particular, $T(X)$ admits a ring structure if and only if $|X| = 1$.

In fact, the second part of Corollary 2.3 was given in [11].

To prove the second part of main result, the following lemma is needed.

Lemma 2.4. *Let X be a nonempty set and $\emptyset \neq Y \subseteq X$. If $|Y| = 1$, then $P(X, Y)$ admits a ring structure.*

Proof. Let $Y = \{a\}$. We clearly have that

$$P(X, Y) = \{0\} \cup \{A_a | \emptyset \neq A \subseteq X\}. \tag{2.1}$$

Let $\mathcal{P}(X)$ be the set of all subsets of X. Define an operation $+$ on $\mathcal{P}(X)$ by

$$A + B = (A \setminus B) \cup (B \setminus A) \quad \text{for all } A, B \in \mathcal{P}(X). \tag{2.2}$$

Then $(\mathcal{P}(X), +)$ is an abelian group having $\emptyset$ as its identity. Define $\phi : P(X, Y) \to \mathcal{P}(X)$ by

$$\phi(0) = \emptyset \quad \text{and} \quad \phi(A_a) = A \quad \text{for all } \emptyset \neq A \subseteq X. \tag{2.3}$$

Then by (2.1), ϕ is a bijection from $P(X, Y)$ onto $\mathcal{P}(X)$, so $(P(X, Y), \oplus)$ is an abelian group with identity 0 where

$$f \oplus g = \phi^{-1}(\phi(f) + \phi(g)) \quad \text{for all } f, g \in P(X, Y). \tag{2.4}$$

It follows from (2.2), (2.3) and (2.4) that for nonempty subsets A, B of X,

$$A_a \oplus B_a = 0 \Leftrightarrow A = B, \tag{2.5}$$

$$A \neq B \Rightarrow A_a \oplus B_a = ((A \setminus B) \cup (B \setminus A))_a. \tag{2.6}$$

Hence (2.1) and (2.5) yield

$$f \oplus f = 0 \quad \text{for all } f \in P(X,Y). \tag{2.7}$$

Next, we shall show that $\circ$ is two-sided distributive over $\oplus$ in $P(X,Y)$. If $f, g, h \in P(X,Y)$ are such that $f = 0$ or $g = 0$ or $h = 0$, then we clearly have that $f(g \oplus h) = fg \oplus fh$ and $(g \oplus h)f = gf \oplus hf$. Next, let A, B, C be nonempty subsets of X.

Case 1: $B_a \oplus C_a = 0$. By (2.5), $B = C$. Thus

$$A_a(B_a \oplus C_a) = 0 = (B_a \oplus C_a)A_a,$$

$$A_aB_a \oplus A_aC_a = \begin{cases} B_a \oplus C_a = 0 & \text{if } a \in A, \\ 0 \oplus 0 = 0 & \text{if } a \notin A, \end{cases}$$

$$B_aA_a \oplus C_aA_a = B_aA_a \oplus B_aA_a = 0 \quad \text{by (2.7)}.$$

Case 2: $B_a \oplus C_a \neq 0$. From (2.5) and (2.6), we have $B \neq C$ and $B_a \oplus C_a = ((B \setminus C) \cup (C \setminus B))_a$, respectively.

$$\begin{aligned} A_a(B_a \oplus C_a) &= A_a((B \setminus C) \cup (C \setminus B))_a \\ &= \begin{cases} ((B \setminus C) \cup (C \setminus B))_a = B_a \oplus C_a & \text{if } a \in A, \\ 0 & \text{if } a \notin A, \end{cases} \\ (B_a \oplus C_a)A_a &= ((B \setminus C) \cup (C \setminus B))_a A_a \\ &= \begin{cases} A_a & \text{if } a \in (B \setminus C) \cup (C \setminus B), \\ 0 & \text{if } a \notin (B \setminus C) \cup (C \setminus B), \end{cases} \\ A_aB_a \oplus A_aC_a &= \begin{cases} B_a \oplus C_a & \text{if } a \in A, \\ 0 & \text{if } a \notin A, \end{cases} \\ B_aA_a \oplus C_aA_a &= \begin{cases} A_a \oplus 0 = A_a & \text{if } a \in B \setminus C, \\ 0 \oplus A_a = A_a & \text{if } a \in C \setminus B, \\ A_a \oplus A_a = 0 & \text{if } a \in B \cap C, \\ 0 \oplus 0 = 0 & \text{if } a \notin B \cup C, \end{cases} \\ &= \begin{cases} A_a & \text{if } a \in (B \setminus C) \cup (C \setminus B), \\ 0 & \text{if } a \notin (B \setminus C) \cup (C \setminus B). \end{cases} \end{aligned}$$

This shows that $\circ$ is two-sided distributive over $\oplus$. Hence $(P(X,Y), \oplus, \circ)$ is a left nearring. Hence the lemma is proved. $\square$

Theorem 2.5. *Let X be a nonempty set and $\emptyset \neq Y \subseteq X$. Then the following statemants hold.*

(i) *The semigroup $P(X,Y)$ admits a right nearring structure.*
(ii) *The semigroup $P(X,Y)$ admits a left nearring structure if and only if $|Y| = 1$.*

Proof. (i) Let θ be a symbol not representing any element of X. For $f \in P(X)$, define $f^* \in T(X \cup \{\theta\})$ as in Proposition 1.3, that is,

$$f^*(x) = \begin{cases} f(x) & \text{if } x \in \mathrm{dom} f, \\ \theta & \text{if } x \in (X \cup \{\theta\}) \smallsetminus \mathrm{dom} f. \end{cases} \tag{2.1}$$

Fron (2.1), we have

$$\mathrm{ran} f^* = \mathrm{ran} f \cup \{\theta\} \quad \text{for all } f \in P(X). \tag{2.2}$$

Define $\phi : P(X) \to T(X \cup \{\theta\})$ by $\phi(f) = f^*$ for all $f \in P(X)$. By Proposition 1.3, ϕ is a monomorphism and

$$\phi(P(X)) = \{f^* | f \in P(X)\} = \{g \in T(X \cup \{\theta\}) | g(\theta) = \theta\}. \tag{2.3}$$

Hence (2.2) and (2.3) yield the fact that

$$\phi(P(X,Y)) = \{f^* | f \in P(X,Y)\} = \{g \in T(X \cup \{\theta\}, Y \cup \{\theta\}) | g(\theta) = \theta\}. \tag{2.4}$$

Note that $\{g \in T(X \cup \{\theta\}, Y \cup \{\theta\}) | g(\theta) = \theta\}$ is a subsemigroupof $T(X \cup \{\theta\}, Y \cup \{\theta\})$. By Proposition 1.2, there is an operation $+$ on $Y \cup \{\theta\}$ such that $(Y \cup \{\theta\}, +)$ is an abelian group with identity θ. From the proof of Theorem 2.1, $(T(X \cup \{\theta\}, Y \cup \{\theta\}), +, \circ)$ is a right nearring where

$$(f+g)(x) = f(x) + g(x) \quad \text{for all } f, g \in T(X \cup \{\theta\}, Y \cup \{\theta\}) \text{ and } x \in X \cup \{\theta\}.$$

If $h, k \in T(X \cup \{\theta\}, Y \cup \{\theta\})$ are such that $h(\theta) = \theta = k(\theta)$, then

$$(h+k)(\theta) = h(\theta) + k(\theta) = \theta + \theta = 0,$$
$$(-h)(\theta) = -h(\theta) = -\theta = \theta.$$

It follows that $\{g \in T(X \cup \{\theta\}, Y \cup \{\theta\}) | g(\theta) = \theta\}$ is a subnearring of $(T(X \cup \{\theta\}, Y \cup \{\theta\}), +, \circ)$. Since ϕ is a monomorphism, by (2.4), we have that the semigroups $P(X,Y)$ and $\{g \in T(X \cup \{\theta\}, Y \cup \{\theta\}) | g(\theta) = \theta\}$ are isomorphic. Consequently $P(X,Y)$ admits a right nearring structure.

(ii) Assume that $|Y| > 1$. Let $a, b \in Y$ be distinct. Then $\begin{pmatrix} a \\ a \end{pmatrix}, \begin{pmatrix} a \\ b \end{pmatrix} \in P(X,Y)$. Suppose that there is an operation $+$ on $P(X,Y)$ such that $(P(X,Y), +, \circ)$ is a left nearring. Then $\begin{pmatrix} a \\ a \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix} = f$ for some $f \in P(X,Y)$. This implies that

$$\begin{pmatrix} a \\ a \end{pmatrix} f = \begin{pmatrix} a \\ a \end{pmatrix} \left(\begin{pmatrix} a \\ a \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix} \right) = \begin{pmatrix} a \\ a \end{pmatrix} + 0 = \begin{pmatrix} a \\ a \end{pmatrix},$$

and

$$\begin{pmatrix} b \\ b \end{pmatrix} f = \begin{pmatrix} b \\ b \end{pmatrix} \left(\begin{pmatrix} a \\ a \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix} \right) = 0 + \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}.$$

It follows that $a = \begin{pmatrix} a \\ a \end{pmatrix}(a) = \left(\begin{pmatrix} a \\ a \end{pmatrix} f \right)(a) = \begin{pmatrix} a \\ a \end{pmatrix} f(a)$ and $b = \begin{pmatrix} a \\ b \end{pmatrix}(a) = \left(\begin{pmatrix} b \\ b \end{pmatrix} f \right)(a) = \begin{pmatrix} b \\ b \end{pmatrix} f(a)$ which imply respectively that $f(a) = a$ and $f(a) = b$. This is a contradiction since $a \neq b$. This proves that if $P(X, Y)$ admits a left nearring structure, then $|Y| = 1$.

The converse follows from Lemma 2.4. □

Theorem 1.5 follows directly from Theorem 2.5

Corollary 2.6. *Let X be a nonempty set. Then the following statements hold.*

(i) *$P(X)$ admits a right nearring structure.*

(ii) *$P(X)$ admits a left nearring structure if and only if $|X| = 1$.*

The following corollary is a direct consequence of Lemma 2.4 and Theorem 2.5.

Corollary 2.7. *Let X be a nonempty set and $\emptyset \neq Y \subseteq X$. Then $P(X, Y)$ admits a ring structure if and only if $|Y| = 1$.*

In particular, $P(X)$ admits a ring structure if and only if $|X| = 1$.

The second part of Corollary 2.7 was also proved directly in [11].

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