Thai Journal of Mathematics : 247-259
Special Issue : Annual Meeting in Mathematics 2019

http://thaijmath.in.cmu.ac.th
Online ISSN 1686-0209

The Characterization of Caterpillars with Multidimension 3

Varanoot Khemmani and Supachoke Isariyapalakul[1]

Department of Mathematics, Faculty of Science,
Srinakharinwirot University, Bangkok, Thailand
e-mail : `varanoot@g.swu.ac.th` (V. Khemmani)
`supachoke.isa@g.swu.ac.th` (S. Isariyapalakul)

Abstract : Let v be a vertex of a connected graph G, and let $W = \{w_1, w_2, ..., w_k\}$ be a set of vertices of G. The *multirepresentation* of v with respect to W is the k-multiset $mr(v|W) = \{d(v, w_1), d(v, w_2), ..., d(v, w_k)\}$. A set W is called a multiresolving set of G if no two vertices of G have the same multirepresentations with respect to W. The *multidimension* of G is the minimum cardinality of a multiresolving set of G. In this paper, we characterize the caterpillars with multidimension 3.

Keywords : caterpillar; multirepresentation; multiresolving set; multidimension.
2010 Mathematics Subject Classification : 05C12.

1 Introduction

The distance $d(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest $u-v$ path in G. For an ordered set $W = \{w_1, w_2, ..., w_k\} \subseteq V(G)$ and a vertex v of G. the k-vector

$$r(v|W) = (d(v, w_1), d(v, w_2), ..., d(v, w_k))$$

is called a *representation* of v with respect to W. If every two distinct vertices of G have distinct representations with respect to W, then the ordered set W is called a *resolving set* of G. A resolving set of G having a minimum cardinality is called

This work was funded by the Faculty of Science, Srinakharinwirot University
[1]`Corresponding author.`

a *minimum resolving set* or a *basis* of G and this cardinality is the *dimension* of G, and is denoted by $\dim(G)$. To illustrate these concepts, consider a connected graph G of Figure 1 with a vertex set $V(G) = \{u, v, w, x, y, z\}$.

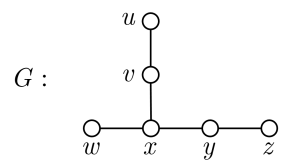

Figure 1: A connected graph G

We consider an ordered set $W = \{u, z\}$. There are six representations of vertices with respect to W:

$$r(u|W) = (0, 4), \quad r(v|W) = (1, 3), \quad r(w|W) = (3, 3),$$
$$r(x|W) = (2, 2), \quad r(y|W) = (3, 1), \quad r(z|W) = (4, 0).$$

Since the representations of two distinct vertices with respect to W are distinct, it follows that W is a resolving set of G. Since there is no 1-resolving set of G, it implies that W is a basis of G, that is, $\dim(G) = 2$.

The concepts of resolving sets and minimum resolving sets have previously appeared in [1], [2] and [3]. Hulme, Shiver and Slater described in [4], [5] and [6] the usefulness of these ideas when working with U.S. sonar and coast guard Loran (Long range aids to navigation) stations. Independently, Harary and Melter [7] discovered these concepts as well. Recently, these concepts were rediscovered by Johnson [8] of the Pharmacia Company while attempting to develop a capability of large datasets of chemical graphs. A basic problem in chemistry is to provide mathematical representations for a set of chemical compounds in a way that gives distinct representations to distinct compounds. The structure of a chemical compound can be represented by a labeled graph whose vertex and edge labels specify the atom and bond types, respectively. Thus, a graph-theoretic interpretation of this problem is to provide representations for the vertices of a graph in such a way that distinct vertices have distinct representations. More applications of these concepts to navigation of robots in networks and other areas are discussed in [9].

The foregoing discussion then gives rise to representations that is like multisets. In this case, we consider those sets W of vertices of connected graphs G for which any two vertices of G having distinct representations with respect to W in term of multisets.

Let $W = \{w_1, w_2, \ldots, w_k\}$ be a set of vertices of a connected graph G. For each vertex v of G, the *multirepresentation* of v with respect to W is a k-multiset, which is denoted by $mr_G(v|W)$ or simply $mr(v|W)$ if the graph G under consideration is clear, and defined by

$$mr(v|W) = \{d(v, w_1), d(v, w_2), \ldots, d(v, w_k)\}.$$

If $mr(x|W) \neq mr(y|W)$ for every pair x, y of distinct vertices of G, then W is called a *multiresolving set* of G. A multiresolving set of G containing a minimum number of vertices is called a *minimum multiresolving set* or a *multibasis* of G. The cardinality of multibasis is a *multidimension* of G, which is denoted by $\dim_M(G)$.

To illustrate these concepts, consider a connected graph G of Figure 1. As we know that the set $W = \{u, z\}$ is a resolving set of G. However, since $mr(v|W) = \{1, 3\} = mr(y|W)$, it follows that W is not a multiresolving set of G. Indeed, the set $W' = \{u, v, z\}$ is a multiresolving set of G with multirepresentations of the vertices of G with respect to W' as

$$\begin{array}{lll} mr(u|W') = \{0,1,4\}, & mr(v|W') = \{0,1,3\}, & mr(w|W') = \{2,3,3\}, \\ mr(x|W') = \{1,2,2\}, & mr(y|W') = \{1,2,3\}, & mr(z|W') = \{0,3,4\}. \end{array}$$

Since there is no multiresolving sets of cardinality 1 or 2, it follows that W' is a multibasis of G, that is $\dim_M(G) = 3$.

Not all connected graphs have a multiresolving set and also $\dim_M(G)$ is not defined for all connected graphs G. For example, the star $K_{1,3}$ has no multiresolving set. Therefore, $\dim_M(K_{1,3})$ is not defined. However, for a connected graph G of order n that $\dim_M(G)$ is defined, every multiresolving set of G is also a resolving set of G, and so

$$1 \leq \dim(G) \leq \dim_M(G) \leq n.$$

For every set W of vertices of a connected graph G, the vertices of G whose multirepresentations with respect to W contain 0, are vertices in W. On the other hand, the multirepresentations of vertices of G that do not belong to W have elements, all of which are positive. Indeed, to determine whether a set W is a multiresolving set of G, the vertex set $V(G)$ can be partitioned into W and $V(G) - W$ to examine whether the vertices in each subset have distinct multirepresentations with respect to W. The multiresolving set was introduced in [10] and further studied in [11] and [12].

2 Preliminaries

Two vertices u and v of a connected graph G are *distance-similar* if $d(u, x) = d(v, x)$ for all $x \in V(G) - \{u, v\}$. Certainly, distance similarity in G is an equivalence relation on $V(G)$. For example, consider a complete bipartite graph $K_{r,s}$ with partite sets U and V. Every pair of vertices in the same partite set are distance-similar. Then the distance-similar equivalence classes in $K_{r,s}$ are its partite sets U and V. The following results were obtained in [10] showing the usefulness of the distance-similar equivalence class to determine the multidimensions of connected graphs.

Theorem 2.1 ([10]). *Let G be a connected graph such that $\dim_M(G)$ is defined. If U is a distance-similar equivalence class in G with $|U| = 2$, then every multiresolving set of G contains exactly one vertex of U.*

Theorem 2.2 ([10]). *If U is a distance-similar equivalence class in a connected graph G with $|U| \geq 3$, then $\dim_M(G)$ is not defined.*

It was shown in [10] and [12] that a path is only a connected graph with multidimension 1, and there is no connected graph with multidimension 2. We state these results in the next theorems.

Theorem 2.3 ([10], [12]). *Let G be a connected graph. Then $\dim_M(G) = 1$ if and only if $G = P_n$, the path of order n.*

Theorem 2.4 ([10], [12]). *A connected graph has no multiresolving set of cardinality 2.*

As we already mentioned, if W is a multiresolving set of a connected graph G, then the multirepresentations of two distinct vertices of G are distinct. This lead us to the fact that W is also a multiresolving set of $G-v$, where v is an end-vertex of G.

Theorem 2.5. *Let G be a connected graph such that $\dim_M(G)$ is defined, and let W be a multiresolving set of G. If v is an end-vertex of G such that $v \notin W$, then W is a multiresolving set of $G - v$.*

Proof. Assume that v is an end-vertex of G. Let $W = \{w_1, w_2, ..., w_k\}$ be a multiresolving set of G that does not contain v. Then

$$mr_G(x|W) = \{d_G(x, w_1), d_G(x, w_2), ..., d_G(x, w_k)\}$$

and

$$mr_G(y|W) = \{d_G(y, w_1), d_G(y, w_2), ..., d_G(y, w_k)\}$$

are not the same for all vertices x and y of G. since v does not belong to W, it follows that

$$mr_{G-v}(x|W) = \{d_{G-v}(x, w_1), d_{G-v}(x, w_2), ..., d_{G-v}(x, w_k)\} = mr_G(x|W)$$

and

$$mr_{G-v}(y|W) = \{d_{G-v}(y, w_1), d_{G-v}(y, w_2), ..., d_{G-v}(y, w_k)\} = mr_G(y|W),$$

that is, $mr_{G-v}(x|W) \neq mr_{G-v}(y|W)$ for all vertices x and y of $G - v$. Hence, W is a multiresolving set of $G - v$. □

The following is an immediate corollary of Theorem 2.5.

Corollary 2.6. *Let G be a connected graph such that $\dim_M(G)$ is defined, and let W be a multiresolving set of G. If $v_1, v_2, ..., v_t \notin W$ are end-vertices of G, then W is a multiresolving set of $G - \{v_1, v_2, ..., v_t\}$.*

Next, we present a useful necessary condition for a set to be a multiresolving set.

Proposition 2.7. *Let T be a tree of order at least 3 containing a vertex u. If W is a multiresolving set of T, then W contains at least one vertex from each of $\deg_T u$ components of $T-u$, with one possible exception.*

Proof. We see that $T-u$ has only one component if and only if u is an end-vertex of T. Then we may assume, to the contrary, that there is a vertex u of degree at least 2 such that $T-u$ has two components X and Y containing no vertex of W. Then there are two vertices x of X and y of Y that are adjacent to u in T. Thus, $d(x,w) = d(u,w)+1 = d(y,w)$ for all vertices w of W. This implies that $mr(x|W) = mr(y|W)$, and so W is not a multiresolving set of T. □

3 The Characterization of Caterpillars with Multidimension 3

A caterpillar is a tree of order at least 3, the removal of whose end-vertices produces a path called the *spine* of the caterpillar. A vertex of the spine of the caterpillar is called a *spine-vertex.* Let T be a caterpillar that $\dim_M(T)$ is defined. Since any two end-vertices that are adjacent to the same spine-vertex of T are distance-similar, it follows by Theorem 2.2 that there are at most two end-vertices that are adjacent to each spine-vertex of T. Therefore, we consider multiresolving sets of such a caterpillar. In order to do this, let us introduce some additional definitions and notation. For integers $s, k_1, k_2, ..., k_s$ with $s \geq 1$, $1 \leq k_1, k_s \leq 2$ and $0 \leq k_2, k_3, ..., k_{s-1} \leq 2$, let $\text{ca}(k_1, k_2, ..., k_s)$ be a caterpillar which is obtained from the spine $(u_1, u_2, ..., u_s)$ by joining k_i end-vertices to the spine-vertex u_i, where $1 \leq i \leq s$. Observe that, if $k_i = 0$, then there is no end-vertex joining to the spine-vertex u_i. Also, if $k_i = 1$, then the spine-vertex u_i is adjacent to an end-vertex which is called the *first end-vertex* v_i of u_i. Furthermore, if $k_i = 2$, then there are two end-vertices joining to u_i that are called the *first* and *second end-vertices* of u_i and denoted by v_i and w_i, respectively. Moreover, let Ψ be a set of all integers i with $k_i = 2$, that is, $\Psi = \{i \in \mathbb{Z} \mid k_i = 2\}$. This is illustrated in Figure 2 for the caterpillar $\text{ca}(1,2,0,2,1,2,2)$ with $\Psi = \{2,4,6,7\}$.

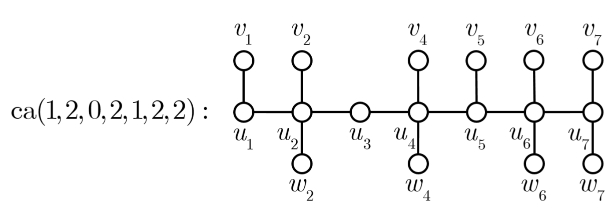

Figure 2: The caterpillar $\text{ca}(1,2,0,2,1,2,2)$

For integer s with $1 \leq s \leq 2$, the caterpillars $\text{ca}(k_1)$ and $\text{ca}(k_1, k_2)$ are shown in Figure 3, where the vertices of multibasis of these caterpillars are indicated by

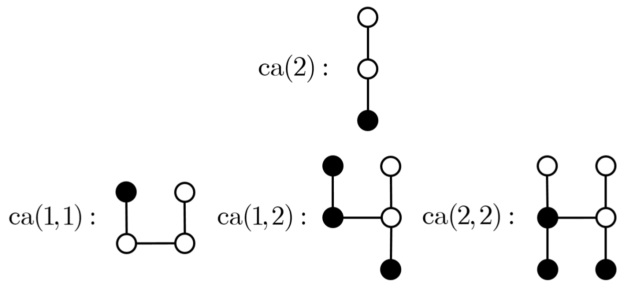

Figure 3: The caterpillars ca(2), ca(1, 1), ca(1, 2) and ca(2, 2)

solid vertices. Notice that $\mathrm{ca}(2) \cong P_3$, $\mathrm{ca}(1,1) \cong P_4$ and $\mathrm{ca}(1,2) \cong \mathrm{ca}(2,1)$. This implies that there is no caterpillar having multidimension 3, where $s = 1$, and there are two distinct caterpillars having multidimension 3, where $s = 2$. For $s = 3$, it is routine to verify that $\mathrm{ca}(1,0,2) \cong \mathrm{ca}(2,0,1)$, $\mathrm{ca}(1,1,1)$, $\mathrm{ca}(1,1,2) \cong \mathrm{ca}(2,1,1)$, $\mathrm{ca}(2,0,2)$ and $\mathrm{ca}(2,1,2)$ are caterpillars having multidimension 3. For $s \geq 4$, we are prepared to establish a characterization of a caterpillar $\mathrm{ca}(k_1, k_2, ..., k_s)$ with multidimension 3. In order to do this, we first present several preliminary results.

Proposition 3.1. *Let s, α, β be integers with $s \geq 4$ and $1 \leq \alpha < \beta \leq s$, and let W be a set of vertices of a caterpillar $\mathrm{ca}(k_1, k_2, ..., k_s)$ containing one of $\{v_1, w_1\}$ and one of $\{v_s, w_s\}$. If $mr(u_\alpha|W) = mr(u_\beta|W)$ or $mr(v_\alpha|W) = mr(v_\beta|W)$, then $1 \leq \alpha \leq \lceil \frac{s}{2} \rceil$ and $\beta = s - \alpha + 1$.*

Proof. (i) Suppose that $mr(u_\alpha|W) = mr(u_\beta|W)$. Without loss of generality, assume that W contains v_1 and v_s. For $1 \leq \alpha < \beta \leq \lceil \frac{s}{2} \rceil$, since $d(u_\alpha, v_s) = s - \alpha + 1$ and $d(u_\beta, v_s) = s - \beta + 1$ are the maximum elements of $mr(u_\alpha|W)$ and $mr(u_\beta|W)$, respectively, it follows that $\alpha = \beta$, which is a contradiction. For $\lceil \frac{s}{2} \rceil + 1 \leq \alpha < \beta \leq s$, since $d(u_\alpha, v_1) = \alpha$ and $d(u_\beta, v_1) = \beta$ are the maximum elements of $mr(u_\alpha|W)$ and $mr(u_\beta|W)$, respectively, it follows that $\alpha = \beta$, a contradiction is produced. Thus, $1 \leq \alpha \leq \lceil \frac{s}{2} \rceil$ and $\lceil \frac{s}{2} \rceil + 1 \leq \beta \leq s$. Moreover, since $d(u_\alpha, v_s) = s - \alpha + 1$ and $d(u_\beta, v_1) = \beta$ are the maximum elements of $mr(u_\alpha|W)$ and $mr(u_\beta|W)$, respectively, it follows that $\beta = s - \alpha + 1$, as we claimed. (ii) can be obtained in a manner similar to that used in the proof of (i). $\square$

Proposition 3.2. *Let s, γ, δ be integers with $s \geq 4$ and $1 \leq \gamma, \delta \leq s$, and let W be a set of vertices of a caterpillar $\mathrm{ca}(k_1, k_2, ..., k_s)$ containing one of $\{v_1, w_1\}$ and one of $\{v_s, w_s\}$. Then*

(i) *if $1 \leq \gamma < \delta \leq s$ and $mr(v_\gamma|W) = mr(u_\delta|W)$, then $1 \leq \gamma \leq \lceil \frac{s}{2} \rceil$ and $\delta = s - \gamma + 2$, and*

(ii) *if $1 \leq \delta \leq \gamma \leq s$ and $mr(v_\gamma|W) = mr(u_\delta|W)$, then $\lceil \frac{s}{2} \rceil + 1 \leq \gamma \leq s$ and $\delta = s - \gamma$.*

Proof. (i) Suppose that $1 \leq \gamma < \delta \leq s$ and $mr(v_\gamma|W) = mr(u_\delta|W)$. Without loss of generality, let us assume that W contains v_1 and v_s. If $1 \leq \gamma < \delta \leq \lceil \frac{s}{2} \rceil$, then $d(v_\gamma, v_s) = s - \gamma + 2$ and $d(u_\delta, v_s) = s - \delta + 1$ are the maximum elements of $mr(v_\gamma|W)$ and $mr(u_\delta|W)$, respectively. Therefore, $\delta = \gamma - 1$, that is, $\gamma > \delta$, which gives a contradiction. If $\lceil \frac{s}{2} \rceil + 1 \leq \gamma < \delta \leq s$, then $d(v_\gamma, v_1) = \gamma + 1$ and $d(u_\delta, v_1) = \delta$ are the maximum elements of $mr(v_\gamma|W)$ and $mr(u_\delta|W)$, respectively. Thus, $\delta = \gamma + 1$. Since $d(v_\gamma, v_s) = s - \gamma + 2$ belongs to $mr(v_\gamma|W)$, there is a vertex w for which $w = u_{2\delta-s-3}$ or $v_{2\delta-s-2}$ or $w_{2\delta-s-2}$ such that $d(u_\delta, w) = s - \gamma + 2$. Moreover, since $d(v_\gamma, w) = d(u_\delta, w) = s - \gamma + 2$, it follows that $mr(v_\gamma|W)$ contains $s - \gamma + 2$'s more than $mr(u_\delta|W)$ does, which is impossible. Therefore, $1 \leq \gamma \leq \lceil \frac{s}{2} \rceil$ and $\lceil \frac{s}{2} \rceil + 1 \leq \delta \leq s$. Moreover, since $d(v_\gamma, v_s) = s - \gamma + 2$ and $d(u_\delta, v_1) = \delta$ are the maximum elements of $mr(v_\gamma|W)$ and $mr(u_\delta|W)$, respectively, it follows that $\delta = s - \gamma + 2$, as we claimed. For (ii), the statement may be proven in the same way as (i), and therefore such proof is omitted. □

An argument similar to the one used in the proof of Propositions 3.1 and 3.2 establishes the following results.

Proposition 3.3. *Let s, α, β be integers with $s \geq 4$ and $1 \leq \alpha < \beta \leq s$, and let W be a set of vertices of a caterpillar* $\mathrm{ca}(k_1, k_2, ..., k_s)$ *containing u_1 and one of $\{v_s, w_s\}$ except v_1 and w_1. If $mr(u_\alpha|W) = mr(u_\beta|W)$ or $mr(v_\alpha|W) = mr(v_\beta|W)$, then $1 \leq \alpha \leq \lceil \frac{s}{2} \rceil$ and $\beta = s - \alpha + 2$.*

Proposition 3.4. *Let s, γ, δ be integers with $s \geq 4$ and $1 \leq \gamma, \delta \leq s$, and let W be a set of vertices of a caterpillar* $\mathrm{ca}(k_1, k_2, ..., k_s)$ *containing u_1 and one of $\{v_s, w_s\}$ except v_1 and w_1. Then*

(i) *if $1 \leq \gamma < \delta \leq s$ and $mr(v_\gamma|W) = mr(u_\delta|W)$, then $1 \leq \gamma \leq \lceil \frac{s}{2} \rceil$ and $\delta = s - \gamma + 3$, and*

(ii) *if $1 \leq \delta \leq \gamma \leq s$ and $mr(v_\gamma|W) = mr(u_\delta|W)$, then $\lceil \frac{s}{2} \rceil + 1 \leq \gamma \leq s$ and $\delta = s - \gamma + 1$.*

For an even integer $s \geq 4$, let T_1 be a caterpillar $\mathrm{ca}(k_1, k_2, ..., k_s)$ such that $\Psi = \{1, r, s\}$, where $r \in \{2, 3, ..., s-1\}$. In particular, the caterpillar $T_1 = \mathrm{ca}(2, 0, 2, 1, 0, 1, 0, 2)$ is shown in Figure 4.

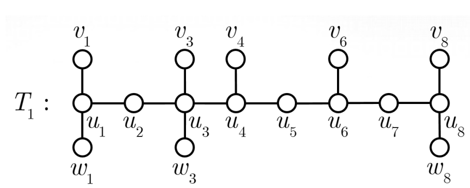

Figure 4: The caterpillar $T_1 = \mathrm{ca}(2, 0, 2, 1, 0, 1, 0, 2)$ with $\Psi = \{1, 3, 8\}$

For an odd integer $s \geq 5$, let T_2 be a caterpillar $\mathrm{ca}(k_1, k_2, ..., k_s)$ such that $\Psi = \{1, r, s\}$, where

$$r \in \begin{cases} \{2,3,...,s-1\} - \{3, \frac{s+1}{2}, s-2\} & \text{if } s \equiv 1 \pmod 4, \\ \{2,3,...,s-1\} - \{3, \frac{s-1}{2}, \frac{s+1}{2}, \frac{s+3}{2}, s-2\} & \text{if } s \equiv 3 \pmod 4. \end{cases} \tag{3.1}$$

For example, the caterpillar $T_2 = \mathrm{ca}(2,0,1,2,0,1,1,0,2)$ is illustrated in Figure 5.

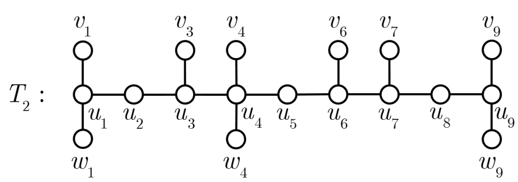

Figure 5: The caterpillar $T_2 = \mathrm{ca}(2,0,1,2,0,1,1,0,2)$ with $\Psi = \{1,4,9\}$

For an odd integer $s \geq 9$, let T_3 be a caterpillar $\mathrm{ca}(k_1, k_2, ..., k_s)$ such that $\Psi = \{1, 3, s\}$ and $k_{\frac{s-1}{2}} = 0$, or $\Psi = \{1, s-2, s\}$ and $k_{\frac{s+3}{2}} = 0$. For an odd integer $s \geq 11$ and $s \equiv 3 \pmod 4$, let T_4 be a caterpillar $\mathrm{ca}(k_1, k_2, ..., k_s)$ such that $\Psi = \{1, \frac{s-1}{2}, s\}$ and $k_{\frac{s+5}{4}} = 0$, or $\Psi = \{1, \frac{s+3}{2}, s\}$ and $k_{\frac{3s-1}{4}} = 0$.

Proposition 3.5. *A caterpillar T_i, where $1 \leq i \leq 4$ has multidimension* 3.

Proof. For each integer i with $1 \leq i \leq 4$, we show that every caterpillar T_i has multidimension 3. We verify this for T_2 only since the proof for T_1, T_3 and T_4 uses an argument similar to the one for T_2. First, we verify that $W = \{w_1, w_r, w_s\}$ is a multiresolving set of T_2, where r satisfies the condition (3.1). Without loss of generality, we may assume that $2 \leq r \leq \lceil \frac{s}{2} \rceil$. The multirepresentations of vertices of W with respect to W are $mr(w_1|W) = \{0, r+1, s+1\}$, $mr(w_r|W) = \{0, r+1, s-r+2\}$ and $mr(w_s|W) = \{0, s-r+2, s+1\}$. Since $r \notin \{1, \frac{s+1}{2}, s\}$, it follows that these 3-multisets are distinct. Next, we claim that $mr(x|W) \neq mr(y|W)$ for all vertices $x, y \in V(T_2) - W$. Suppose, contrary to our claim, that $mr(x|W) = mr(y|W)$ for some vertices $x, y \in V(T_2) - W$. We consider three cases.

Case 1. x and y are spine-vertices.

Let $x = u_\alpha$ and $y = u_\beta$, where $1 \leq \alpha < \beta \leq s$. Then by Proposition 3.1, $1 \leq \alpha \leq \lceil \frac{s}{2} \rceil$ and $\beta = s - \alpha + 1$. Thus, $mr(u_\beta|W) = \{s-\beta+1, \beta-r+1, \beta\} = \{\alpha, s-\alpha-r+2, s-\alpha+1\}$. Since $mr(u_\alpha|W) = \{\alpha, |\alpha - r|+1, s-\alpha+1\}$, it follows that $|\alpha - r| + 1 = s - \alpha - r + 2$. If $\alpha \geq r$, then $2\alpha = s+1$, and so $\alpha = \beta$ which is impossible. If $\alpha < r$, then $r = \frac{s-1}{2}$, a contradiction.

Case 2. x and y are first end-vertices.

Let $x = v_\alpha$ and $y = v_\beta$, where $1 \leq \alpha < \beta \leq s$. Then by Proposition 3.1, $1 \leq \alpha \leq \lceil \frac{s}{2} \rceil$ and $\beta = s - \alpha + 1$. Thus, $mr(v_\beta|W) = \{s-\beta+2, \beta-r+2, \beta+1\} =$

$\{\alpha+1, s-\alpha-r+3, s-\alpha+2\}$. Since $mr(v_\alpha|W) = \{\alpha+1, |\alpha-r|+2, s-\alpha+2\}$, it follows that $|\alpha-r|+2 = s-\alpha-r+3$. If $\alpha \geq r$, then $2\alpha = s+1$, and so $\alpha = \beta$, which cannot occur. If $\alpha < r$, then $r = \frac{s-1}{2}$, that is also a contradiction.

Case 3. x is a first end-vertex and y is a spine-vertex.

Let $x = v_\gamma$ and $y = u_\delta$, where $1 \leq \gamma, \delta \leq s$. We consider two subcases.

Subcase 3.1. $1 \leq \gamma < \delta \leq s$.

Then by Proposition 3.2 (i), $1 \leq \gamma \leq \lceil \frac{s}{2} \rceil$ and $\delta = s-\gamma+2$. Thus, $mr(u_\delta|W) = \{s-\delta+1, \delta-r+1, \delta\} = \{\gamma-1, s-\gamma-r+3, s-\gamma+2\}$. Since $mr(v_\gamma|W) = \{\gamma+1, |\gamma-r|+2, s-\gamma+2\}$, it follows that $|\gamma-r|+2 = \gamma-1$ and $\gamma+1 = s-\gamma-r+3$. If $\gamma \geq r$, then $r = 3$, which is impossible. If $\gamma < r$, then $s = 4(\gamma-2)+3$, that is, $s \equiv 3 \pmod 4$. Also, we obtain that $2r = s-1$, and then $r = \frac{s-1}{2}$, which is a contradiction.

Subcase 3.2. $1 \leq \delta \leq \gamma \leq s$.

Then by Proposition 3.2 (ii), $\lceil \frac{s}{2} \rceil + 1 \leq \gamma \leq s$ and $\delta = s-\gamma$. Thus, $mr(v_\gamma|W) = \{s-\gamma+2, \gamma-r+2, \gamma+1\} = \{\delta+2, s-\delta-r+2, s-\delta+1\}$. Since $mr(u_\delta|W) = \{\delta, |\delta-r|+1, s-\delta+1\}$, it follows that $|\delta-r|+1 = \delta+2$ and $\delta = s-\delta-r+2$. Consequently, $|\delta-r| = s-\delta-r+3$. If $\delta \geq r$, then $2\delta = s+3$, which cannot occur. If $\delta < r$, then $2r = s+3$, a contradiction.

Therefore, $mr(x|W) \neq mr(y|W)$ for all vertices $x, y \in V(T_2) - W$, that is, W is a multiresolving set of T_2 and so $\dim_M(T_2) \leq 3$. Since T_2 is not a path, it follows by Theorems 2.3 and 2.4 that $\dim_M(T_2) \geq 3$. Hence, $\dim_M(T_2) = 3$. □

The following corollary is an immediate consequence of Proposition 3.5.

Corollary 3.6. *Let T be a caterpillar $\mathrm{ca}(k_1, k_2, ..., k_s)$ such that $T \cong T_i$, where $1 \leq i \leq 4$ with $\Psi = \{1, r, s\}$. Then W is a multibasis of T if and only if $W = \{x_1, x_r, x_s\}$, where $x_i \in \{v_i, w_i\}$ for $i = 1, r, s$.*

For an integer $s \geq 4$, let T_5 be a caterpillar $\mathrm{ca}(k_1, k_2, ..., k_s)$ such that $\Psi = \{p, s\}$ or $\Psi = \{1, q\}$, where $1 \leq p < q \leq s$.

Proposition 3.7. *A caterpillar T_5 has multidimension* 3.

Proof. First, suppose that $\Psi = \{p, s\}$, where $1 \leq p \leq s-1$. Since T_5 is not a path, it follows by Theorems 2.3 and 2.4 that $\dim_M(T_5) \geq 3$. We consider two cases.

Case 1. $p = 1$.

We show that $W = \{u_1, w_1, w_s\}$ is a multiresolving set of T_5. The multirepresentations of vertices of W with respect to W are $mr(u_1|W) = \{0, 1, s\}$, $mr(w_1|W) = \{0, 1, s+1\}$ and $mr(w_s|W) = \{0, s, s+1\}$. Thus, these 3-multisets are distinct. Next, we claim that $mr(x|W) \neq mr(y|W)$ for all vertices $x, y \in V(T_5) - W$. Assume, contrary to our claim, that $mr(x|W) = mr(y|W)$ for some vertices $x, y \in V(T_5) - W$. We consider three subcases.

Subcase 1.1. x and y are spine-vertices.

Let $x = u_\alpha$ and $y = u_\beta$, where $1 \leq \alpha < \beta \leq s$. Then by Proposition 3.1, $1 \leq \alpha \leq \lceil \frac{s}{2} \rceil$ and $\beta = s-\alpha+1$. Thus, $mr(u_\beta|W) = \{s-\beta+1, \beta-1, \beta\} = \{\alpha, s-\alpha, s-\alpha+1\}$. Since $mr(u_\alpha|W) = \{\alpha, \alpha-1, s-\alpha+1\}$, it follows that $\alpha-1 = s-\alpha$ and so $\alpha = \beta$, which is impossible.

Subcase 1.2. x and y are first end-vertices.
Let $x = v_\alpha$ and $y = v_\beta$, where $1 \leq \alpha < \beta \leq s$. Then by Proposition 3.1, $1 \leq \alpha \leq \lceil \frac{s}{2} \rceil$ and $\beta = s - \alpha + 1$. Thus, $mr(v_\beta|W) = \{s - \beta + 2, \beta, \beta + 1\} = \{\alpha + 1, s - \alpha + 1, s - \alpha + 2\}$. Since $mr(v_\alpha|W) = \{\alpha + 1, \alpha, s - \alpha + 2\}$, it follows that $\alpha = s - \alpha + 1$ and so $\alpha = \beta$, this is also a contradiction.

Subcase 1.3. x is a first end-vertex and y is a spine-vertex.
Let $x = v_\gamma$ and $y = u_\delta$, where $1 \leq \gamma, \delta \leq s$. We consider two subcases.

Subcase 1.3.1. $1 \leq \gamma < \delta \leq s$.
Then by Proposition 3.2 (i), $1 \leq \gamma \leq \lceil \frac{s}{2} \rceil$ and $\delta = s - \gamma + 2$. Since $mr(u_\delta|W) = \{s-\delta+1, \delta-1, \delta\} = \{\gamma-1, s-\gamma+1, s-\gamma+2\}$ and $mr(v_\gamma|W) = \{\gamma+1, \gamma, s-\gamma+2\}$, it follows that $mr(v_\gamma|W) \neq mr(u_\delta|W)$, which is impossible.

Subcase 1.3.2. $1 \leq \delta \leq \gamma \leq s$.
Then by Proposition 3.2 (ii), $\lceil \frac{s}{2} \rceil + 1 \leq \gamma \leq s$ and $\delta = s - \gamma$. Since $mr(v_\gamma|W) = \{s - \gamma + 2, \gamma, \gamma + 1\} = \{\delta + 2, s - \delta, s - \delta + 1\}$ and $mr(u_\delta|W) = \{\delta, \delta - 1, s - \delta + 1\}$, it follows that $mr(v_\gamma|W) \neq mr(u_\delta|W)$, this is also a contradiction.
Therefore, $mr(x|W) \neq mr(y|W)$ for all vertices $x, y \in V(T_5) - W$, that is, W is a multiresolving set of T_5. Hence, $\dim_M(T_5) \leq 3$, and so $\dim_M(T_5) = 3$, where $p = 1$.

Case 2. $p \geq 2$.
We consider two subcases.

Subcase 2.1. s is even.
With the aid of Theorem 2.5 and Corollary 3.6, since $T_5 \cong T_1 - w_1$ and $W = \{v_1, w_p, w_s\}$ is a multiresolving set of T_1, it follows that W is a multiresolving set of T_5. Therefore, $\dim_M(T_5) \leq 3$, and so $\dim_M(T_5) = 3$, where $p \geq 2$ and s is even.

Subcase 2.2. s is odd.
We consider two subcases.

Subcase 2.2.1. $p = 2$.
By Theorem 2.5 and Corollary 3.6, since $T_5 \cong T_2 - w_1$ and $W = \{v_1, w_p, w_s\}$ is a multiresolving set of T_2, it follows by Theorem 2.5 that W is a multiresolving set of T_5. Therefore, $\dim_M(T_5) \leq 3$, and so $\dim_M(T_5) = 3$, where $p = 2$ and s is odd.

Subcase 2.2.2. $p \geq 3$.
Let $W = \{u_1, w_p, w_s\}$. The multirepresentations of vertices of W with respect to W are $mr(u_1|W) = \{0, p, s\}$, $mr(w_p|W) = \{0, p, s - p + 2\}$ and $mr(w_s|W) = \{0, s - p + 2, s\}$. Thus, these 3-multisets are distinct. Next, we claim that $mr(x|W) \neq mr(y|W)$ for all vertices $x, y \in V(T_5) - W$. Suppose, contrary to our claim, that $mr(x|W) = mr(y|W)$ for some vertices $x, y \in V(T_5) - W$. We consider three subcases.

Subcase 2.2.2.1. x and y are spine-vertices.
Let $x = u_\alpha$ and $y = u_\beta$, where $1 \leq \alpha < \beta \leq s$. Then by Proposition 3.3, $1 \leq \alpha \leq \lceil \frac{s}{2} \rceil$ and $\beta = s - \alpha + 2$. Thus, $mr(u_\beta|W) = \{s - \beta + 1, |\beta - p| + 1, \beta - 1\} = \{\alpha - 1, |\beta - p| + 1, s - \alpha + 1\}$. Since $mr(u_\alpha|W) = \{\alpha - 1, |\alpha - p| + 1, s - \alpha + 1\}$, it follows that $|\alpha - p| + 1 = |\beta - p| + 1$. If $p \leq \alpha$ or $\beta \leq p$, then $\alpha = \beta$, which is impossible. If $\alpha < p < \beta$, then $s = 2p - 2$, contradicting the fact that s is odd.

Subcase 2.2.2.2. x and y are first end-vertices.

Let $x = v_\alpha$ and $y = v_\beta$, where $1 \le \alpha < \beta \le s$. Then by Proposition 3.3, $1 \le \alpha \le \lceil \frac{s}{2} \rceil$ and $\beta = s - \alpha + 2$. Thus, $mr(v_\beta|W) = \{s - \beta + 2, |\beta - p| + 2, \beta\} = \{\alpha, |\beta - p| + 2, s - \alpha + 2\}$. Since $mr(v_\alpha|W) = \{\alpha, |\alpha - p| + 2, s - \alpha + 2\}$, it follows that $|\alpha - p| + 2 = |\beta - p| + 2$. By the same argument as the proof in Subcase 2.2.2.1., we obtain a contradiction.

Subcase 2.2.2.3. x is a first end-vertex and y is a spine-vertex.
Let $x = v_\gamma$ and $y = u_\delta$, where $1 \le \gamma, \delta \le s$. There are two possibilities:

1) $1 \le \gamma < \delta \le s$.
Then by Proposition 3.4 (i), $1 \le \gamma \le \lceil \frac{s}{2} \rceil$ and $\delta = s - \gamma + 3$. Thus, $mr(u_\delta|W) = \{s - \delta + 1, |\delta - p| + 1, \delta - 1\} = \{\gamma - 2, |\delta - p| + 1, s - \gamma + 2\}$. Since $mr(v_\gamma|W) = \{\gamma, |\gamma - p| + 2, s - \gamma + 2\}$, it follows that $|\gamma - p| + 2 = \gamma - 2$ and $\gamma = |\delta - p| + 1$. Consequently, $|\gamma - p| + 3 = |\delta - p|$. If $p \le \gamma$, then $2\gamma = s$, contradicting the fact that s is odd. If $\gamma < p < \delta$, then $2p = s$, a contradiction. If $\delta \le p$, then $2\gamma - 6 = s$, this is also a contradiction.

2) $1 \le \delta \le \gamma \le s$.
Then by Proposition 3.4 (ii), $\lceil \frac{s}{2} \rceil + 1 \le \gamma \le s$ and $\delta = s - \gamma + 1$. Thus, $mr(v_\gamma|W) = \{s - \gamma + 2, |\gamma - p| + 2, \gamma\} = \{\delta + 1, |\gamma - p| + 2, s - \delta + 1\}$. Since $mr(u_\delta|W) = \{\delta - 1, |\delta - p| + 1, s - \delta + 1\}$, it follows that $|\delta - p| + 1 = \delta + 1$ and $\delta - 1 = |\gamma - p| + 2$. Consequently, $|\delta - p| = |\gamma - p| + 3$. If $p < \delta$, then $s = 2\gamma + 2$, contradicting the fact that s is odd. If $\delta \le p \le \gamma$, then $s = 2p - 4$, a contradiction. If $\gamma < p$, then $s = 2\delta + 2$, this is also a contradiction.
Therefore, $\dim_M(T_5) \le 3$, and so $\dim_M(T_5) = 3$, where $p \ge 3$ and s is odd.
Similarly, for $\Psi = \{1, q\}$, where $2 \le q \le s$, $\dim_M(T_5) = 3$ can be proven in the same manner as well. □

For an integer $s \ge 4$, let T_6 be a caterpillar $\text{ca}(k_1, k_2, ..., k_s)$ such that $\Psi = \{r\}$, where $r \in \{1, 2, ..., s\}$. For an integer $s \ge 4$, let T_7 be a caterpillar $\text{ca}(k_1, k_2, ..., k_s)$ such that $\Psi = \emptyset$ and $k_r = 1$, where $r \in \{2, 3, ..., s - 1\}$. Combining Theorem 2.5 and Proposition 3.7, we arrive yet another result.

Proposition 3.8. *A caterpillar T_i, where $6 \le i \le 7$ has multidimension* 3.

Caterpillars with multidimension 3 are completely characterized, as we present next.

Theorem 3.9. *For an integer $s \ge 4$, let T be a caterpillar ca$(k_1, k_2, ..., k_s)$. Then T has multidimension* 3 *if and only if $T \cong T_i$, where $i \in \{1, 2, ..., 7\}$.*

Proof. The preceding results provide the sufficient condition for a caterpillar T having multidimension 3. To show the necessary condition, suppose that T has a multidimension 3. By Theorem 2.1, it implies that $|\Psi| \le 3$. For $|\Psi| = 0$, there is an integer r with $2 \le r \le s - 1$ such that $k_r = 1$, for otherwise T is a path, contradicting the fact that $\dim_M(T) = 3$. Hence, $T \cong T_7$. For $|\Psi| = 1$, obviously, $T \cong T_6$. It remains therefore only to consider $|\Psi| = 2$ and $|\Psi| = 3$.

For $|\Psi| = 2$, we claim that Ψ contains at least one of $\{1, s\}$. Suppose, contrary to our claim, that Ψ contains neither 1 nor s. Let $\Psi = \{r_1, r_2\}$, where $2 \le r_1 <$

$r_2 \leq s-1$. By Theorem 2.1, every multibasis of T contains exactly one vertex of $\{v_{r_1}, w_{r_1}\}$, say w_{r_1}. Since there are $\deg_T u_{r_1} = 4$ distinct components of $T - u_{r_1}$, it follows by Proposition 2.7 that there is a vertex of a multibasis W belonging to the component containing the spine-vertex u_{r_1-1}. Similarly, since there are $\deg_T u_{r_2} = 4$ distinct components of $T - u_{r_2}$, there is a vertex of W belonging to the component containing the spine-vertex u_{r_2+1}. Therefore, W contains at least four vertices, this is a contradiction. Thus, Ψ contains at least one of $\{1, s\}$, that is, $T \cong T_5$.

For $|\Psi| = 3$, we show that Ψ contains both 1 and s. Assume, to the contrary, that Ψ does not contain 1 or s, say 1. Let $\Psi = \{r_1, r_2, r_3\}$, where $2 \leq r_1 < r_2 < r_3 \leq s$. Then $W = \{w_{r_1}, w_{r_2}, w_{r_3}\}$ is a multibasis of T. Notice that $\deg_T u_{r_1} = 4$, that is, there are four distinct components of $T - u_{r_1}$. However, both w_{r_2} and w_{r_3} must belong to the same component containing the spine-vertex u_{r_1+1}, contradicting Proposition 2.7 that w_{r_1}, w_{r_2} and w_{r_3} cannot belong to the same component of $T - u_{r_1}$. Thus, Ψ contains 1 and s. We may assume without loss of generality that $\Psi = \{1, r, s\}$ with $2 \leq r \leq \lceil \frac{s}{2} \rceil$. Then $W = \{w_1, w_r, w_s\}$ is a multibasis of T. If s is even, then $T \cong T_1$. We may assume that s is odd. If $r = \lceil \frac{s}{2} \rceil$, then $mr(w_1|W) = mr(w_s|W)$, which is impossible. Thus $2 \leq r \leq \frac{s-1}{2}$. Next, we consider two cases according to whether s is congruent to 1 or 3 modulo 4.

Case 1. $s \equiv 1 \pmod 4$.
If $r \neq 3$, then $T \cong T_2$. For $r = 3$, since $r \leq \frac{s-1}{2}$, it follows that $s \geq 9$. Next, we claim that $k_{\frac{s-1}{2}} = 0$. Suppose, contrary to our claim, that $k_{\frac{s-1}{2}} \geq 1$. Then $mr(v_{\frac{s-1}{2}}|W) = \{\frac{s-3}{2}, \frac{s+1}{2}, \frac{s+5}{2}\} = mr(u_{\frac{s+5}{2}}|W)$, contradicting the fact that W is a multibasis of T. Hence, $k_{\frac{s-1}{2}} = 0$, and so $T \cong T_3$.

Case 2. $s \equiv 3 \pmod 4$.
If $r \neq 3, \frac{s-1}{2}$, then $T \cong T_2$. For $r = 3$, we claim that $k_{\frac{s-1}{2}} = 0$. Suppose, contrary to our claim, that $k_{\frac{s-1}{2}} \geq 1$. Then $mr(v_{\frac{s-1}{2}}|W) = \{\frac{s-3}{2}, \frac{s+1}{2}, \frac{s+5}{2}\} = mr(u_{\frac{s+5}{2}}|W)$, contradicting the fact that W is a multibasis of T, as we claimed. Hence, $s \geq 11$, and so $T \cong T_3$. For $r = \frac{s-1}{2} \geq 4$, since $r \leq \frac{s-1}{2}$, it follows that $s \geq 11$. Next, we claim that $k_{\frac{s+5}{4}} = 0$. Suppose, contrary to our claim that $k_{\frac{s+5}{4}} \geq 1$. Then $mr(v_{\frac{s+5}{4}}|W) = \{\frac{s+1}{4}, \frac{s+9}{4}, \frac{3s+3}{4}\} = mr(u_{\frac{3s+3}{4}}|W)$, contradicting the fact that W is a multibasis of T. Hence, $k_{\frac{s+5}{4}} = 0$, and so $T \cong T_4$. $\square$

4 Final Remarks

For an integer $s \geq 2$, let T be a caterpillar $\mathrm{ca}(k_1, k_2, ..., k_s)$ of order n such that $\Psi \neq \emptyset$ and $\dim_M(T)$ is defined. It then follows by Theorem 2.1 that

$$|\Psi| \leq \dim_M(T) \leq n - |\Psi|.$$

Moreover, by Corollary 3.6, caterpillars T_1, T_2, T_3 and T_4 also illustrate the sharpness of this lower bound. It would be interesting to determine whether this upper bound is sharp or not.

References

[1] G. Chartrand, L. Eroh, M.A. Johnson, O.R. Oellermann, Resolvability in graphs and the metric dimension of a graph, Discrete Applied Mathematics 105 (2000) 99–113.

[2] P.J. Slater, Leaves of trees, Congressus Numerantium 14 (1988) 549–559.

[3] P.J. Slater, Dominating and reference sets in a graph, Journal of Mathematical and Physical Sciences 22 (4) (1988) 445–455.

[4] B.L. Hulme, A.W. Shiver, P.J. Slater, FIRE: a subroutine for fire protection network analysis, SAND 81-1261, Sandia National Laboratories, Albuquerque, 1981.

[5] B.L. Hulme, A.W. Shiver, P.J. Slater, Computing minimum cost fire protection, SAND 820809, Sandia National Laboratories, Albuquerque, 1982.

[6] B.L. Hulme, A.W. Shiver, P.J. Slater, A boolean algebraic analysis of fire protection, North-Holland Mathematics Studies 95 (C) (1981) 215–227.

[7] F. Harary, R.A. Melter, On the metric dimension of a graph, Ars Combinatoria 2 (1976) 191–195.

[8] M. Johnson, Browsable structure-activity datasets, Advances in Molecular Similarity 2 (1999) 153–170.

[9] S. Khuller, B. Rsghavachari, A. Rosenfeld, Localization in graphs, CS-TR-3326, University of Maryland, Maryland, 1994.

[10] V. Saenpholphat, On multiset dimension in graphs, Academic SWU. 1 (2009) 193–202.

[11] V. Khemmani, S. Isariyapalakul, The multiresolving sets of graphs with prescribed multisimilar equivalence classes, Int. J. Math. Math. Sci. (2018) Article ID 8978193.

[12] R. Simanjuntak, T. Vetrík, P.B. Mulia, The multiset dimension of graphs, Discrete Applied Mathematics, 2017.

(Received 13 June 2019)
(Accepted 24 December 2019)