



Domination Game on Powers of Cycles

Nattakritta Chantarachada and Chalermpong Worawannotai¹

Department of Mathematics, Faculty of Science
Silpakorn University, Nakhon Pathom 73000, Thailand
e-mail : augaig1537@gmail.com (N. Chantarachada)
worawannotai_c@silpakorn.edu (C. Worawannotai)

Abstract : The *domination game* played on a graph G consists of two players, *Dominator* and *Staller*, who alternate taking turns choosing a vertex from G such that whenever a vertex is chosen, at least one additional vertex is *dominated*. Playing a vertex will make all vertices in its closed neighborhood dominated. The game ends when all vertices are dominated i.e. the chosen vertices form a dominating set. Dominator's goal is to finish the game as soon as possible, and Staller's goal is to prolong it as much as possible. The *game domination number* is the total number of chosen vertices after the game ends when Dominator and Staller play the game by using optimal strategies. In this paper, we obtain the game domination numbers of powers of cycles and find optimal strategies for Dominator and Staller.

Keywords : domination game; game domination number; power of cycles.

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1 Introduction

A set S of vertices of a graph G is a *dominating set* if every vertex not in S is adjacent to some vertex of S . The *domination number* of a graph G is the number of vertices in a minimum dominating set for G , denoted by $\gamma(G)$.

There are many game variations of domination [1, 2, 3, 4, 5]. In this paper we study the domination game introduced in 2010 by Brešar, Klavžar and Rall

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¹Corresponding author.

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[4], where the original idea of the game is attributed to Henning (2003, personal communication). The *domination game* played on a graph G consists of two players, *Dominator* and *Staller*, who alternate taking turns choosing a vertex from G such that whenever a vertex is chosen, at least one additional vertex is *dominated*. Playing a vertex will make all vertices in its closed neighborhood dominated. The game ends when all vertices are dominated i.e. the chosen vertices form a dominating set. Dominator's goal is to finish the game as soon as possible, and Staller's goal is to prolong it as much as possible. The *game domination number* is the size of the final dominating set when both players play optimally, denoted by $\gamma_g(G)$ when Dominator starts the game and by $\gamma'_g(G)$ when Staller starts the game.

Brešar, Klavžar and Rall [4] gave a bound of game domination number in terms of domination number: for any graph G , $\gamma(G) \leq \gamma_g(G) \leq 2\gamma(G) - 1$. They also studied the difference between the two types of game domination numbers of a graph. Later, Kinnersley, West and Zamani [6] improved upon this result and showed that the difference is at most 1 i.e. for any graph G , $|\gamma_g(G) - \gamma'_g(G)| \leq 1$.

Domination game played on various families of graphs have been studied. In 2013, Zamani [6, 7] determined the game domination numbers of paths and cycles, and Brešar and Klavžar [8] proved a lower bound of the game domination number of a tree in terms of its order and maximum degree. In 2015, Bujtás [9] proved a lower bound of the game domination number of a certain families of forests. Dorbec, Košmrlj, and Renault [10] showed how the game domination number of the union of two no-minus graphs corresponds to the game domination numbers of the initial graphs. This result led to another proof of the game domination numbers of paths and cycles [11], and the game domination numbers of a graph constructed from 1-sum of paths [12]. In 2018, Ruksasakchai, Onphaeng, and Worawannotai [13] determined the game domination numbers of a disjoint union of paths and cycles.

In this paper, we determine the game domination numbers of powers of cycles and find optimal strategies for Dominator and Staller.

2 Preliminaries

In this section we recall some additional definitions and a useful observation.

For a graph G , the s -th power of G , denoted by G^s , is the graph with the same vertex set as G and two vertices are adjacent if their distance in G is at most s . Thus, the s -power of a cycle on n vertices is denoted by C_n^s . For convenience, we let the vertex set of C_n^s be $\{0, 1, 2, \dots, n-1\}$ so that vertices i, j in C_n^s are adjacent if and only if $|i - j| \leq s$ or $|j - i| \leq s$ where the differences are taken modulo n .

A *partially dominated graph* is a graph G with a declaration that some vertices have already been dominated initially. We extend the notion of dominating sets to partially dominated graphs as follows. Let G be a partially dominated graph. A set S of vertices of graph G is a *dominating set* if every undominated vertex not in S is adjacent to some vertex of S . From this, the notion of domination numbers and

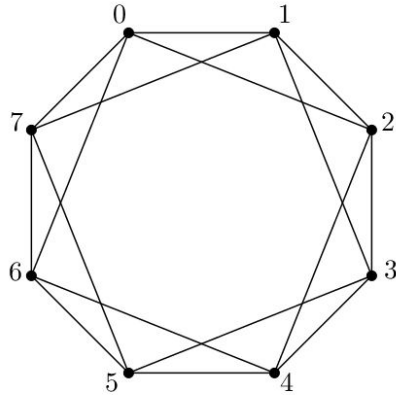


Figure 1: C_8^2

the game domination numbers extend naturally. Note that as the game progresses the graph becomes a partially dominated graph with fewer undominated vertices.

We denote the open neighborhood of a vertex v of a graph G by $N_G(v)$ and its closed neighborhood by $N_G[v]$. We simply write $N(v)$ and $N[v]$ if the graph is understood.

A vertex u of a partially dominated graph G is *saturated* if every vertex in $N[u]$ is dominated. The *residual graph* of G is the partially dominated graph obtained from G by removing all saturated vertices and all edges joining dominated vertices. Since removing such vertices and edges does not affect the game, the game domination numbers of a partially dominated graph and its residual graph are the same.

Next, we give an important observation that can be used for establishing bounds for the game domination numbers. Recall that Dominator’s goal is to finish the game as soon as possible, and Staller’s goal is to prolong it as much as possible. If Dominator has a strategy, possibly suboptimal, that can end the game within a certain number of moves or Staller has a strategy, possibly suboptimal, that can prolong the game to at least a certain number of moves, then a bound for game domination numbers can be established as follows.

Observation 2.1. *Let G be a graph, the following statements hold.*

- (i) *For Dominator-start game, if Dominator has a strategy that ensures both players together end the game within k moves, then $\gamma_g(G) \leq k$.*
- (ii) *For Staller-start game, if Dominator has a strategy that ensures both players together end the game within k moves, then $\gamma'_g(G) \leq k$.*

- (iii) For Dominator-start game, if Staller has a strategy that ensures both players together end the game using at least k moves, then $\gamma_g(G) \geq k$.
- (iv) For Staller-start game, if Staller has a strategy that ensures both players together end the game using at least k moves, then $\gamma'_g(G) \geq k$. \square

Observation 2.1 can be used to prove a bound of a game domination number by presenting an appropriate Dominator's strategy or a Staller's strategy.

3 Main results

In this section, we determine the game domination numbers of powers of cycles C_n^s . To avoid triviality, we assume $n > 2s + 1$. For completeness, we also give the domination numbers of the graphs here.

Theorem 3.1. For positive integers n and s , $\gamma(C_n^s) = \lceil n/(2s + 1) \rceil$.

Proof. Let $G = C_n^s$ and let S be a minimum dominating set of G . For each vertex v in G , we have $|N[v]| = 2s + 1$. So any vertex in S dominates at most $2s + 1$ new vertices. Hence $\gamma(G) = |S| \geq \lceil n/(2s + 1) \rceil$.

Let $n = (2s + 1)q + r$ where q is a nonnegative integer and $0 \leq r \leq 2s$. Let $S' = \{(2s + 1)i + s \mid 0 \leq i \leq q - 1\} \subseteq V(G)$ and $S'' = S' \cup \{u\} \subseteq V(G)$ where $u = \min\{n - 1, (2s + 1)q + s\}$.

If $2s + 1 \mid n$, then an arbitrary vertex v has the form $v = (2s + 1)i + j$ for some integers i and j where $0 \leq i \leq q - 1$ and $0 \leq j \leq 2s$. So v is adjacent to $(2s + 1)i + s \in S'$. Hence S' is a dominating set of G . Therefore $\gamma(G) \leq |S'| = q = \lceil n/(2s + 1) \rceil$.

If $2s + 1 \nmid n$, then an arbitrary vertex v has the form $v = (2s + 1)i + j$ for some integers i and j where $0 \leq i \leq q$ and $0 \leq j \leq 2s$. If $i \leq q - 1$, then v is adjacent to $(2s + 1)i + s \in S'$. If $i = q$, then v is adjacent to $u \in S''$. Hence S'' is a dominating set of G . Therefore $\gamma(G) \leq |S''| = q + 1 = \lceil n/(2s + 1) \rceil$.

Therefore $\gamma(G) = \lceil n/(2s + 1) \rceil$. \square

Now, we determine the game domination numbers of C_n^s . At any point during the game, we can keep track of the dominated vertices by considering the subgraph of the original graph induced by the dominated vertices. The components of the induced subgraph are called *dominated components*.

Theorem 3.2. Let $G = C_n^s$. If $n = (2s + 2)q + r$ where q is a nonnegative integer and $0 \leq r \leq 2s + 1$, then we have

(i) $\gamma_g(G) = 2q + \lceil r \neq 0 \rceil$

(ii) $\gamma'_g(G) = 2q + \lceil r = 2s + 1 \rceil$

where $[p] = 1$ if the statement p is true and $[p] = 0$ if p is false. Moreover, an optimal strategy for Staller is to always make a move that dominates exactly one new vertex (except the move that starts the game) and an optimal strategy for Dominator is to always make a move that dominates as many new vertices as possible without creating a new dominated component (except the move that starts the game).

Proof. First, we prove the lower bounds for (i) and (ii). Consider a Staller's strategy where he always makes a move to dominates exactly one new vertex (except when he plays to start the game).

We show that Staller can always follows this strategy. Suppose the game has been started but has not finished. Without loss of generality, there are vertices i and $i + 1$ of G where i is dominated but $i + 1$ is not. Then, the vertices $i, i - 1, i - 2, \dots, i - 2s$ are dominated so Staller can play vertex $i - s + 1$ to dominate exactly one new vertex, that is vertex $i + 1$.

This strategy guarantees that in every round, except for the first round of the game started by Staller, no more than $2s + 2$ new vertices are dominated because Dominator can dominate at most $2s + 1$ vertices in each move.

The lower bound for (i). In each round, Dominator and Staller together can dominate at most $2s + 2$ new vertices. The first $2q$ moves dominate at most $(2s + 2)q$ vertices. Since $n = (2s + 2)q + r$, at least $2q + [r \neq 0]$ moves are needed to end the game.

By Observation 2.2(iii), we have $\gamma_g(G) \geq 2q + [r \neq 0]$ which proves the lower bound for (i).

The lower bound for (ii). In each round (except the first round), Staller and Dominator together can dominate at most $2s + 2$ new vertices. For the first round, they can dominate at most $4s + 2$ new vertices. The first $2q - 1$ and $2q$ moves dominate at most $(2s + 2)q - 1$ and $(2s + 2)q + 2s$ vertices, respectively. Since $n = (2s + 2)q + r$, at least $2q + [r = 2s + 1]$ moves are needed to end the game.

By Observation 2.2(iv), we have $\gamma'_g(G) \geq 2q + [r = 2s + 1]$ which proves the lower bound for (ii).

Next, we prove the upper bounds for (i) and (ii). Consider a Dominator's strategy where he always makes a move to dominates as many new vertices as possible without creating a new dominated component (except when he plays to start the game).

Note that if a non-starting move that creates a new dominated component exists, then there is a move that dominates the same number of new vertices $(2s + 1)$ without creating a new dominated component. Therefore, Dominator can always follow this strategy.

Let M be the total number of moves when the game has finished and let $k = \lfloor M/2 \rfloor$. When $k = 1$, the upper bounds hold trivially. Therefore, we assume $k \geq 2$.

The upper bound for (i). Let $d_1, s_1, d_2, s_2, \dots, d_k, s_k, (d_{k+1})$ be the sequence of moves by Dominator and Staller from start to finish. Let $f(d_i)$ (respectively $f(s_i)$) denote the number of newly dominated vertices when Dominator plays d_i

(respectively when Staller plays s_i).

In rounds 1 to $k - 1$, if Staller makes α moves that create new dominated components, then Staller can force Dominator to dominate fewer than $2s + 1$ new vertices for at most α times. Therefore, $\sum_{j=1}^{k-1} (f(s_j) + f(d_{j+1})) \geq (2s + 2)(k - 1)$.

Since $f(d_1) = 2s + 1$ and $f(s_k) \geq 1$, we have $\sum_{j=1}^k (f(d_j) + f(s_j)) \geq (2s + 2)k$. We divide the argument into two cases according to the player who finishes the game.

Case DS1: the game is finished by Staller (the last move is s_k). Then $n = \sum_{j=1}^k (f(d_j) + f(s_j)) \geq (2s + 2)k$. Therefore, $k \leq n/(2s + 2)$. Since k is an integer, we have $k \leq \lfloor n/(2s + 2) \rfloor$. Now,

$$M = 2k \leq 2\lfloor n/(2s + 2) \rfloor + [r \neq 0].$$

Case DS2: the game is finished by Dominator (the last move is d_{k+1}). Since $f(d_{k+1}) \geq 1$, we have $n = \left(\sum_{j=1}^k (f(d_j) + f(s_j))\right) + f(d_{k+1}) \geq (2s + 2)k + 1$. Therefore, $k \leq (n - 1)/(2s + 2)$. Since k is an integer, we have $k \leq \lfloor (n - 1)/(2s + 2) \rfloor$. Now,

$$\begin{aligned} M &= 2k + 1 \leq 2(\lfloor (n - 1)/(2s + 2) \rfloor) + 1 \\ &= \begin{cases} 2\lfloor n/(2s + 2) \rfloor - 1 & \text{if } r = 0 \\ 2\lfloor n/(2s + 2) \rfloor + 1 & \text{if } r \neq 0 \end{cases} \\ &\leq 2\lfloor n/(2s + 2) \rfloor + [r \neq 0]. \end{aligned}$$

From the above cases and by Observation 2.1(i), we have $\gamma_g(G) \leq M \leq 2q + [r \neq 0]$ which proves the upper bound for (i).

The upper bound for (ii). Let $s_1, d_1, s_2, d_2, \dots, s_k, d_k, s_{k+1}$ be the sequence of moves by Staller and Dominator from start to finish. Let $f(s_i)$ (respectively $f(d_i)$) denote the number of newly dominated vertices when Staller plays s_i (respectively when Dominator plays d_i). We divide the argument into two cases according to the player who finishes the game.

Case SD1: the game is finished by Dominator (the last move is d_k). In rounds 2 to $k - 1$, if Staller makes α moves that create new dominated components, then Staller can force Dominator to dominate fewer than $2s + 1$ new vertices for at most α times. Therefore, $\sum_{j=2}^{k-1} (f(s_j) + f(d_j)) \geq (2s + 2)(k - 2)$.

Since $k \geq 2$, we have $f(d_1) = 2s + 1$. From $f(s_1) = 2s + 1$, $f(s_k) \geq 1$ and $f(d_k) \geq 1$, we have $n = \sum_{j=1}^k (f(s_j) + f(d_j)) \geq (2s + 2)k$. Therefore, $k \leq n/(2s + 2)$. Since k is an integer, we have $k \leq \lfloor n/(2s + 2) \rfloor$. Now,

$$M = 2k \leq \lfloor n/(2s + 2) \rfloor + [r = 2s + 1].$$

Case SD2: the game is finished by Staller (the last move is s_{k+1}).

In rounds 2 to k , if Staller makes α moves that create new dominated components, then Staller can force Dominator to dominate fewer than $2s + 1$ new vertices for at most α times. Therefore $\sum_{j=2}^k (f(s_j) + f(d_j)) \geq (2s + 2)(k - 1)$.

Since $k \geq 2$, we have $f(d_1) = 2s + 1$. From $f(s_1) = 2s + 1$ and $f(s_{k+1}) \geq 1$, we have $n = \left(\sum_{j=1}^k (f(s_j) + f(d_j))\right) + f(s_{k+1}) \geq (2s + 2)k + 2s + 1$. Therefore, $k \leq (n - 2s - 1)/(2s + 2)$. Since k is an integer, we have $k \leq \lfloor (n - 2s - 1)/(2s + 2) \rfloor$. Now,

$$\begin{aligned} M &= 2k + 1 \leq 2(\lfloor (n - 2s - 1)/(2s + 2) \rfloor) + 1 \\ &= \begin{cases} 2\lfloor n/(2s + 2) \rfloor + 1 & \text{if } r = 2s + 1 \\ 2\lfloor n/(2s + 2) \rfloor - 1 & \text{if } r \neq 2s + 1 \end{cases} \\ &\leq 2\lfloor n/(2s + 2) \rfloor + [r = 2s + 1]. \end{aligned}$$

From the above cases and by Observation 2.1(ii), we have $\gamma'_g(G) \leq M \leq 2q + [r = 2s + 1]$ which proves the upper bound for (ii). \square

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