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On some Algrebraic Structures of AG*-groupoids

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Abstract : An AG*-groupoid is an AG-groupoid S satisfying the identity $(ab)c = b(ac)$ for all $a, b, c \in S$. In this paper, we study some properties of AG*-groupoids. Moreover, we construct a congruence relation on a cancellative AG*-groupoid.

Keywords : AG-groupoid; AG*-groupoid; cancellative; regular; congruence.
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1 Introduction and preliminaries

By a *groupoid* $(S, \cdot)$ we mean a nonempty set S on which a binary operation $\cdot$ is defined. We say that S is an *AG-groupoid (Abel-Grassmann's groupoid)* if $\cdot$ is left invertive, that is, $(ab)c = (cb)a$ for all $a, b, c \in S$. The notion of an AG-groupoid was first introduced by Kazim and Naseeruddin in 1977 and they have called it a *left almost semigroup* (*LA-semigroup*) [1]. Such a groupoid satisfies the medial law: $(ab)(cd) = (ac)(bd)$ for all $a, b, c, d \in S$ [2]. In fact, if S is an AG-groupoid with left identity, then S satisfies the paramedical law: $(ab)(cd) = (db)(ca)$ for all $a, b, c, d \in S$ [3]. If an AG-groupoid satisfies the identity: $(ab)c = b(ac)$ for all $a, b, c \in S$, then it is called *AG*-groupoid*. It is well known that every AG*-groupoid satisfies the paramedical law. Both AG-groupoids and AG*-groupoids have been wildly studied. Some properties of AG*-groupoids were investigated in [4]. In

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[5], a description of a fully regular AG*-groupoid was presented. Other algebraic properties of AG-groupoids and AG*-groupoids can be found in [1, 3, 6, 7]. In this paper, we study some algebraic structures of AG*-groupoids. Futhermore, we define a commutative congruence on a cancellative AG*-groupoid.

An element a of a groupoid S is called *left* (*right*) *cancellative* if for every $x, y \in S$, $ax = ay$ ($xa = ya$) implies $x = y$. An element a of a groupoid S is called *cancellative* if it is both left and right cancellative. An AG-groupoid S is called a *left cancellative* (*right cancellative, cancellative*) *AG-groupoid* if every element of S is left cancellative (right cancellative, cancellative).

We first present some propositions on AG-groupoids most of which will be used later.

Theorem 1.1 ([3]). *Let S be an AG-groupoid. If a is right cancellative of S, then a is left cancellative. Hence every right cancellative element of S is cancellative.*

Theorem 1.2 ([3]). *Let S be an AG-groupoid with left identity. Every left cancellative element of S is also right cancellative.*

Theorem 1.3 ([7]). *Let S be an AG-groupoid with left identity. Then S is commutative if and only if S is associative.*

An element a of an AG-groupoid S is called *3-band* if $a = (aa)a$.

The following results show that a left cancellative element of an AG-groupoid is right cancellative if it is 3-band.

Proposition 1.4. *Let S be an AG-groupoid. If a is left cancellative and 3-band, then a is right cancellative.*

Proof. Suppose that a is a left cancellative element of S and $a = (aa)a$. Let $x, y \in S$ be such that $xa = ya$. Then

$$ax = ((aa)a)x = (xa)(aa) = (ya)(aa) = ((aa)a)y = ay.$$

Since a is left cancellative, we have $x = y$. Therefore a is a right cancellative element of S. □

The following is an immediate consequence of Proposition 1.4.

Corollary 1.5. *Let a be an element of an AG-groupoid S such that $a^2 = a$. If a is left cancellative, then a is also right cancellative.*

Proposition 1.6. *Let S be an AG-groupoid. If a is a left cancellative element of S and $a = bc$ for some $b, c \in S$, then b and c are left and right cancellative elements of S, respectively.*

Proof. Suppose that a is a left cancellative element of S and $a = bc$ for some $b, c \in S$. Let $x, y \in S$ be such that $xc = yc$. Then

$$ax = (bc)x = (xc)b = (yc)b = (bc)y = ay.$$

Since a is left cancellative, we have $x = y$. This shows that c is right cancellative. Let $x, y \in S$ be such that $bx = by$. Then

$$a(xc) = (bc)(xc) = (bx)(cc) = (by)(cc) = (bc)(yc) = a(yc).$$

Since a is left cancellative and c is right cancellative, we deduce that $x = y$. Hence b is left cancellative. □

An element a of AG-groupoid S is called a *regular element* of S if $a = (ax)a$ for some $x \in S$.

Proposition 1.7. *Let S be an AG-groupoid. If a and b are regular elements of S, then ab is a regular element of S. In particular, the set of all regular elements of S becomes an AG-subgroupoid of S if it is nonempty.*

Proof. Suppose that a and b are regular elements of S. Then $a = (ax)a$ and $b = (by)b$ for some $x, y \in S$. Since

$$ab = ((ax)a)((by)b) = ((ax)(by))(ab) = ((ab)(xy))(ab),$$

it follows that ab is a regular element of S. □

2 Main Results

We first study some properties of cancellative elements of an AG*-groupoid.

Theorem 2.1. *Let a be an element of an AG*-groupoid S. Then the following statements are equivalent.*

(*i*) *a is a left cancellative element of S.*

(*ii*) *a is a right cancellative element of S.*

(*iii*) *a is a cancellative element of S.*

Proof. (*i*) ⇒ (*ii*) Suppose that a is a left cancellative element of S. Let $x, y \in S$ be such that $xa = ya$. Since

$$a(a(ay)) = a((aa)y) = a((ya)a) = a((xa)a) = a((aa)x) = a(a(ax))$$

and by assumption, we deduce that $x = y$. Therefore a is right cancellative.

(*ii*) ⇒ (*iii*) It is clearly by Theorem 1.1.

(*iii*) ⇒ (*i*) Obvious. □

Theorem 2.2. *Let a and b be elements of an AG*-groupoid S. Then a and b are cancellative if and only if ab is cancellative. In particular, the set of all cancellative elements of S is an AG*-subgroupoid of S if it is nonempty.*

Proof. Suppose that a and b are cancellative elements of S. Let $x, y \in S$ be such that $x(ab) = y(ab)$. This implies that $(ax)b = x(ab) = y(ab) = (ay)b$. By cancellativity of b and a, we deduce that $x = y$. Therefore ab is right cancellative. By Theorem 2.1, ab is cancellative of S.

Conversely, it follows directly from Proposition 1.6 and Theorem 2.1. □

Immediately we adapt the statement by using Theorem 2.2 to obtain to following corollary.

Corollary 2.3. *Let S be an AG*-groupoid such that $S^2 = S$. If a is a cancellative element of S, then a is a product of two cancellative elements of S.*

Proposition 2.4. *Let S be an AG*-groupoid and $a, b \in S$. If a is a left cancellative element of S and $a = ab$, then $ab = ba$ and $b^2 = b$.*

Proof. Suppose that a is left cancellative and $a = ab$. Then $ab = (ab)b = b(ab) = ba$. Since

$$ab^2 = a(bb) = (ab)(bb) = (ba)(bb) = (bb)(ab) = (bb)a = (ab)b = ab,$$

we have $b^2 = b$ by assumption. □

Proposition 2.5. *Let S be an AG*-groupoid. If a is a left cancellative element of S and $a^2 = a$, then a is a left identity of S.*

Proof. Let $b \in S$. Since $ab = (aa)b = a(ab)$ and a is left cancellative, we deduce that $b = ab$. This shows that a is a left identity of S. □

In fact, if e is a right identity of an AG-groupoid S, then $ab = (ae)b = (be)a = ba$ for all $a, b \in S$. Hence S is commutative. If e is a left identity of S, then e is right cancellative. To prove this, let $a, b \in S$ be such that $ae = be$. Then $a = ea = (ee)a = (ae)e = (be)e = (ee)b = eb = b$.

Lemma 2.6. *Let e be an element of an AG*-groupoid S. Then e is a right identity of S if and only if e is a left identity of S. In this case, S is commutative.*

Proof. As mentioned, if e is a right identity of S, then S is commutative. Hence e is a left identity of S.

For the converse, assume that e is a left identity of S. From the above observation, e is right cancellative. Let $a \in S$. Since $(ae)e = e(ae) = ae$, we deduce that $ae = a$. This shows that e is a right identity of S. □

Note that Lemma 2.6 does not hold for AG-groupoid. The counterexample is given as follows.

Example 2.7. Let $S = \{a, b, c, d, e\}$ with the following Cayley table as shown in Table 1.

$\cdot$	a	b	c	d	e
a	a	a	a	a	a
b	a	b	c	d	e
c	a	e	b	c	d
d	a	d	e	b	d
e	a	c	d	e	b

Table 1.

By routine calculation to prove that S is an AG-groupoid but not an AG*-groupoid since $(cd)e \neq d(ce)$. We see that b is a left identity element in S but not right identity.

The next corollary follows directly from Lemma 2.6 and Theorem 1.3

Corollary 2.8. *If S is an AG*-groupoid with left identity, then S is a commutative semigroup.*

The quoted results will be indispensable for our proof.

Theorem 2.9. [4] *Let S be an AG*-groupoid. Then the following statements hold:*

(*i*) $a^2(bc) = (a^2b)c$ *for all* $a, b, c \in S$.

(*ii*) $(ab)c^2 = a(bc^2)$ *for all* $a, b, c \in S$.

(*iii*) $(ab^2)c = a(b^2c)$ *for all* $a, b, c \in S$.

Theorem 2.10. *Let a be an element of an AG*-groupoid S with $a^2 = a$ and let*

$$Q_a = \{x \in S \mid ax = x\}.$$

Then

(*i*) Q_a *is a commutative monoid.*

(*ii*) Q_a *has the identity element.*

(*iii*) Q_a *is an ideal of* S.

Proof. Since $a \in Q_a$, it is a nonempty subset of S. Let $x, y \in Q_a$. Then $ax = x$ and $ay = y$, and hence $a(xy) = (aa)(xy) = (ax)(ay) = xy$. So $xy \in Q_a$. By virtue of Theorem 2.9 (*ii*), we have

$$xy = (ax)(ay) = (yx)(aa) = x(y(aa)) = (xy)(aa) = (ay)(ax) = yx.$$

These prove that Q_a is a commutative semigroup having a as its identity. Hence (*i*) and (*ii*) hold. Let $s \in S$ and $x \in Q_a$. Then

$$a(sx) = (aa)(sx) = (as)(ax) = (as)x = (xs)a = s(xa) = sx$$

which implies that $sx \in Q_a$. Also, we have $a(xs) = (xa)s = xs$. Hence (*iii*) holds. □

An element a of an AG-groupoid S is called:

- a *left regular element* of S if $a = xa^2$ for some $x \in S$.
- a *right regular element* of S if $a = a^2x$ for some $x \in S$.
- a *completely regular element* of S if a is regular, left regular and right regular.
- a *(2,2)-regular element* of S if $a = (a^2x)a^2$ for some $x \in S$.
- a *weakly regular element* of S if $a = (ax)(ay)$ for some $x, y \in S$.
- a *left quasi regular element* of S if $a = (xa)(ya)$ for some $x, y \in S$.
- an *intra-regular element* of S if $a = (xa^2)y$ for some $x, y \in S$.
- a *strongly regular element* of S if $a = (ax)a$ and $ax = xa$ for some $x \in S$.

Next, to show that regular, left regular, right regular, completely regular, (2,2)-regular, weakly regular, left quasi regular, intra-regular and strongly regular coincide in any AG*-groupoids. The following lemmas are needed.

Lemma 2.11. *Let S be an AG*-groupoid. Then $(ab)(ba) = (ba)(ab)$ for all $a, b \in S$.*

Proof. Let $a, b \in S$. Then $(ab)(ba) = ((ba)b)a = b((ba)a) = b((aa)b) = (b(aa))b = ((ab)a)b = (ba)(ab)$. □

Lemma 2.12. *Let S be an AG*-groupoid and $a \in S$. If $a = (ax)a$ for some $x \in S$, then ax is an idempotent of S.*

Proof. Suppose that $a = (ax)a$ for some $x \in S$. Then

$$ax = ((ax)a)x = (x(aa))x = (aa)(xx) = (ax)(ax).$$

This shows that ax is idempotent. □

Theorem 2.13. *Let a be an element of an AG*-groupoid S. Then the following statements are equivalent.*

(i) *a is regular of S.*

(ii) *a is left regular of S.*

(iii) *a is right regular of S.*

(iv) *a is completely regular of S.*

(v) *a is (2,2)-regular of S.*

(vi) *a is weakly regular of S.*

(vii) *a is left quasi regular of S.*

(viii) *a is intra-regular of S.*

(*ix*) *a is strongly regular of S.*

Proof. Only sample proofs are necessary.

$(ii) \Rightarrow (iii)$ Suppose that a is left regular of S. Then $a = xa^2$ for some $x \in S$. From Lemma 2.11, we have

$$a = x(aa) = (ax)a = ((xa)(ax))a = ((ax)(xa))a = (xa)((ax)a) = (((ax)a)a)x = a^2x.$$

Hence a is right regular.

$(iii) \Rightarrow (iv)$ Suppose that a is right regular of S. Then there exists $x \in S$ such that $a = a^2x$. From Theorem 2.9, we have

$$a = (aa)x = (((aa)x)a)x = ((aa)(xa))x = ((ax)(aa))x = (ax)((aa)x)$$
$$= (((aa)x)x)a = (ax)a = xa^2.$$

This shows that a is regular and left regular. Thus a is completely regular.

$(iv) \Rightarrow (v)$ Suppose that a is completely regular of S. Then a is right regular. Hence $a = a^2x$ for some $x \in S$. From Theorem 2.9, we have

$$a = (aa)x = (xa)a = (x(a^2x))a = ((xa^2)x)a = (a^2(xx))a = (a^2(xx))(a^2x)$$
$$= ((a^2x)(xx))a^2 = (a^2(x(xx)))a^2.$$

This shows that a is (2,2)-regular.

$(v) \Rightarrow (vi)$ Suppose that a is $(2,2)$-regular of S. Then $a = (a^2x)a^2$ for some $x \in S$. Since $a = (a^2x)a^2 = ((aa)x)(aa) = (a(ax))(aa)$, we have that a is weakly regular of S.

$(vii) \Rightarrow (viii)$ Suppose that a is left quasi regular of S. Then $a = (xa)(ya)$ for some $x, y \in S$. Since $a = (xa)(ya) = (aa)(yx) = (y(aa))x = (ya^2)x$, we have that a is intra-regular.

$(viii) \Rightarrow (i)$ Suppose that a is intra-regular. Then $a = (xa^2)y$ for some $x, y \in S$. Since $a = (xa^2)y = x(a^2y) = (a^2x)y = (yx)a^2 = (yx)(aa) = (a(yx))a$, we have that a is regular.

$(i) \Rightarrow (ix)$ Suppose that a is regular of S. Then $a = (ax)a$ for some $x \in S$. It suffices to show that $ax = xa$. By Lemma 2.12, we have that ax is idempotent of S. Then $ax = (ax)(ax) = (xx)(aa) = x(x(aa)) = x((ax)a) = xa$.

Hence the theorem is completely proved. □

Lemma 2.12 and Theorem 2.13 are not true in an AG-groupoid shown in Table 2.

Example 2.14. Let $S = \{a, b, c, d\}$ and the binary operation $\cdot$ defined on S as follows:

$\cdot$	a	b	c	d
a	b	b	d	d
b	b	b	b	b
c	a	b	c	d
d	a	b	a	b

Table 2.

It is a routine matter to verify that S is an AG-groupoid not an AG*-groupoid since $(ac)d \neq c(ad)$ and we see that a is regular since $a = (ad)a$. But a is not right regular of S and ad is not idempotent of S.

To characterize the regular elements of an AG*-groupoid, the following theorems are shown.

Theorem 2.15. *Let S be a cancellative AG*-groupoid and $a, b \in S$. If ab is regular, then a and b are regular of S.*

Proof. Let ab be regular of S. By Theorem 2.13, ab is right regular of S. Then $ab = ((ab)(ab))x$ for some $x \in S$. Since

$$ab = ((ab)(ab))x = (b(a(ab)))x = (x(a(ab)))b$$

and by cancellativity of b, we have $a = x(a(ab))) = (ax)(ab)$. This implies that a is weakly regular of S. Hence a is regular. Since

$$ab = ((ax)(ab))b = ((aa)(xb))b = (aa)((xb)b) = a(a((xb)b))$$

and by cancellativity of a, we have that $b = a((xb)b) = a((bb)x) = (a(bb))x$. This shows that b is intra-regular of S. By Theorem 2.13, b is regular. □

Corollary 2.16. *Let S be a cancellative AG*-groupoid and $a \in S$. If $a = (ax)a$ for some $x \in S$, then x is regular of S.*

Proof. Suppose that $a = (ax)a$ for some $x \in S$. By Lemma 2.12, ax is idempotent of S which implies that ax is regular. It follows directly from Theorem 2.15 that x is regular. □

As consequence of Proposition 1.7 and Theorem 2.15, the following result follows immediately.

Corollary 2.17. *Let S be a cancellative AG*-groupoid and $a, b \in S$. Then a and b are regular if and only if ab is regular.*

Finally, we define a relation ρ on a cancellative AG*-groupoid S by

$$a\rho b \quad \text{if and only if} \quad ab = ba$$

for all $a, b \in S$. Then we have

Theorem 2.18. *Let S be a cancellative AG*-groupoid. Then the following statements hold:*

(i) *ρ is an equivalence relation.*

(ii) *ρ is a congruence.*

(iii) *S/ρ is a cancellative AG*groupoid.*

(*iv*) S/ρ *is a commutative AG*groupoid.*

Proof. (*i*) Clearly, ρ is reflexive and symmetry. Let $a, b, c \in S$ be such that $(a, b), (b, c) \in \rho$. Then $ab = ba$ and $bc = cb$. Since

$$(ac)b = c(ab) = c(ba) = (bc)a = (cb)a = (ab)c = (ba)c = (ca)b$$

and by cancellativity of b, we have $ac = ca$. This shows that ρ is transitive. Hence ρ is an equivalence relation on S.

(*ii*) Let $a, b \in S$ be such that $(a, b) \in \rho$. To show that $(ac, bc), (ca, cb) \in \rho$ for all $c \in S$. Let $c \in S$. Then $(ac)(bc) = (ab)(cc) = (ba)(cc) = (bc)(ac)$ and $(ca)(cb) = (cc)(ab) = (cc)(ba) = (cb)(ca)$. These show that $(ac, bc), (ca, cb) \in \rho$. Hence ρ is a congruence.

(*iii*) Let $a, b, c \in S$ be such that $(a\rho)(b\rho) = (a\rho)(c\rho)$. Then $(ab, ac) \in \rho$. To show that $b\rho = c\rho$, let $x \in b\rho$. Then $xb = bx$ and hence $(xb)a = (bx)a$. Since

$$(ab)x = (xb)a = (bx)a = (ax)b = x(ab),$$

we deduce that $x \in ab\rho = ac\rho$. Then $x(ac) = (ac)x$. Since

$$(cx)a = (ax)c = x(ac) = (ac)x = (xc)a.$$

and by cancellativity of a, we have $cx = xc$. It shows that $x \in c\rho$. That is, $b\rho \subseteq c\rho$. Similary, we also have $c\rho \subseteq b\rho$. Hence $b\rho = c\rho$. This proves that $a\rho$ is left cancellative of S/ρ. By Theorem 2.1, $a\rho$ is cancellative of S/ρ.

(*iv*) Let $a\rho, b\rho \in S/\rho$. By Lemma 2.11, we have that $(ab)(ba) = (ba)(ab)$. It follows that $(ab)\rho(ba)$. Therefore $(a\rho)(b\rho) = (b\rho)(a\rho)$.

□

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