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The Existence Theorem for a Coincidence Point of Some Admissible Contraction Mappings in a Generalized Metric Space

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Abstract : In the current study, we focus on finding a coincidence point of two certain mappings that are defined on a generalized metric space. These mappings are assumed to have some admissible contraction property of a specific kind based on the generalized metric. Our main results provide the sufficient conditions to the existence of a coincidence point of the mappings. Further, we give an example satisfying all conditions to support our main theorem. Finally, the result is applied to the integral equation. Specifically, we obtain some assumptions to guarantee the existence of a solution when the equation is homogeneous.

Keywords : fixed points; common fixed points; coincidence points; generalized metric spaces.
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1 Introduction

In recent research, fixed point theory is one of the most popular area that has interested a large group of researchers. The common goal is to develop known results to the better results. To be more specific, there are research that study the concept of fixed points of mappings on spaces with different methods of measurement. On top of that, mappings that have been used to discover their fixed points have a contraction property in some sense. Furthermore, a common fixed point and coincidence point of two mappings have also been brought into consideration to generalize the concept of a fixed point of a mapping.

In our work, we are interested in exploring a coincidence point of some contraction mappings defined on a generalized metric space. We give sufficient conditions to the theorem of the existence of a coincidence point of such mappings. In addition, a common fixed point of mappings also exists with some extra condition on coincidence points. Furthermore, we provide an example to support our result. Finally, we apply our theorem to some integral equation problem. The outcome is obtained as the existence theorem of a solution when the equation is homogeneous.

2 Preliminaries

2.1 Generalized Metric Spaces

Jleli and Samet [1] defined a JS-metric space. There are several spaces that are covered by the JS-metric space (see [2, 3, 4]). As a consequence, any results on JS-metric spaces are also valid for those spaces.

Definition 2.1 ([1])**.** Let (X, D) be a JS-metric space, and let $\{x_n\}$ be a sequence in X.

(i) $\{x_n\}$ is said to D**-converge** to $x \in X$ if $\lim_{n\to\infty} D(x_n, x) = 0$.

(ii) $\{x_n\}$ is said to be D**-Cauchy** if $\lim_{m,n\to\infty} D(x_n, x_m) = 0$.

(iii) (X, D) is said to be D**-complete** if any D-Cauchy sequence in X D-converges to some element in X.

Proposition 2.1 ([1])**.** *Let (X, D) be a JS-metric space. If a sequence $\{x_n\}$ in X D-converges to x and y for some $x, y \in X$, then x and y must be the same element.*

Definition 2.2 ([1])**.** Let (X, D) be a JS-metric space. A function $f : X \to X$ is said to be **continuous at** $x_0 \in X$ if

$$\lim_{n\to\infty} D(x_n, x_0) = 0 \text{ implies } \lim_{n\to\infty} D(fx_n, fx_0) = 0.$$

Then f is said to be **continuous** if it is continuous at each x in X.

From this idea, there were many research articles related to JS-metric spaces (see [5, 6, 7]). Inspired by all of the above, we define generalized metric spaces as follows.

Definition 2.3. Let X be a nonempty set. A function $D : X \times X \to [0, +\infty]$ is said to be a **generalized metric** on X if the following conditions hold.

(D_1) For any $x, y \in X$, if $D(x, y) = 0$, then $x = y$;

(D_2) For any $x, y \in X$, $D(x, y) = D(y, x)$;

(D_3) There is a real number $C > 0$ such that for any $x, y \in X$,

$$D(x, y) \leq C \limsup_{n\to\infty} D(x_n, y_n),$$

where $\{x_n\}$ and $\{y_n\}$ are sequences in X that D-converge to x and y, respectively.

Then we say that (X, D) is a **generalized metric space**.

The essential observation is that a generalized metric space is also a JS-metric space. Analogously, definitions and statements related to JS-metric spaces are also true for generalized metric spaces.

2.2 Contraction Mappings

Not only types of spaces, but also types of contraction mappings are concerned in the study of fixed point theory. In 2012, Samet et al. [8] considered mappings called α-admissible and showed the existing results of a fixed point for α-ψ-contractions. After that, Karapinar [9] generalized these mappings to be triangular α-admissible. Later, in the setting of common fixed points and coincidence points, these concepts were extended to be used for two mappings as follows.

Definition 2.4. Let (X, D) be a generalized metric space, and let f, g be self-mapping on X. Given that $\alpha : X \times X \to [0, \infty)$ is a function, f is said to be **triangular-(α, D)-admissible w.r.t.** g if, for all $x, y, z \in X$, the following conditions hold.

(i) If $\alpha(gx, gy) \geq 1$, then $\alpha(fx, fy) \geq 1$ and $D(gx, gy) < \infty$;

(ii) If $\alpha(x, z) \geq 1$ and $\alpha(z, y) \geq 1$, then $\alpha(x, y) \geq 1$.

To obtain contractive property of mappings, Geraghty [10] suggested the class of all functions $\theta : [0, \infty) \to [0, 1)$ such that for any $\{t_n\}$ in $[0, \infty]$, if $\theta(t_n) \to 1$, then $t_n \to 0$. Then define contractive mappings according to such θ.

Motivated by Geraghty, we consider the class Θ whose elements are all functions θ defined above but extended the domain to the extended interval $[0, \infty]$. We now introduce a new class of contractions as follows.

Definition 2.5. Let (X, D) be a generalized metric space, and let f, g be self-mappings on X. Given that $\alpha : X \times X \to [0, \infty)$ is a function, the pair (f, g) is said to be an **admissible Geraghty M-contraction** if the following conditions hold.

(i) f is triangular-(α, D)-admissible w.r.t. g;

(ii) There exists a function $\theta \in \Theta$ such that for all $x, y \in X$,

$$\alpha(gx, gy)D(fx, fy) \leq \theta(M(gx, gy))M(gx, gy), \tag{2.1}$$

where $M(gx, gy) = \max\{D(gx, gy), D(gx, fx), D(gy, fy)\}$.

We are almost ready to present the next section. Nevertheless, there is a property of the two contraction mappings needed to obtain our main result.

Definition 2.6. Let (X, D) be a generalized metric space. The self-mappings f and g on X are said to be **D-compatible** if $\lim_{n\to\infty} D(gfx_n, fgx_n) = 0$, where $\{x_n\}$ is a sequence in X such that $\{fx_n\}$ and $\{gx_n\}$ D-converge to the same limit.

3 Existence of a Coincidence Point

To begin this section, let us provide some notations for convenience. Let (X, D) be a generalized metric space, and let f, g be self-mappings on X. Given that $\alpha : X \times X \to [0, \infty)$, denote the set of all coincidence points of mappings f and g of X by

$$C(f, g) = \{u \in X : fu = gu\},$$

and the set of all common fixed points of mappings f and g by

$$Cm(f, g) = \{u \in X : fu = gu = u\}.$$

Finally, for any sequence $\{x_n\}$ in X and $n \in \mathbb{N}$, denote

$$\beta(D, f, x_n) = \sup\{D(fx_{n+i}, fx_{n+j}) : i, j \in \mathbb{N}\}.$$

Lemma 3.1. *Let (X, D) be a generalized metric space, and let $\alpha : X \times X \to [0, \infty)$ be a function. Assume that f and g are self-mappings on X such that (f, g) is an admissible Geraghty M-contraction. Then for any $x, y \in C(f, g)$, we obtain the following.*

(i) If $\alpha(gx, gx) \geq 1$, then $D(gx, gx) = 0$.

(ii) If $\alpha(gx, gy) \geq 1$, then $gx = gy$.

Proof. (*i*) Let $x \in C(f,g)$ such that $\alpha(gx,gx) \geq 1$. Since f is triangular-(α, D)-admissible w.r.t. g, $D(gx,gx) < \infty$. Note that

$$M(gx,gx) = \max\{D(gx,gx), D(gx,fx), D(gx,fx)\} = D(gx,gx) < \infty.$$

Since (f,g) is an admissible Geraghty M-contraction,

$$\begin{aligned} D(gx,gx) &\leq \alpha(gx,gx)D(gx,gx) \\ &= \alpha(gx,gx)D(fx,fx) \\ &\leq \theta(D(gx,gx))D(gx,gx) \end{aligned}$$

for some $\theta \in \Theta$. From the fact that $0 \leq \theta(t) < 1$ for any $t \in [0,\infty]$, it can be concluded that $D(gx,gx) = 0$.

(*ii*) Let $x, y \in C(f,g)$ such that $\alpha(gx,gy) \geq 1$. Since f is triangular-(α, D)-admissible w.r.t. g, $\alpha(gx,gx) \geq 1, \alpha(gy,gy) \geq 1$ and $D(gx,gy) < \infty$. Since $fx = gx$ and $fy = gy$, by (*i*), we have that $D(gx,fx) = D(gy,fy) = 0$. Then

$$M(gx,gy) = \max\{D(gx,gy), D(gx,fx), D(gy,fy)\} = D(gx,gy) < \infty.$$

Next, since (f,g) is an admissible Geraghty M-contraction,

$$\begin{aligned} D(gx,gy) &\leq \alpha(gx,gy)D(gx,gy) \\ &= \alpha(gx,gy)D(fx,fy) \\ &\leq \theta(M(gx,gy))M(gx,gy) \\ &= \theta(D(gx,gy))D(gx,gy) \end{aligned}$$

for some $\theta \in \Theta$. Thus, $D(gx,gy) = 0$. Hence, $gx = gy$.

□

Theorem 3.2. *Let (X,D) be a D-complete generalized metric space. Given that $\alpha : X \times X \to [0,\infty)$ is a function, let f and g be self-mappings on X such that (f,g) is an admissible Geraghty M-contraction. Suppose that all of the following hold.*

(*i*) *$f(X) \subseteq g(X)$;*

(*ii*) *There exists $x_0 \in X$ such that $\alpha(gx_0, fx_0) \geq 1$ and $\beta(D,f,x_0) < \infty$;*

(*iii*) *f and g are continuous;*

(*iv*) *f and g are D-compatible.*

Then we have that $C(f,g) \neq \emptyset$.

Proof. Since $f(X) \subseteq g(X)$ and $f(x_0) \in g(X)$, there is a sequence $\{x_n\} \in X$ such that $gx_{n+1} = fx_n$ for all $n \in \mathbb{N}$. If $gx_{n_0} = gx_{n_0+1}$ for some $n_0 \in \mathbb{N}$, then

$x_{n_0} \in C(f,g)$ and so we are done. Assume that $gx_n \neq gx_{n+1}$ for all $n \in \mathbb{N}$. Then $D(gx_n, gx_{n+1}) > 0$ for all $n \in \mathbb{N}$.

From the assumption (ii), we get that $\alpha(gx_0, gx_1) = \alpha(gx_0, fx_0) \geq 1$. Since f is triangular-(α, D)-admissible w.r.t. g, $\alpha(gx_1, gx_2) = \alpha(fx_0, fx_1) \geq 1$ and $D(gx_0, gx_1) < \infty$. By continuing in this manner, we obtain that

$$\alpha(gx_n, gx_{n+1}) \geq 1 \text{ and } D(gx_n, gx_{n+1}) < \infty \quad \text{for all } n \in \mathbb{N} \cup \{0\}.$$

Next, we will show that $\lim_{n\to\infty} D(gx_n, gx_{n+1}) = 0$. Let $n \in \mathbb{N}$. Consider

$$\begin{aligned} D(gx_{n+1}, gx_{n+2}) &\leq \alpha(gx_n, gx_{n+1}) D(gx_{n+1}, gx_{n+2}) \\ &= \alpha(gx_n, gx_{n+1}) D(fx_n, fx_{n+1}) \\ &\leq \theta(M(gx_n, gx_{n+1})) M(gx_n, gx_{n+1}) \\ &< M(gx_n, gx_{n+1}), \end{aligned} \tag{3.1}$$

where

$$\begin{aligned} M(gx_n, gx_{n+1}) &= \max\{D(gx_n, gx_{n+1}), D(gx_n, fx_n), D(gx_{n+1}, fx_{n+1})\} \\ &= \max\{D(gx_n, gx_{n+1}), D(gx_{n+1}, gx_{n+2})\}. \end{aligned}$$

If $M(gx_n, gx_{n+1}) = D(gx_{n+1}, gx_{n+2})$, then, by (3.1),

$$D(gx_{n+1}, gx_{n+2}) < D(gx_{n+1}, gx_{n+2}).$$

This is a contradiction. Thus,

$$M(gx_n, gx_{n+1}) = D(gx_n, gx_{n+1}). \tag{3.2}$$

This implies that

$$D(gx_{n+1}, gx_{n+2}) < D(gx_n, gx_{n+1}).$$

Since n is arbitrary, above statements hold for any $n \in \mathbb{N}$. Then the sequence $\{D(gx_n, gx_{n+1})\}$ is nonnegative and decreasing. Therefore, $\{D(gx_n, gx_{n+1})\}$ is convergent. Suppose on the contrary that $\lim_{n\to\infty} D(gx_n, gx_{n+1}) \neq 0$. From (3.1) and (3.2), we have that

$$\frac{D(gx_{n+1}, gx_{n+2})}{D(gx_n, gx_{n+1})} = \frac{D(gx_{n+1}, gx_{n+2})}{M(gx_n, gx_{n+1})} \leq \theta(M(gx_n, gx_{n+1})) < 1.$$

It follows that $\lim_{n\to\infty} \theta(M(gx_n, gx_{n+1})) = 1$. Since $\theta \in \Theta$,

$$\lim_{n\to\infty} D(gx_n, gx_{n+1}) = \lim_{n\to\infty} M(gx_n, gx_{n+1}) = 0,$$

a contradiction. Thus,

$$\lim_{n\to\infty} D(gx_n, gx_{n+1}) = 0. \tag{3.3}$$

Now, we claim that $\{gx_n\}$ is a D-Cauchy sequence. Suppose that this is not the case. That is, there exists $\epsilon > 0$ such that for any $k \in \mathbb{N}$, there are subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ satisfying $D(gx_{n_k}, gx_{m_k}) \geq \epsilon$ for $m_k \geq n_k \geq k$. It is easy to show that $\alpha(gx_n, gx_m) \geq 1$ and so $D(gx_n, gx_m) < \infty$ for any $n, m \in \mathbb{N}$. Let $k \in \mathbb{N}$. Consider

$$\begin{aligned}\alpha(gx_{n_k-1}, gx_{m_k-1})D(gx_{n_k}, gx_{m_k}) =& \alpha(gx_{n_k-1}, gx_{m_k-1})D(fx_{n_k-1}, fx_{m_k-1}) \\ \leq& \theta(M(gx_{n_k-1}, gx_{m_k-1}))M(gx_{n_k-1}, gx_{m_k-1}),\end{aligned}$$

where

$$\begin{aligned}M(gx_{n_k-1}, gx_{m_k-1}) = \max\{&D(gx_{n_k-1}, gx_{m_k-1}), D(gx_{n_k-1}, fx_{n_k-1}), \\ &D(gx_{m_k-1}, fx_{m_k-1})\}.\end{aligned}$$

If $M(gx_{n_k-1}, gx_{m_k-1})$ is equal to either $D(gx_{n_k-1}, fx_{n_k-1})$ or $D(gx_{m_k-1}, fx_{m_k-1})$, then, by (3.3), $\lim_{k\to\infty} D(gx_{n_k}, gx_{m_k}) = 0$. This contradicts to the fact that $\{gx_n\}$ is not a D-Cauchy sequence. Thus, $M(gx_{n_k-1}, gx_{m_k-1}) = D(gx_{n_k-1}, gx_{m_k-1})$. As a consequence, we can conclude that

$$D(gx_{n_k}, gx_{m_k}) \leq \theta(D(gx_{n_k-1}, gx_{m_k-1}))D(gx_{n_k-1}, gx_{m_k-1}).$$

By repeating the same steps, it follows that

$$D(gx_{n_k-i}, gx_{m_k-i}) \leq \theta(D(gx_{n_k-i-1}, gx_{m_k-i-1}))D(gx_{n_k-i-1}, gx_{m_k-i-1}),$$

where $i = 0, 1, 2, \ldots, n_k - 1$. Therefore,

$$D(gx_{n_k}, gx_{m_k}) \leq \prod_{i=1}^{n_k} \theta(D(gx_{n_k-i}, gx_{m_k-i}))D(gx_0, gx_{m_k-n_k}).$$

Let $i_k \in \{1, 2, \ldots, n_k\}$ such that

$$\theta(D(gx_{n_k-i_k}, gx_{m_k-i_k})) = \max\{\theta(D(gx_{n_k-i}, gx_{m_k-i})) : 1 \leq i \leq n_k\}.$$

Define $\eta = \limsup_{k\to\infty}\{\theta(D(gx_{n_k-i_k}, gx_{m_k-i_k}))\}$. Since $\beta(D, f, x_0) < \infty$, if $\eta < 1$, $\lim_{k\to\infty} D(gx_{n_k}, gx_{m_k}) = 0$ which is a contradiction. Thus, $\eta = 1$. Without loss of generality, we assume that $\lim_{k\to\infty} \theta(D(gx_{n_k-i_k}, gx_{n_k+m_k-i_k})) = 1$. By the definition of the function θ, we simply get $\lim_{k\to\infty} D(gx_{n_k-i_k}, gx_{n_k+m_k-i_k}) = 0$. Then there exists $k_0 \in \mathbb{N}$ such that

$$D(gx_{n_{k_0}-i_{k_0}}, gx_{n_{k_0}+m_{k_0}-i_{k_0}}) < \frac{\epsilon}{2}.$$

Consider

$$\begin{aligned}\epsilon &\leq D(gx_{n_{k_0}}, gx_{m_{k_0}}) \\ &\leq \prod_{j=1}^{i_{k_0}} \theta(D(gx_{n_{k_0}-j}, gx_{m_{k_0}-j}))D(gx_{n_{k_0}-i_{k_0}}, gx_{m_{k_0}-i_{k_0}}) \\ &< \frac{\epsilon}{2}.\end{aligned}$$

This is a contradiction. As a result, $\{gx_n\}$ is a D-Cauchy sequence.

Since (X, D) is D-complete, there is $u \in X$ such that

$$\lim_{n\to\infty} D(fx_n, u) = \lim_{n\to\infty} D(gx_n, u) = 0.$$

It remains to show that u is a coincidence point of f and g. Since f and g are D-compatible, $\lim\limits_{n\to\infty} D(gfx_n, fgx_n) = 0$. Then, by the continuity of f and g, $\{gfx_n\}$ D-converges to gu and $\{fgx_n\}$ D-converges to fu. Moreover, from the definition of D, there exists $C > 0$ such that

$$D(gu, fu) \leq C \limsup_{n\to\infty} D(gfx_n, fgx_n).$$

Therefore, $D(gu, fu) = 0$ and so $fu = gu$. Hence, $u \in C(f, g)$. □

Theorem 3.3. *Let (X, D) be a D-complete generalized metric space. Given that $\alpha : X \times X \to [0, \infty)$ is a function, let f and g be self-mappings on X such that (f, g) is an admissible Geraghty M-contraction. Suppose that all of the following hold.*

(i) $f(X) \subseteq g(X)$;

(ii) *There exists $x_0 \in X$ such that $\alpha(gx_0, fx_0) \geq 1$ and $\beta(D, f, x_0) < \infty$;*

(iii) *f and g are continuous;*

(iv) *f and g commute.*

Then we have that $C(f, g) \neq \emptyset$.

Proof. From the proof of Theorem 3.2, by using the assumptions (i)-(iii), we obtain the sequences $\{fx_n\}$ and $\{gx_n\}$ such that $\{gfx_n\}$ D-converges to gu and $\{fgx_n\}$ D-converges to fu for some $u \in X$. Since f and g commute,

$$\{gfx_n\} = \{fgx_n\}.$$

Therefore, $gu = fu$. This means that u is a coincidence point of f and g. □

Corollary 3.4. *In either Theorems 3.2 or 3.3, if $\alpha(gx, gy) \geq 1$ for any $x, y \in C(f, g)$, then $Cm(f, g) \neq \emptyset$.*

Proof. Assume that $\alpha(gx, gy) \geq 1$ for any $x, y \in C(f, g)$. Let $u \in C(f, g)$, and let $c = gu = fu \in X$. Consider that $gc = gfu = fgu = fc$. Thus, c is another coincidence point of f and g. By the assumption, we have that $\alpha(gu, gc) \geq 1$. Then, by Lemma 3.1, we can conclude that $fc = gc = gu = c$. Hence, c is in fact a common fixed point of f and g. □

Example 3.1. Let $X = [0,1]$, and let D be a generalized metric defined by

$$D(x,y) = \begin{cases} 3(x+y), & x \neq 0 \text{ and } y \neq 0, \\ \dfrac{x}{3}, & y = 0, \\ \dfrac{y}{3}, & x = 0. \end{cases}$$

Then (X, D) is a D-complete generalized metric space. Next, suppose that

$$\alpha(x,y) = \begin{cases} 1 & \text{if } x \neq 0 \text{ or } y = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, define the self-mappings f and g on X by

$$f(x) = \frac{x}{2x+30} \quad \text{and} \quad g(x) = \frac{x}{10}.$$

We will show that f and g have a coincidence point using Theorem 3.2. First, it is straightforward to prove that f and g are continuous and $f(X) \subseteq g(X)$. In addition, there is $x_0 = 0 \in X$ such that $\alpha(g(0), f(0)) = \alpha(0,0) \geq 1$ and $\beta(D, f, 0) < \infty$. Note that $D(f0, fu) = \frac{1}{3}\left(\frac{u}{2u+30}\right) < \infty$ for any $u \in X$.

Now, consider the following claims.

Claim 1: f is triangular-(α, D)-admissible w.r.t. g.

Let $x, y, z \in X$. Assume that $\alpha(gx, gy) \geq 1$. Then, $gx \neq 0$ or $gy = 0$. That is, $x \neq 0$ or $y = 0$. Thus, $fx \neq 0$ or $fy = 0$. Therefore, $\alpha(fx, fy) \geq 1$, and it is easy to see that $D(gx, gy) = D(\frac{x}{10}, \frac{y}{10}) < \infty$.

Next, assume that $\alpha(x,z) \geq 1$ and $\alpha(z,y) \geq 1$. It can be observed that if $z = 0$, then $y = 0$, and if $z \neq 0$, then $x \neq 0$. That is, $x \neq 0$ or $y = 0$. Therefore, $\alpha(x,y) \geq 1$. Thus, f is triangular-(α, D)-admissible w.r.t. g.

Claim 2: (f, g) is an admissible Geraghty M-contraction.

Let $x, y \in X$. If $\alpha(gx, gy) < 1$, we are done. Assume that $\alpha(gx, gy) \geq 1$. Let $\theta(t) = \dfrac{1}{2}$ for any $t \in [0, \infty]$. Consider the following cases.

Case 1: $gy = 0$. Since $\alpha(gx, gy) = 1$, $fy = 0$. Then

$$\begin{aligned} \alpha(gx, gy)D(fx, fy) &= D(fx, fy) \\ &= D(\frac{x}{2x+30}, 0) \\ &= \frac{1}{3}\left(\frac{x}{2x+30}\right) \\ &\leq \frac{1}{2}\left(\frac{1}{3}\left(\frac{x}{10}\right)\right) \\ &= \theta(M(gx, gy))D(gx, gy) \\ &\leq \theta(M(gx, gy))M(gx, gy). \end{aligned}$$

Case 2: $gy \neq 0$. Since $\alpha(gx, gy) = 1$, $gx \neq 0$. Then

$$\begin{aligned}
\alpha(gx,gy)D(fx,fy) &= D(fx,fy) \\
&= D(\frac{x}{2x+30}, \frac{y}{2y+30}) \\
&= 3\left(\frac{x}{2x+30} + \frac{y}{2y+30}\right) \\
&\leq \frac{1}{2}\left(3\left(\frac{x}{10} + \frac{y}{10}\right)\right) \\
&= \theta(M(gx,gy))D(gx,gy) \\
&\leq \theta(M(gx,gy))M(gx,gy).
\end{aligned}$$

Therefore, Claim 2 is obtained.

Our task is now to show that f and g are D-compatible, let $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} gx_n = \lim_{n\to\infty} fx_n$. That is, $\lim_{n\to\infty} \frac{x_n}{10} = \lim_{n\to\infty} \frac{x_n}{2x_n+30}$. Therefore, this limit must be 0. As a result, we get that

$$\lim_{n\to\infty} D(gfx_n, fgx_n) = 0.$$

Finally, we can conclude that f and g are D-compatible. Hence, by Theorem 3.2, $C(f,g) \neq \emptyset$. In fact, 0 is a coincidence point of f and g.

4 Application

In this section, we provide an application to the integral equation problem. Let $T > 0$. Consider the following integral equation.

$$x(t) = \int_0^T p(t,s,x(s))ds + b(t) \tag{4.1}$$

for $t \in [0,T]$. To apply our result to the equation (4.1), let $X = C([0,T],\mathbb{R})$. Define $D : X \times X \to [0,\infty]$ by

$$D(x,y) = \max_{t\in[0,T]} |x(t)| + \max_{t\in[0,T]} |y(t)|$$

for any $x, y \in C([0,T],\mathbb{R})$. It can be shown that (X, D) is a D-complete generalized metric space. Then we have the following theorem.

Theorem 4.1. *According to* (4.1), *assume that all of the following hold.*

(i) $p : [0,T] \times [0,T] \times \mathbb{R} \to \mathbb{R}$ *is a continuous function;*

(ii) there is a real number $k \in (0,1)$ *such that for any* $x, y \in \mathbb{R}$, *if* $x \leq y$, *then* $p(t,s,x) \leq p(t,s,y)$ *and*

$$|p(t,s,x)| + |p(t,s,y)| \leq \frac{k}{T}\left(|x| + |y|\right),$$

where $s, t \in [0,T]$*;*

(iii) there is $x_0 \in X$ such that

$$x_0(t) \geq \int_0^T p(t,s,x_0(s))ds$$

and $\beta(D, \int_0^T p(t,s,x(s))ds, x_0) < \infty$, where $(x_0, x_1, x_2, \dots)$ is a sequence defined by $x_n(t) = \int_0^T p(t,s,x_{n-1}(s))ds$ for each $n \in \mathbb{N}$ and $t \in [0,T]$.

Then a solution to the integral equation (4.1) *exists when it is homogeneous.*

Proof. Let f and g be mappings defined on X by

$$fx(t) = \int_0^T p(t,s,x(s))ds, \quad \text{and} \quad gx(t) = x(t)$$

for any $x \in X$ and $t \in [0,T]$. Then we have that f and g are continuous functions such that $f(X) \subseteq g(X)$. Moreover, f and g commute.

Next, define a function $\alpha : X \times X \to [0,\infty)$ by

$$\alpha(x,y) = \begin{cases} 1 & \text{if } x(t) \geq y(t) \text{ for any } t \in [0,T], \\ \\ 0 & \text{otherwise.} \end{cases}$$

We will show that (f,g) is an admissible Geraghty M-contraction. It is easy to see that for any $x,y,z \in X$, if $\alpha(x,z) \geq 1$ and $\alpha(z,y) \geq 1$, then $\alpha(x,y) \geq 1$. Then we have to only show that f is triangular-(α, D)-admissible w.r.t. g. Let $x,y \in X$ such that $\alpha(gx,gy) \geq 1$ and $t \in [0,T]$. Then $gx(t) \geq gy(t)$. Therefore, $x(t) \geq y(t)$. Thus, by the assumption (ii), we have that $p(t,s,x) \geq p(t,s,y)$. Then

$$\begin{aligned} fx(t) &= \int_0^T p(t,s,x(s))ds \\ &\geq \int_0^T p(t,s,y(s))ds \\ &= fy(t). \end{aligned}$$

It follows that $\alpha(fx,fy) \geq 1$. Consider that $D(gx,gy) = D(x,y) < \infty$ since $x,y \in C([0,T],\mathbb{R})$. Then we can conclude that f is triangular-(α, D)-admissible w.r.t. g.

Note that if $\alpha(gx,gy) < 1$, then the inequality (2.1) is true. Assume that $\alpha(gx,gy) \geq 1$. That is, $x(t) \geq y(t)$ for all $t \in [0,T]$. By assumption (ii), for any

$t \in [0, T]$, consider

$$\begin{aligned} &|fx(t)| + |fy(t)| \\ &\leq \int_0^T (|p(t, s, x(s))| + |p(t, s, y(s))|) ds \\ &\leq \frac{k}{T} \int_0^T (|x(s)| + |y(s)|) ds \\ &\leq k \left(\max_{t \in [0,T]} |gx(t)| + \max_{t \in [0,T]} |gy(t)| \right). \end{aligned}$$

Thus, the inequality (2.1) is satisfied for $\theta(t) = k$ for $t \in [0, \infty]$. Therefore, (f, g) is an admissible Geraghty M-contraction.

From Theorem 3.3, we obtain a coincidence point of f and g. Hence, this point is a solution to the integral equation (4.1) if it is homogeneous. □

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