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# Strong Convergence of the Shrinking Projection Method for the Split Equilibrium Problem and an Infinite Family of Relatively Nonexpansive Mappings in Banach spaces

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**Abstract :** In this paper, we use the shrinking projection method to prove a strong convergence theorem for finding a common solution of the split equilibrium problem and fixed point problem of a relatively quasi−nonexpansive mapping. Consequently, our main theorem can apply to find a common solution of the split equilibrium problem and common fixed point problem for an infinite family of relatively nonexpansive mappings in Banach spaces.

**Keywords :** split equilibrium problem; equilibrium problem; relatively quasi-nonexpansive; relatively nonexpansive; common fixed point; shrinking projection method; Banach space.

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# 1 Introduction

In 1994, Censor and Elfving [1] studied *the split feasibility problem* in two Hilbert spaces $H_1$ and $H_2$ which is to find $z \in H_1$ such that $z \in C \cap A^{-1}Q$, where $C$ and $Q$ are nonempty closed and convex subsets of Hilbert spaces $H_1$ and $H_2$, respectively and $A : H_1 \to H_2$ is a bounded linear operator. Furthermore, if $C \cap A^{-1}Q$ is nonempty, then $z \in C \cap A^{-1}Q$ is equivalent to

$$z = P_C(I - \lambda A^*(I - P_Q)A)z, \tag{1.1}$$

where $\lambda > 0$ and $P_C$ is the metric projection of $H_1$ onto $C$. Thus, many authors used such results to studied the split feasibility problem in Hilbert spaces; see, for instance [2, 3, 4, 5]. The result of (1.1) was extended to Banach spaces by Takahashi [6, 7]. Since then, many author have been investigating the split feasibility problem in Banach spaces (see [8, 9, 10, 11] and the reference therein). Let $S : H_1 \to H_1$ and $T : H_2 \to H_2$ be any mappings, *the split common fixed point problem* [12, 13] is to find $z \in H_1$ such that $z \in F(S) \cap A^{-1}F(T)$, where $F(S)$ and $F(T)$ are the fixed point sets of $S$ and $T$, respectively. In 2016, Takahashi [14] studied the split common fixed point problem in two Banach spaces, see also [15, 16].

Let $F : C \times C \to \mathbb{R}$ be a bifunction. *The equilibrium problem* for $F$ is to find $z \in C$ such that

$$F(z, y) \geq 0, \tag{1.2}$$

for all $y \in C$. The set of all solutions of the problem (1.2) is denoted by $EP(F)$.

In 1955, Nikaido and Isoda [17] first used the inequality in convex game models. In 1972, Fan [18] proved existence theorems for $EP(F)$. Moreover, many problems in physics, economics and others can be reduced to find a solution of the problem (1.2). After the works of [19, 20, 21, 22], the equilibrium problem has been investigated by many authors (see [23, 24, 25, 26, 27, 28, 29, 30] and the references therein).

In 2012, He [31] considered *the split equilibrium problem* in Hilbert spaces. Let $F_1 : C \times C \to \mathbb{R}, F_2 : Q \times Q \to \mathbb{R}$ be two bifunctions and $A : H_1 \to H_2$ be a bounded linear operator. The split equilibrium problem is to find $x^* \in C$ such that

$$F_1(x^*, x) \geq 0, \forall x \in C \text{ and } y^* = Ax^* \in Q \text{ such that } F_2(y^*, y) \geq 0, \forall y \in Q. \tag{1.3}$$

The authors also introduced an iterative algorithm to find a solution of the split equilibrium problem. Also, they introduced the following an iterative algorithm to find a solution of (1.3) involing $A^*$ is the adjoint of $A$. The split equilibrium problem and fixed point problems has been studied in Hilbert spaces by many authors; see [32, 33, 34] and the references therein.

In 2017, Guo et al. [35] considered the split equilibrium problem in Banach spaces defined as : let $E_1, E_2$ be two Banach spaces and $C, Q$ be nonempty closed and convex subsets of $E_1$ and $E_2$, respectively. Let $A : E_1 \to E_2$ be a bounded linear operator. Let $F : C \times C \to \mathbb{R}$ and $H : Q \times Q \to \mathbb{R}$ be two bifunctions. Let

$\Omega$ denote the set of solutions of the split equilibrium problem on $F$ and $H$, that is,

$$\Omega = \{z \in C : z \in EP(F), Az \in EP(H)\}.$$

The authors proved a strong convergence theorem as follows:

Let $E_1$ be a uniformly smooth and uniformly convex Banach space and $E_2$ be a uniformly smooth, strictly convex and reflexive Banach space. Let $A : E_1 \to E_2$ be a linear and continuous operator. Let $C$ and $Q$ be nonempty closed and convex subsets of $E_1$ and $E_2$, respectively. Let $S : C \to C$ be a relatively nonexpansive mapping and $F : C \times C \to \mathbb{R}, H : Q \times Q \to \mathbb{R}$ be two bifunctions satisfying the conditions (A1)-(A4) with $\Omega \cap F(S) \neq \emptyset$. Define a sequence $\{x_n\}$ by the following manner:

$$\begin{cases} take\, x_1 = x \in E, find\, v \in E_1\, such\, that\, Av \in Q, \\ V_n = \{x \in E_1 : \|x - v\| \leq n\}, \\ U_n = \{x \in V_n : Ax \in Q\}, \\ F(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jx_n\rangle \geq 0, \forall y \in C, \\ H(Az_n, Ay) + \frac{1}{s_n}\langle y - z_n, Jz_n - Ju_n\rangle \geq 0, \forall y \in U_n, \\ y_n = J^{-1}(\alpha_n Ju_n + (1 - \alpha_n)JS\Pi_C z_n), \\ C_n = \{z \in C : \phi(z, y_n) \leq \phi(z, x_n)\}, \\ D_n = \cap_{i=1}^n C_i \\ x_{n+1} = \Pi_{D_n} x, \end{cases} \tag{1.4}$$

for each $n \geq 1$, where $\{r_n\} \subset [r, \infty)$ with $r > 0$, $\{s_n\} \subset [s, \infty)$ with $s > 0$ and $\alpha_n \subset (0, 1)$. Then the sequence $\{x_n\}$ defined by (1.4) converges strongly to a point $\Pi_{\Omega \cap F(S)} x$, where $\Pi_{\Omega \cap F(S)}$ is the generalized projection of $E_1$ onto $\Omega \cap F(S)$.

The algorithm (1.4) does not involve with the adjoint $A^*$ of the operator A and the norm $\|A\|$, which are quite difficult to compute, but involve only the operator $A$. Furthermore, they also prove a weak convergence theorem for the set of solution of the split equilibrium problem and fixed point problem for a relatively nonexpansive mapping in Banach spaces. Using this idea, Inthakon and Niyamosot [36] also proved strong and weak convergence theorems for the split equilibrium problem and common fixed point problem for two relatively nonexpansive mappings in Banach spaces.

In 2008, Takahashi et al. [37] proved a strong convergence theorem for nonexpansive mapping by using the shrinking projection method as follows: Let $H$ be a Hilbert space and let $C$ be a nonempty closed convex subset of $C$. Let $T$ be a nonexpansive mapping of $C$ into itself such that $F(T) \neq \emptyset$ and let $x_0 \in H$. For $C_1 = C$ and $u_1 = P_{C_1} x_0$, define a sequence $\{u_n\}$ of $C$ as follows:

$$\begin{cases} y_n = \alpha_n u_n + (1 - \alpha_n)Tu_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|u_n - z\|\}, \\ u_{n+1} = P_{C_{n+1}} x_0, n \in \mathbb{N}, \end{cases} \tag{1.5}$$

where $0 \leq \alpha_n \leq a < 1$ for all $n \in \mathbb{N}$. Then, $\{u_n\}$ converges strongly to $z_0 = P_{F(T)} x_0$.

Furthermore, studying strong convergence by the shrinking projection method has been used widely in Banach spaces; see for instance [38, 39] and the references therein.

In this paper, we focus on using the shrinking projection method to prove a strong convergence theorem for finding a common solution of the split equilibrium problem and fixed point problem of a relatively quasi−nonexpansive mapping. Consequently, our main theorem can apply to find a common solution of the split equilibrium problem and common fixed point problem for an infinite family of relatively nonexpansive mappings in Banach spaces.

# 2 Preliminaries

Let $E$ be a Banach space and let $E^*$ denote the dual of E. We denote the value of $x^*$ at $x$ by $\langle x, x^*\rangle$. Then the *duality mapping* $J$ on $E$ defined by

$$J(x) = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\}$$

for every $x \in E$. By the Hahn-Banach theorem, $J(x)$ is nonempty.

Let $S(E)$ be the unit sphere centered at the origin of $E$. A Banach space $E$ is said to be *strictly convex* if $\|(x + y)/2\| < 1$ wherever $x, y \in S(E)$ and $x \neq y$. The modulus $\delta$ of convexity of $E$ is defined by

$$\delta(\epsilon) = \inf\{1 - \frac{\|x + y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \epsilon\}$$

for every $\epsilon$ with $0 \leq \epsilon \leq 2$. A Banach space is said to be *uniformly convex* if $\delta(\epsilon) > 0$ for every $\epsilon > 0$. For $x \in E$ and $f \in E^*$ define $i\,(x)\,(f) = f\,(x)$. We know that $i\,(x) \in E^{**}$ and that the mapping $i : X \to E^{**}$ is an isometric isomorphism, called the canonical embedding of $E$ into $E^{**}$. If $i\,(E) = E^{**}$, then $E$ is said to be *reflexive.* A uniformly convex Banach space is strictly convex and reflexive. Then the space $E$ is said to be *smooth* if the limit

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists for all $x, y \in S(E)$. The norm of $E$ is also said to be *uniformly Gâteaux differentiable* if for all $y \in S(E)$, the limit (2.1) attains uniformly for $x \in S(E)$. The norm of $E$ is said to be *Fréchet differentiable* if for each $x \in S(E)$, the limit (2.1) is attained uniformly for $y \in S(E)$. The norm of $E$ is said to be *uniformly Fréchet differentiable* (and $E$ is said to be *uniformly smooth*) if the limit (2.1) is attained uniformly for $(x, y)$ in $S(E) \times S(E)$. We know that $E$ is smooth if and only if $J$ is a single-valued mapping of $E$ into $E^*$. We also know that $E$ is *reflexive* if and only if $J$ is surjective, and $E$ is strictly convex if and only if $J$ is one-to-one.

Let $E$ be a smooth, strictly convex and reflexive Banach space and $C$ be a nonempty closed and convex subset of $E$. Let $\phi$ be the function on $E \times E$ defined by

$$\phi(x, y) = \|y\|^2 - 2\langle x, Jy\rangle + \|x\|^2,$$

for all $x, y \in E$. From the definition of $\phi$, we have that

$$(\|x\| - \|y\|)^2 \leq \phi(x,y) \leq (\|x\| + \|y\|)^2,$$

for $x, y \in E$. In 1996, Alber [40] defined the *generalized projection* $\Pi_C$ from $E$ onto $C$ as $\Pi_C(x) = \arg\min_{y\in C} \phi(x,y)$, for all $x \in E$. If $E$ is a Hilbert space, then $\phi(x,y) = \|x-y\|^2$ and $\Pi_C$ is the metric projection $P$ of $X^E$ onto $C$.

Let $C$ be a closed and convex subset of $E$ and let $T$ be a mapping from $C$ into itself. A point $p$ in $C$ is said to be an *asymptotic fixed point* of $T$ [41] if $C$ contains a sequence $\{x_n\}$ which converges weakly to $p$ such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. The set of asymptotic fixed points of $T$ is denoted by $\hat{F}(T)$. We say that the mapping $T$ is called *relatively nonexpansive* [42, 43] if the following conditions are satisfied:

(R1) $F(T) \neq \emptyset$,

(R2) $\phi(p, Tx) \leq \phi(p,x)$, for each $x \in C, p \in F(T)$,

(R3) $F(T) = \hat{F}(T)$.

If $T$ satisfies (R1) and (R2), then $T$ is called *relatively quasi-nonexpansive* or *quasi-$\phi$- nonexpansive*. It is obvious that the class of relatively quasi-nonexpansive mappings is more general than the class of relatively nonexpansive mappings.

It is known from [43] that if $E$ be a strictly convex and smooth Banach space, let $C$ be a closed convex subset of $E$, and let $T$ be a relatively nonexpansive mapping from $C$ into itself. Then $F(T)$ is closed and convex. Furthermore, since the condition (R3) is not required in the proof of [43], we can concluded that the fixed point set of relatively quasi-nonexpansive mapping is closed and convex.

In 2008, Kohsaka and Takahashi [44] proved the following result for a countable family of relatively nonexpansive mappings.

**Lemma 2.1** ([44]). *Let $C$ be a nonempty closed and convex subset of a uniformly convex and uniformly smooth Banach space $E$. Let $\{T_i : C \to E\}_{i=1}^{\infty}$ be a sequence of relatively nonexpansive mappings such that $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Suppose that $\{\alpha_i\}_{i=1}^{\infty} \subset (0,1)$ and $\{\beta_i\}_{i=1}^{\infty} \subset (0,1)$ are sequences such that $\sum_{i=1}^{\infty} \alpha_i = 1$ and $\mathcal{U} : C \to E$ is defined by*

$$\mathcal{U}x = J^{-1}\left(\sum_{i=1}^{\infty} \alpha_i \left(\beta_i Jx + (1-\beta_i)JT_i x\right)\right) \text{ for each } x \in C.$$

*Then $\mathcal{U}$ is relatively nonexpansive and $F(\mathcal{U}) = \bigcap_{i=1}^{\infty} F(T_i)$.*

In 2010, Nilsrakoo and Saejung [45] also proved the following result.

**Lemma 2.2** ([45]). *Let $C$ be a nonempty closed and convex subset of a uniformly convex and uniformly smooth Banach space $E$. Let $\{T_i : C \to E\}_{i=1}^{\infty}$ be a sequence of relatively nonexpansive mappings such that $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Suppose that $\{\alpha_i\}_{i=1}^{\infty} \subset (0,1)$ and $\{\beta_i\}_{i=1}^{\infty} \subset (0,1)$ are sequences such that $\sum_{i=1}^{\infty} \alpha_i = 1$ and $\mathcal{S} : C \to E$ is defined by*

$$\mathcal{S}x = J^{-1}\left(\sum_{i=1}^{\infty} \alpha_i J T_i x\right) \text{ for each } x \in C.$$

*Then $\mathcal{S}$ is relatively nonexpansive and $F(\mathcal{S}) = \bigcap_{i=1}^{\infty} F(T_i)$.*

By the way, for solving the equilibrium problem, let us assume that a bifunction $F$ satisfies the following conditions:
(A1) $F(x,x) = 0$ for all $x \in C$;
(A2) $F$ is monotone, i.e. $F(x,y) + F(y,x) \leq 0$ for all $x, y \in C$;
(A3) for all $x, y, z \in C, \limsup_{t \downarrow 0} F(tz + (1-t)x, y) \leq F(x,y)$;
(A4) for all $x \in C, y \mapsto F(x,y)$ is convex and lower semicontinuous.
The following lemma is due to Takahashi and Zembayashi [30].

**Lemma 2.3** ([30]). *Let $C$ be a closed and convex subset of a uniformly smooth, strictly convex and reflexive Banach space $E$, and let $F$ be a bifunction from $C \times C \to \mathbb{R}$ satisfying (A1) - (A4). For $r > 0$ and $x \in E$, define a mapping $T_r^F : E \to C$ by*

$$T_r^F(x) = \{z \in C : F(z,y) + \frac{1}{r}\langle y - z, Jz - Jy\rangle \geq 0, \forall y \in C\},$$

*for all $x \in E$ Then $T_r^F$ is well-defined and the followings hold:*

*(1) $T_r^F$ is single-valued;*
*(2) $T_r^F$ is firmly nonexpansive-type mapping, i.e., for all $x, y \in E$,*

$$\langle T_r^F x - T_r^F y, JT_r^F x - JT_r^F y\rangle \leq \langle T_r^F x - T_r^F y, Jx - Jy\rangle;$$

*(3) $F(T_r^F) = EP(F)$;*
*(4) $EP(F)$ is closed and convex.*

The following results let us know more about the generalized projections.

**Lemma 2.4** ([40, 44]). *Let $C$ be a nonempty closed and convex subset of a smooth, strictly convex and reflexive Banach space $E$. Then*

$$\phi(x, \Pi_C y) + \phi(\Pi_C y, y) \leq \phi(x,y),$$

*for all $x \in C$ and $y \in E$.*

**Lemma 2.5** ([40, 44]). *Let $C$ be a nonempty closed and convex subset of a smooth, strictly convex and reflexive Banach space $E$. Then, for any $x \in E$ and $z \in C$ we have*

$$z = \Pi_C x \Leftrightarrow \langle y - z, Jx - Jz \rangle \leq 0,$$

*for all $y \in C$.*

The following results also play the important role in our main theorems.

**Lemma 2.6** ([44]). *Let $E$ be a smooth and uniformly convex Banach space. Suppose that $\{x_n\}$ and $\{y_n\}$ are the sequences in $E$ such that either $\{x_n\}$ or $\{y_n\}$ is bounded. If $\lim_{n\to\infty} \phi(x_n, y_n) = 0$, then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

**Lemma 2.7** ([30]). *Let $C$ be a closed convex subset of a smooth, strictly convex and reflexive Banach space $E$, $F$ be a bifunction from $C \times C \to \mathbb{R}$ satisfying the conditions (A1)-(A4) and let $r > 0$. Then, for any $x \in E$ and $q \in F(T_r^F)$,*

$$\phi(q, T_r^F x) + \phi(T_r^F x, x) \leq \phi(q, x).$$

# 3 Main Results

We use the shrinking projection method to prove strong convergence theorem as follows.

**Theorem 3.1.** *Let $E_1$ be a uniformly smooth and uniformly convex Banach space and $E_2$ be a uniformly smooth, strictly convex and reflexive Banach space. Let $A : E_1 \to E_2$ be a linear and continuous operator. Let $C$ and $Q$ be nonempty closed and convex subsets of $E_1$ and $E_2$, respectively. Assume that $S : C \to C$ be a relatively quasi-nonexpansive mapping and $F : C \times C \to \mathbb{R}, H : Q \times Q \to \mathbb{R}$ be two bifunctions satisfying the conditions (A1)-(A4) with $\Omega \cap F(S) \neq \emptyset$. Let $C_1 = C$ and define a sequence $\{x_n\}$ by the following manner:*

$$\begin{cases} take\, x_1 = x \in E, find\, v \in E_1\, such\, that\, Av \in Q, \\ V_n = \{x \in E_1 : \|x - v\| \leq n\}, \\ U_n = \{x \in V_n : Ax \in Q\}, \\ F(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jx_n \rangle \geq 0, \forall y \in C, \\ H(Az_n, Ay) + \frac{1}{s_n}\langle y - z_n, Jz_n - Ju_n \rangle \geq 0, \forall y \in U_n, \\ y_n = J^{-1}(\alpha_n Ju_n + (1 - \alpha_n) JS\Pi_C z_n), \\ C_{n+1} = \{z \in C_n : \phi(z, y_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x, \end{cases} \tag{3.1}$$

*for each $n \geq 1$, where $\{r_n\} \subset [r, \infty)$ with $r > 0$, $\{s_n\} \subset [s, \infty)$ with $s > 0$ and $\alpha_n \subset (0, 1)$. Then the sequence $\{x_n\}$ defined by (3.1) converges strongly to a point $\Pi_{\Omega \cap F(S)} x$, where $\Pi_{\Omega \cap F(S)}$ is the generalized projection of $E_1$ onto $\Omega \cap F(S)$.*

*Proof.* For each $n \geq 1$, we can see that $v$ is contained in $V_n$ and $U_n$. Therefore $V_n$ and $U_n$ are nonempty. By the definiton of $V_n$, we have $V_n$ is closed. Since $A$ is linear and continuous, $V_n$ is convex and $U_n$ is closed and convex. It is obvious that $C_1 = C$ is closed and convex. Suppose that $C_k$ is closed and convex for some $k \in \mathbb{N}$. For $z \in C_k$, we see that

$$\phi(z, y_k) \leq \phi(z, x_k) \Leftrightarrow \|y_k\|^2 - \|x_k\|^2 - 2\langle z, Jy_k - Jx_k\rangle \leq 0.$$

This implies that $C_{k+1}$ is closed and convex, and hence $C_n$ is closed and convex for each $n \geq 1$. Next, we show that $x_n$ is well defined. Let $G(x, y) = H(Ax, Ay)$ for all $x, y \in U_n$. Since $A$ is linear and continuous, then $G$ is a bifunction from $U_n \times U_n$ into $\mathbb{R}$ satisfying (A1)-(A4). Moreover, for each $n \geq 1$, we can rewrite

$$H(Az_n, Ay) + \tfrac{1}{s_n}\langle y - z_n, Jz_n - Ju_n\rangle \geq 0,$$

as

$$G(x, y) + \frac{1}{s_n}\langle y - z_n, Jz_n - Ju_n\rangle \geq 0, \quad \text{for all} \quad y \in U_n.$$

Let $p \in \Omega \cap F(S)$ so we have $p \in EP(F)$ and $Ap \in EP(H)$. By Lemma 2.3, we have $p \in F(T^F_{r_n})$ and hence $p = T^F_{r_n}p$. Since $Ap \in EP(H), H(Ap, z) \geq 0$ for all $z \in Q$. Since $Az \in Q$ for all $z \in U_n, H(Ap, Az) \geq 0$ for all $z \in U_n$. It follows that $G(p, z) \geq 0$ for all $z \in U_n$ which implies that $p \in EP(G)$. By Lemma 2.3, we have $p = T^G_{s_n}p$ and hence $p \in C$. Let $u_n = T^F_{r_n}x_n$ and $z_n = T^G_{s_n}u_n$. By Lemma 2.7 and $p \in F(T^F_{r_n})$, we have

$$\begin{aligned} \phi(p, T^F_{r_n}x_n) + \phi(T^F_{r_n}x_n, x_n) &\leq \phi(p, x_n) \\ \phi(p, u_n) + \phi(u_n, x_n) &\leq \phi(p, x_n). \end{aligned}$$

Thus,

$$\phi(p, u_n) \leq \phi(p, x_n) - \phi(u_n, x_n),$$

and hence

$$\phi(p, u_n) \leq \phi(p, x_n). \tag{3.2}$$

Since $p \in F(S)$ and $S$ is relatively quasi-nonexpansive mapping, we have

$$\phi(p, S\Pi_C z_n) \leq \phi(p, \Pi_C z_n).$$

Furthermore, we have from Lemma 2.4 that

$$\phi(p, \Pi_C z_n) \leq \phi(p, u_n).$$

On the other hand, since $p \in EP(F)$, we can apply Lemma 2.3 and Lemma 2.7 to get that

$$\phi(p, z_n) \leq \phi(p, u_n).$$

It follows from (3.2) that

$$\phi(p, S\Pi_C z_n) \leq \phi(p, x_n). \tag{3.3}$$

Thus,

$$\begin{aligned}
\phi(p, y_n) &= \phi(p, J^{-1}(\alpha_n J u_n + (1-\alpha_n) J S \Pi_C z_n)) \\
&= \|p\|^2 - 2\langle p, \alpha_n J u_n + (1-\alpha_n) J S \Pi_C z_n)\rangle \\
&\quad + \|\alpha_n u_n + (1-\alpha_n) J S \Pi_C z_n)\|^2 \\
&\leq \|p\|^2 - 2\langle p, \alpha_n J u_n\rangle - 2\langle p, (1-\alpha_n) J S \Pi_C z_n)\rangle + \alpha_n \|u_n\|^2 \\
&\quad + (1-\alpha_n)\|J S \Pi_C z_n)\|^2 \\
&= \alpha_n \phi(p, u_n) + (1-\alpha_n)\phi(p, S\Pi_C z_n)) \\
&\leq \alpha_n \phi(p, x_n) + (1-\alpha_n)\phi(p, x_n) \\
&= \phi(p, x_n).
\end{aligned}$$

Therefore, $p \in C_n$ for each $n \geq 1$ and hence $C_n$ is nonempty. It follows that $\Omega \cap F(S) \subset C_n$ for each $n \geq 1$ which implies that $\{x_n\}$ is well-defined. For each $n \geq 1$, we have from Lemma 2.4 that

$$\begin{aligned}
\phi(x_{n+1}, x) &= \phi(\Pi_{C_{n+1}} x, x) \\
&\leq \phi(z, x) - \phi(z, \Pi_{C_{n+1}} x) \\
&\leq \phi(z, x), \quad \forall z \in C_{n+1}.
\end{aligned}$$

Since $\Omega$ and $F(S)$ are nonempty closed and convex, $\Omega \cap F(S)$ is closed and convex. Let $x^* = \Pi_{\Omega \cap F(S)} x$, one has $x^* \in \Omega \cap F(S) \subset C_{n+1}$ and

$$\phi(x_{n+1}, x) \quad \leq \quad \phi(x^*, x).$$

Therefore $\{\phi(x_n, x)\}$ is bounded which implies that $\{x_n\}$ is bounded. It follows that $\{u_n\}$ and $\{z_n\}$ are also bounded. Since $x_{n+2} = \Pi_{C_{n+2}} x \in C_{n+2} \subset C_{n+1}$, we have

$$\phi(x_{n+1}, x) \leq \phi(x_{n+2}, x).$$

Thus, we can conclude that the limit of $\{\phi(x_n, x)\}$ exists.
For each $m \geq 1$, since $x_{n+m} \in C_{n+m} \subset C_{n+m-1}$ and Lemma 2.4, we have

$$\begin{aligned}
\phi(x_{n+m}, x_n) &= \phi(x_{n+m}, \Pi_{C_n} x) \\
&\leq \phi(x_{n+m}, x) - \phi(\Pi_{C_n} x, x) \\
&= \phi(x_{n+m}, x) - \phi(x_n, x).
\end{aligned}$$

From the existence of $\lim\limits_{n\to\infty} \phi(x_n, x)$, we have

$$\lim_{n\to\infty} \phi(x_{n+m}, x_n) = 0, \quad \text{for each } \ m \geq 1. \tag{3.4}$$

It follows from Lemma 2.6 that

$$\lim_{n\to\infty} \|x_n - x_{n+m}\| = 0, \quad \text{for each } \ m \geq 1. \tag{3.5}$$

Thus, the sequence $\{x_n\}$ is Cauchy. Therefore, there exists $q \in C$ such that $x_n \to q$ as $n \to \infty$. Finally, we show that $q = \Pi_{\Omega \cap F(S)} x$. Indeed, we have from $x_{n+1} = \Pi_{C_{n+1}} x$, $\Omega \cap F(S) \subset C_{n+1}$ and Lemma 2.5 that

$$\langle y - x_{n+1}, Jx - Jx_{n+1}\rangle \leq 0, \quad \text{for all } \ y \in \Omega \cap F(S). \tag{3.6}$$

By letting $n \to \infty$ in (3.6) and noting that $x_n \to q$, we have

$$\langle y - q, Jx - Jq \rangle \leq 0, \quad \text{for all } \ y \in \Omega \cap F(S).$$

Therefore, we can conclude from Lemma 2.5 that

$$q = \Pi_{\Omega \cap F(S)} x$$

and the proof is complete. □

Since every relatively nonexpansive mapping is relatively quasi− nonexpansive mapping, Theorem 3.1 is also true when $S$ is a relavetively nonexpansive mapping and hence we can apply Lemma 2.1 and Theorem 3.1 to get a strong convergence theorem for finding an element in $\Omega \cap \bigcap_{i=1}^{\infty} F(T_i)$ , where $T_i$ are relatively nonexpansive mappings in Banach spaces as follows.

**Theorem 3.2.** *Let $E_1$ be a uniformly smooth and uniformly convex Banach space and $E_2$ be a uniformly smooth, strictly convex and reflexive Banach space. Let $A : E_1 \to E_2$ be a linear and continuous operator. Let $C$ and $Q$ be nonempty closed and convex subsets of $E_1$ and $E_2$, respectively. Assume that $\{T_i : C \to C\}_{i=1}^{\infty}$ be a sequence of relatively nonexpansive mappings and $F : C \times C \to \mathbb{R}, H : Q \times Q \to \mathbb{R}$ be two bifunctions satisfying the conditions (A1)-(A4) with $\Omega \cap \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Define $S : C \to C$ by $Sx = J^{-1}\left(\sum_{i=1}^{\infty} \alpha_i \left(\beta_i Jx + (1 - \beta_i) JT_i x\right)\right)$ for each $x \in C$, where $\{\alpha_i\}_{i=1}^{\infty} \subset (0, 1)$ and $\{\beta_i\}_{i=1}^{\infty} \subset (0, 1)$ are sequences such that $\sum_{i=1}^{\infty} \alpha_i = 1$. Let $C_1 = C$ and define a sequence $\{x_n\}$ by the following manner:*

$$\begin{cases} take\, x_1 = x \in E, find\, v \in E_1 \, such\, that\, Av \in Q, \\ V_n = \{x \in E_1 : \|x - v\| \leq n\}, \\ U_n = \{x \in V_n : Ax \in Q\}, \\ F(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jx_n \rangle \geq 0, \forall y \in C, \\ H(Az_n, Ay) + \frac{1}{s_n}\langle y - z_n, Jz_n - Ju_n \rangle \geq 0, \forall y \in U_n, \\ y_n = J^{-1}(\alpha_n Ju_n + (1 - \alpha_n) JS\Pi_C z_n) \\ C_{n+1} = \{z \in C_n : \phi(z, y_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x, \end{cases} \tag{3.7}$$

*for each $n \geq 1$, where $\{r_n\} \subset [r, \infty)$ with $r > 0$, $\{s_n\} \subset [s, \infty)$ with $s > 0$. Then the sequence $\{x_n\}$ defined by (3.7) converges strongly to a point $\Pi_{\Omega \cap F(S)} x$, where $\Pi_{\Omega \cap F(S)}$ is the generalized projection of $E_1$ onto $\Omega \cap F(S)$ and $F(S) = \bigcap_{i=1}^{\infty} F(T_i)$.*

Furthermore, Lemma 2.2 and Theorem 3.1 also allow us to get the following result.

**Theorem 3.3.** *Let $E_1$ be a uniformly smooth and uniformly convex Banach space and $E_2$ be a uniformly smooth, strictly convex and reflexive Banach space. Let $A : E_1 \to E_2$ be a linear and continuous operator. Let $C$ and $Q$ be nonempty closed and convex subsets of $E_1$ and $E_2$, respectively. Assume that $\{T_i : C \to C\}_{i=1}^{\infty}$ be a sequence of relatively nonexpansive mappings and $F : C\times C \to \mathbb{R}$, $H : Q\times Q \to \mathbb{R}$ be two bifunctions satisfying the conditions (A1)-(A4) with $\Omega \cap \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Define $\mathcal{S} : C \to C$ by $\mathcal{S}x = J^{-1}\left(\sum_{i=1}^{\infty} \alpha_i JT_i x\right)$ for each $x \in C$, where $\{\alpha_i\}_{i=1}^{\infty} \subset (0, 1)$ is a sequence such that $\sum_{i=1}^{\infty} \alpha_i = 1$. Let $C_1 = C$ and define a sequence $\{x_n\}$ by the following manner:*

$$\begin{cases} take\, x_1 = x \in E, find\, v \in E_1\, such\, that\, Av \in Q, \\ V_n = \{x \in E_1 : \|x - v\| \leq n\}, \\ U_n = \{x \in V_n : Ax \in Q\}, \\ F(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jx_n\rangle \geq 0, \forall y \in C, \\ H(Az_n, Ay) + \frac{1}{s_n}\langle y - z_n, Jz_n - Ju_n\rangle \geq 0, \forall y \in U_n, \\ y_n = J^{-1}(\alpha_n Ju_n + (1 - \alpha_n)JS\Pi_C z_n) \\ C_{n+1} = \{z \in C_n : \phi(z, y_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x, \end{cases} \tag{3.8}$$

*for each $n \geq 1$, where $\{r_n\} \subset [r, \infty)$ with $r > 0$, $\{s_n\} \subset [s, \infty)$ with $s > 0$. Then the sequence $\{x_n\}$ defined by (3.8) converges strongly to a point $\Pi_{\Omega\cap F(S)}x$, where $\Pi_{\Omega\cap F(S)}$ is the generalized projection of $E_1$ onto $\Omega \cap F(S)$ and $F(S) = \bigcap_{i=1}^{\infty} F(T_i)$.*

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