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Limit Distribution Functions for Sums of the Reciprocals of a Power of Tangent of Random Variables

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Abstract : Let $r \in [\frac{1}{2}, \infty)$ and (X_n) be a sequence of independent continuous random variables such that $\mathrm{Im}(X_n) \subseteq \mathbb{R} - \left\{\frac{j\pi}{2} | j \in \mathbb{Z}\right\}$. This paper provides the sufficient conditions guaranteeing the existence of real constants $(A_n), (A_n(r))$ and $(B_n(r))$ such that the sequences of the distribution functions of $\frac{1}{n}\sum_{k=1}^{n} \frac{1}{\tan X_k} - A_n$ and $\frac{1}{B_n(r)}\sum_{k=1}^{n} \frac{1}{|\tan X_k|^r} - A_n(r)$ converge.

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1 Introduction

The Central Limit Theorem (CLT) is one of well-known theorem in probability theory and this equipment is always used for many applications in mathematical

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statistics, applied mathematics, biostatistics and statistical physics. The theorem may be said simply as follows [1]:

"There is no need to know very much about the actual distribution of the variables, as long as there are enough instances of them their sum can be treated as normally distributed."

Since 1770, it sparked starting during Laplace tried to find the way for solving the problem about meteor inclination angles. But the deviations between the mean of the data and the theoretical value, there still are problems. He cannot get an accurate result without approximation. The process of finding an approximation induced Laplace to form this theorem and the characteristic functions was key to prove. The classical version of CLT [2] states that:

Let $(X_n), n = 1, 2, 3, ...$ be a sequence of independent, identically distributed random variables. Suppose that $S_n = \frac{1}{n}\sum_{k=1}^{n} X_k$ and $Z_n = \frac{S_n - E(S_n)}{\sqrt{Var(S_n)}}$. Then the distribution of Z_n tends to the standard normal distribution as $n \to \infty$.

In 1988, Shapiro [3] considered the other forms of S_n that is sums of the reciprocals of random variables as follows:

$$S_n = \frac{1}{n}\sum_{k=1}^{n}\frac{1}{X_k} \quad \text{and} \quad S_{n,r} = \frac{1}{n^r}\sum_{k=1}^{n}\frac{1}{|X_k|^r} - A_n(r)$$

where $A_n(r)$ are real constants. For $r > \frac{1}{2}$, Shapiro showed that the distribution functions of S_n and $S_{n,r}$ converge to a Cauchy distribution function and a stable law with exponent less than two, respectively. Termwuttipong [4] fulfilled the other case, $0 < r \leq \frac{1}{2}$, and showed that the limit distribution function is a normal distribution function.

Twelve years later, Neammanee [5] considered the convergence of the distribution functions of

$$\frac{1}{B_n}\sum_{k=1}^{n}\frac{1}{\ln X_k} - A_n \quad \text{and} \quad \frac{1}{B_n(r)}\sum_{k=1}^{n}\frac{1}{|\ln X_k|^r} - A_n(r)$$

for $r > 0$.

In 2002, Neammanee [6] extended his previous work with a continuous function g from subset A of $\mathbb{R}$ into $\mathbb{R}$ which satisfied the following conditions:

1. there exists an $a \in A$ such that $g(a) = 0$,
2. g is strictly monotone on $A \cap (-\infty, a]$ and $A \cap [a, \infty)$,
3. g' exists and continuous on $(a - \delta^*, a) \cup (a, a + \delta^*)$ for some $\delta^* > 0$ and $g'(a)$ exists and positive.

He considered the convergence of the distribution functions of

$$\frac{1}{n}\sum_{k=1}^{n}\frac{1}{g(X_k)}-A_n \quad \text{and} \quad \frac{1}{n^r}\sum_{k=1}^{n}\frac{1}{|g(X_k)|^r}-A_n(r)$$

for $r>0$. The above function g can automatically generalized the results of Shapiro [3], Termwuttipong [4] and Neammanee [5].

Siricheon [7] and Neammanee [8] changed the characteristic of g to a periodic function that its graph cuts X-axis at infinitely points that is a sine function:

$$\frac{1}{n}\sum_{k=1}^{n}\frac{1}{\sin(X_k)}-A_n \quad \text{and} \quad \frac{1}{n^r}\sum_{k=1}^{n}\frac{1}{|\sin(X_k)|^r}-A_n(r)$$

for $r>\frac{1}{2}$.

It is suspicious that the infinitely many numbers of x-intercepts does not affect the weak convergence and we may see that functions in those articles ([4–6]) are unbounded. Then we consider another periodic function, a tangent function that merges together with infinitely many numbers of x-intercepts as the same behavior of the sine function and unbounded property like logarithm function.

2 Main Theorems

Let (X_n) be a sequence of independent continuous random variables such that $\mathrm{Im}(X_n)\subseteq\mathbb{R}-\left\{\frac{j\pi}{2}\mid j\in\mathbb{Z}\right\}$ for every n and let f_n and F_n be the probability density function and the distribution function of X_n, respectively. Define $X_{nk}=\frac{1}{n\tan X_k}$ and $X_{nk}^r=\frac{1}{n^r|\tan X_k|^r}$ for $r>\frac{1}{2}$, these are our main results.

Theorem 2.1. *Assume that*

(*i*) *there exists* $p>1$ *such that* $\{(|j^p|+1)f_k \ : k\in\mathbb{N}, j\in\mathbb{Z}\}$ *is uniformly equicontinuous, i.e. for any* $\epsilon>0$ *there exists* $\delta>0$ *such that for real numbers* x *and* y *if* $|x-y|<\delta$, *then* $|(|j^p|+1)f_k(x)-(|j^p|+1)f_k(y)|<\epsilon$ *for every integer* j *and positive integer* k,

(*ii*) $\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\sum_{j\in\mathbb{Z}}f_k(j\pi)=L$ *for some* $L\geq 0$.

Then

1. *there exists a sequence of real numbers* (A_n) *such that the distribution function of* $\sum_{k=1}^{n}X_{nk}-A_n$ *converges, as* $n\to\infty$, *to the distribution function* F *where* $F=1_{[0,\infty)}$ *if* $L=0$ *and* $F\sim Cau(L\pi)$ *if* $L>0$,

2. *there exists a sequence of real numbers* $(A_n(r))$ *such that the distribution function of* $\sum_{k=1}^{n}X_{nk}^r-A_n(r)$ *converges, as* $n\to\infty$, *to the distribution function* F^r *if* $L=0$ *then* $F^r=1_{[0,\infty)}$, *otherwise* F^r *is a stable distribution function with characteristics exponent* $\frac{1}{r}$.

Theorem 2.2. *Under the condition (i), (ii) in Theorem 2.1 and suppose that there exist a real number M and $n_0 \in \mathbb{N}$ such that $\sum_{j\in\Lambda} f_k(j\pi) \leq M(k \ln k)$ for $k \geq n_0$, then there exist sequences of real constants (A_n) and (B_n) such that the distribution function of the sums $\frac{1}{B_n}\sum_{k=1}^{n} \frac{1}{\sqrt{|\tan(X_k)|}} - A_n$ converge weakly to Φ, where Φ is the standard normal distribution function and the finite set $\Lambda = \{j \in \mathbb{Z} | j\pi \in (a,b)\}$ and $a\pi <$ $Im(X_n) < b\pi$ and $a, b \in \mathbb{R}$.*

3 Proof of Main Theorems

First, we begin this section by introducing Theorem A and Theorem B as the important tools that using to prove Theorem 2.1 and Theorem 2.2, respectively. Next, the lemmas that relate through our discussion are provided and the proof of Theorem 2.1 and Theorem 2.2 will be the last part of each subsections.

Theorem A ([9, p. 116]) *In order that for some suitably chosen constants A_n the distribution functions of sums $X_{n1} + X_{n2} + \cdots + X_{nk_n} - A_n$ of independent infinitesimal random variables, i.e. for any $\epsilon > 0$, $\lim_{n\to\infty} \sup_{1\leq k\leq k_n} P(|X_{nk}| \geq \epsilon) = 0$, converge to a limit if there exist non-decreasing functions M and N, defined on the intervals $(-\infty, 0)$ and $(0, +\infty)$, respectively with $M(-\infty) = 0$ and $N(+\infty) = 0$ and a constant $\sigma \geq 0$ such that*

A1. $\lim_{n\to+\infty} \sum_{k=1}^{k_n} F_{nk}(x) = M(x)$, *for a continuity point x of M,*

A2. $\lim_{n\to+\infty} \sum_{k=1}^{k_n} [F_{nk}(x) - 1] = N(x)$, *for a continuity point x of N,*

A3.
$$\lim_{\epsilon\to 0^+} \liminf_{n\to\infty} \sum_{k=1}^{k_n} \left\{ \int_{|x|<\epsilon} x^2\, dF_{nk}(x) - \left(\int_{|x|<\epsilon} x\, dF_{nk}(x) \right)^2 \right\}$$
$$= \lim_{\epsilon\to 0^+} \limsup_{n\to\infty} \sum_{k=1}^{k_n} \left\{ \int_{|x|<\epsilon} x^2\, dF_{nk}(x) - \left(\int_{|x|<\epsilon} x\, dF_{nk}(x) \right)^2 \right\}$$
$$= \sigma^2,$$
where F_{nk} denotes the distribution function of X_{nk}.

A constant A_n may be chosen according to

$$A_n = \sum_{k=1}^{k_n} \int_{|x|<\tau} x\, dF_{nk}(x) - \gamma(\tau)$$

where $-\tau$ and τ are continuity points of M and N, respectively. Note that the *formula of*

$$\gamma(\tau) = \gamma + \int_{|u|<\tau} udG(u) - \int_{|u|\geq\tau} \frac{1}{u} dG(u)$$

where the details of the constant γ and the function G are in [9, p.76-77].

Theorem B ([10, p. 97]) *Let (X_n) be a sequence of independent random variables. Then there exist sequences of real constants (A_n) and (B_n) such that $B_n > 0$, the distribution functions of the sums $\frac{1}{B_n}(X_1 + X_2 + \cdots + X_n) - A_n$ converge weakly to Φ and a sequence (X_{nk}) ; $k = 1, 2, ..., n,\ n = 1, 2, ...$ is infinitesimal where $X_{nk} = \frac{X_k}{B_n}$ if and only if there exists a sequence of real constants (c_n) such that $\lim\limits_{n\to\infty} c_n = \infty$ and*

B1. $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \int_{|x|\geq c_n} dF_k(x) = 0,$

B2. $\lim\limits_{n\to\infty} \frac{1}{c_n^2} \sum\limits_{k=1}^{n} \left\{ \int_{|x|<c_n} x^2\, dF_k(x) - \left(\int_{|x|<c_n} x\, dF_k(x) \right)^2 \right\} = \infty$

where F_k denotes the distribution function of X_k. The constants A_n and B_n can be chosen by

$$B_n^2 = \sum_{k=1}^{n} \left\{ \int_{|x|>c_n} x^2\, dF_k(x) - \left(\int_{|x|>c_n} x\, dF_k(x) \right)^2 \right\}$$

and

$$A_n = \frac{1}{B_n} \sum_{k=1}^{n} \int_{|x|>c_n} x\, dF_k(x).$$

Next, we give auxiliary results for proving the theorem (Theorem 2.1).

Lemma 3.1. *For any $x \in \mathbb{R}$ and $r > \frac{1}{2}$, the distributions of X_{nk} and X_{nk}^r are represented as follows:*

1. $F_{nk}(x) = \begin{cases} \sum_{j\in\mathbb{Z}} \left[F_k(j\pi) - F_k\left(j\pi - \frac{\pi}{2}\right)\right] & \text{if } x = 0 \\ \sum_{j\in\mathbb{Z}} \left[F_k(j\pi) - F_k\left(j\pi + \tan^{-1}\frac{1}{nx}\right)\right] & \text{if } x < 0 \\ \sum_{j\in\mathbb{Z}} \left[F_k(j\pi) - F_k\left(j\pi + \tan^{-1}\frac{1}{nx}\right)\right] + 1 & \text{if } x > 0, \end{cases}$

2. $F^r_{nk}(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 + \sum_{j\in\mathbb{Z}} \left[F_k\left(j\pi - \tan^{-1}\frac{1}{nx^{\frac{1}{r}}}\right) - F_k\left(j\pi + \tan^{-1}\frac{1}{nx^{\frac{1}{r}}}\right)\right] & \text{if } x > 0\ . \end{cases}$

Proof. Note that, $F_{nk}(0) = P(\tan X_k < 0) = \sum_{j\in\mathbb{Z}} P\left(j\pi - \frac{\pi}{2} < X_k < j\pi\right)$ $= \sum_{j\in\mathbb{Z}} \left[F_k(j\pi) - F_k\left(j\pi - \frac{\pi}{2}\right)\right]$. If $x < 0$, we have $F_{nk}(x) = P\left(\frac{1}{nx} \leq \tan X_k < 0\right)$ $= \sum_{j\in\mathbb{Z}} P\left(j\pi + \tan^{-1}\frac{1}{nx} \leq X_k < j\pi\right) = \sum_{j\in\mathbb{Z}} \left[F_k(j\pi) - F_k\left(j\pi + \tan^{-1}\frac{1}{nx}\right)\right]$, and if $x > 0$, $F_{nk}(x) = F_{nk}(0) + (F_{nk}(x) - F_{nk}(0)) = F_{nk}(0) + P(0 < X_{nk} \leq x) =$ $F_{nk}(0) + P\left(\tan X_k \geq \frac{1}{nx}\right) = F_{nk}(0) + 1 - P\left(\tan X_k < \frac{1}{nx}\right) = F_{nk}(0) + 1 -$ $P\left(\bigcup_{j\in\mathbb{Z}}\{j\pi - \frac{\pi}{2} < X_k < j\pi + \tan^{-1}(\frac{1}{nx})\}\right) = \sum_{j\in\mathbb{Z}} \left[F_k(j\pi) - F_k\left(j\pi - \frac{\pi}{2}\right)\right] + (1+$ $\sum_{j\in\mathbb{Z}} \left[F_k\left(j\pi - \frac{\pi}{2}\right) - F_k\left(j\pi + \tan^{-1}\frac{1}{nx}\right)\right]) = 1 + \sum_{j\in\mathbb{Z}} \left[F_k(j\pi) - F_k\left(j\pi + \tan^{-1}\frac{1}{nx}\right)\right]$.
By the same arguments, the closed form of F^r_{nk} follows directly. □

Lemma 3.2. *For any $x \in \mathbb{R}$ and $k \in \mathbb{N}$, the distribution function of $Y_k = \frac{1}{\sqrt{|\tan(X_k)|}}$ is represented as follows:*

$$G_k(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 + \sum_{j\in\Lambda}[F_k(j\pi - \arctan(\frac{1}{x^2})) - F_k(j\pi + \arctan(\frac{1}{x^2}))] & \text{if } x > 0 \end{cases}$$

where Λ defined as in Theorem 2.2.

Proof. We can use the same manner as Lemma 3.1 to prove this lemma. □

Lemma 3.3. *Under the assumption (i) and (ii) in Theorem 2.1 the double sequences (X_{nk}) and (X_{nk}^r), $k = 1, 2, ..., n; n = 1, 2, ...$ have the infinitesimal property for any positive real number r.*

Proof. By the assumption (ii), we have $\lim_{n\to\infty} \frac{1}{n} \sum_{j\in\mathbb{Z}} f_n(j\pi) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \sum_{j\in\mathbb{Z}} f_k(j\pi) - \lim_{n\to\infty} \left(\frac{n-1}{n}\right)\left(\frac{1}{n-1}\right) \sum_{k=1}^{n-1} \sum_{j\in\mathbb{Z}} f_k(j\pi) = L - L = 0$. So, for any $\epsilon > 0$, there exists $n_1 \in \mathbb{N}$ such that $\frac{1}{k} \sum_{j\in\mathbb{Z}} f_k(j\pi) < \epsilon$ for all $k > n_1$ and $\lim_{n\to\infty} \frac{1}{n} \sum_{j\in\mathbb{Z}} f_{n-l}(j\pi) = (\lim_{n\to\infty} \frac{n-l}{n}) (\lim_{n\to\infty} \frac{1}{n-l} \sum_{j\in\mathbb{Z}} f_{n-l}(j\pi)) = 0$ for all $l \in \mathbb{N}$. Since $\{\frac{1}{n} \sum_{j\in\mathbb{Z}} f_{n-l}(j\pi) \mid l = 1, 2, ..., n-1\} = \{\frac{1}{n} \sum_{j\in\mathbb{Z}} f_k(j\pi) \mid k = 1, 2, ..., n-1\}$ for any $n \in \mathbb{N}$, we have $\lim_{n\to\infty} \frac{1}{n} \sum_{j\in\mathbb{Z}} f_k(j\pi) = 0$ for all $k \in \mathbb{N}$. Thus, for each $k = 1, 2, ..., n_1$, there exists a natural number n_2 such that $n_2 > n_1$ and $\frac{1}{n} \sum_{j\in\mathbb{Z}} f_k(j\pi) < \epsilon$ for $n > n_2$. This leads to $\max_{1\le k\le n} \frac{1}{n} \sum_{j\in\mathbb{Z}} f_k(j\pi) < \epsilon$, for all $n > n_2$, i.e., $\lim_{n\to\infty} \max_{1\le k\le n} \frac{1}{n} \sum_{j\in\mathbb{Z}} f_k(j\pi) = 0$. Now we are ready to show that the double sequence (X_{nk}) has the infinitesimal property. Let $\epsilon > 0$, $n \in \mathbb{N}$ and $k \in \{1, 2, ..., n\}$ be given. By Lemma 3.1, we see that $P(|X_{nk}| \ge \epsilon) = \sum_{j\in\mathbb{Z}} [F_k(j\pi + \tan^{-1} \frac{1}{n\epsilon}) - F_k(j\pi - \tan^{-1} \frac{1}{n\epsilon})]$. We note from the assumpion (i) that, for $k \in \mathbb{N}$ and $j \in \mathbb{Z}$, there exists $\delta > 0$ such that for any $x \in \mathbb{R}$ if $|x - j\pi| < \delta$ then $|(|j^p| + 1)f_k(x) - (|j^p| + 1)f_k(j\pi)| < 1$. By the result of $\lim_{n\to\infty} \tan^{-1} \frac{1}{n\epsilon} = 0$, it implies that there exists $n_0 \in \mathbb{N}$, $\tan^{-1} \frac{1}{n\epsilon} < \delta$ for $n > n_0$. Apply the Mean Value Theorem with the function $F_k|_{[j\pi - \tan^{-1} \frac{1}{n\epsilon}\ ,\ j\pi + \tan^{-1} \frac{1}{n\epsilon}]}$, we have $F_k\left(j\pi + \tan^{-1} \frac{1}{n\epsilon}\right) - F_k\left(j\pi - \tan^{-1} \frac{1}{n\epsilon}\right) = (2\tan^{-1} \frac{1}{n\epsilon}) f_k(c_{nk}^j)$ where $c_{nk}^j - j\pi \in \left(-\tan^{-1} \frac{1}{n\epsilon}\ ,\ \tan^{-1} \frac{1}{n\epsilon}\right)$. The infinitesimal property of (X_{nk}) follows directly from the fact that $\lim_{n\to\infty} \tan^{-1} \frac{1}{n\epsilon} = 0$, the series of $\sum_{j\in\mathbb{Z}} \frac{1}{|j^p| + 1}$ converges and $0 \le \max_{1\le k\le n} P(|\ X_{nk}\ | \ge \epsilon) \le (2\tan^{-1} \frac{1}{n\epsilon})[\sum_{j\in\mathbb{Z}} \frac{1}{|j^p| + 1} + \max_{1\le k\le n} \sum_{j\in\mathbb{Z}} f_k(j\pi)]$. Next, for any $r > \frac{1}{2}$, we have $\max_{1\le k\le n} P(|\ X_{nk}^r\ | \ge \epsilon) = \max_{1\le k\le n} P(|\ X_{nk}\ | \ge \epsilon^{\frac{1}{r}})$. Hence a double sequence $(X_{nk}^r), n = 1, 2, ...; k = 1, 2, 3, ..., n$ is infinitesimal because (X_{nk}) is infinitesimal. □

Lemma 3.4. *Assume (i) and (ii) in Theorem 2.1 hold. For each $r > \frac{1}{2}$, let M and $M_r : (-\infty, 0) \to \mathbb{R}$ be defined by $M(x) = -\frac{L}{x}$ and $M_r(x) = 0$. For any $x < 0$,*

1. $\lim_{n\to\infty} \sum_{k=1}^{n} F_{nk}(x) = -\dfrac{L}{x}$ *and*
2. $\lim_{n\to\infty} \sum_{k=1}^{n} F^r_{nk}(x) = 0.$

Proof. 1. Let $x < 0$ be given. Since $\lim_{n\to\infty}(n \tan^{-1}\dfrac{1}{nx}) = \dfrac{1}{x}$ and assumption (ii), we have $\lim_{n\to\infty}(-n\ \tan^{-1}\dfrac{1}{nx})\left|\dfrac{1}{n}\sum_{k=1}^{n}\sum_{j\in\mathbb{Z}} f_k(j\pi) - L\right| = \lim_{n\to\infty}\left|(-n\ \tan^{-1}\dfrac{1}{nx})L + \dfrac{L}{x}\right| = 0$. Let $\epsilon > 0$, $j \in \mathbb{Z}$ and $n \in \mathbb{N}$. By the same argument of Lemma 3.3, we apply the Mean Value Theorem with the function $F_k|_{\left[j\pi+\tan^{-1}\frac{1}{nx}\ ,\ j\pi\right]}$, then there exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$, $F_k(j\pi) - F_k(j\pi + \tan^{-1}\dfrac{1}{nx}) = (-\tan^{-1}\dfrac{1}{nx})f_k(c^j_{nk})$ where $c^j_{nk} - j\pi \in (\tan^{-1}\frac{1}{nx}, 0)$ for any $k \in \{1, 2, ..., n\}$. So under the assumption (i) and the boundedness of a sequence $(n\tan^{-1}\frac{1}{nx})$, we can conclude that $\lim_{n\to\infty}\left|n\ \tan^{-1}\dfrac{1}{nx}\right|\dfrac{1}{n}\sum_{k=1}^{n}\sum_{j\in\mathbb{Z}}|f_k(c^j_{nk}) - f_k(j\pi)| = 0$. By Lemma 3.1 and

$$\begin{aligned}
\left|\sum_{k=1}^{n} F_{nk}(x) + \frac{L}{x}\right| &= \left|\sum_{k=1}^{n}\left[\sum_{j\in\mathbb{Z}}[F_k(j\pi) - F_k(j\pi + \tan^{-1}\frac{1}{nx})]\right] + \frac{L}{x}\right| \\
&\leq \left|\sum_{k=1}^{n}\left[(-n\ \tan^{-1}\frac{1}{nx})\frac{1}{n}\sum_{j\in\mathbb{Z}}(f_k(c^j_{nk}) - f_k(j\pi))\right]\right| \\
&+ \left|\sum_{k=1}^{n}\left[(-n\ \tan^{-1}\frac{1}{nx})\frac{1}{n}\sum_{j\in\mathbb{Z}} f_k(j\pi)\right] - (-n\ \tan^{-1}\frac{1}{nx})L\right| \\
&+ \left|(-n\ \tan^{-1}\frac{1}{nx})L + \frac{L}{x}\right|,
\end{aligned}$$

we have $\lim_{n\to\infty}\sum_{k=1}^{n} F_{nk}(x) = -\dfrac{L}{x}$.

2. By Lemma 3.1, it is obvious that $\lim_{n\to\infty}\sum_{k=1}^{n} F^r_{nk}(x) = 0.$ □

Lemma 3.5. *Assume (i) and (ii) in Theorem 2.1 hold. For each* $r > \frac{1}{2}$*, let* N *and* $N_r : (0, \infty) \to \mathbb{R}$ *be defined by* $N(x) = -\frac{L}{x}$ *and* $N_r(x) = -\frac{2L}{x^{\frac{1}{r}}}$*. For any* $x > 0$*,*

1. $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} [F_{nk}(x) - 1] = -\frac{L}{x}$ *and*

2. $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} [F_{nk}^{r}(x) - 1] = -\frac{2L}{x^{\frac{1}{r}}}$.

Proof. We can proved 1. and 2. by using the same way as Lemma 3.4 applying the Mean Value Theorem with $F_k|_{[j\pi\ ,\ j\pi+\tan^{-1}\frac{1}{nx}]}$ and $F_k|_{\left[j\pi-\tan^{-1}\frac{1}{nx^{\frac{1}{r}}}\ ,\ j\pi+\tan^{-1}\frac{1}{nx^{\frac{1}{r}}}\right]}$, respectively, for any $n \in \mathbb{N},\ k \in \{1, 2, ..., n\},\ x > 0$ and $r > \frac{1}{2}$.

□

Lemma 3.6. *Under the assumption (i), (ii) in Theorem 2.1 and let* $r > \frac{1}{2}$*. Then the following limits hold :*

1. $\lim\limits_{\epsilon\to 0^+} \lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \left\{ \int_{|x|<\epsilon} x^2\, dF_{nk}(x) - \left(\int_{|x|<\epsilon} x\, dF_{nk}(x) \right)^2 \right\} = 0$ *and*

2. $\lim\limits_{\epsilon\to 0^+} \lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \left\{ \int_{|x|<\epsilon} x^2\, dF_{nk}^{r}(x) - \left(\int_{|x|<\epsilon} x\, dF_{nk}^{r}(x) \right)^2 \right\} = 0.$

Proof. 1. From Lemma 3.1, [8, p.186] and the fundamental theorem of calculus, we get

$$\begin{aligned} 0 &\le \sum_{k=1}^{n} \left\{ \int_{|x|<\epsilon} x^2\, dF_{nk}(x) - \left(\int_{|x|<\epsilon} x\, dF_{nk}(x) \right)^2 \right\} \\ &\le \sum_{k=1}^{n} \left\{ \int_{-\epsilon}^{0^-} x^2 dF_{nk}(x) + \int_{0^+}^{\epsilon} x^2 dF_{nk}(x) \right\} \\ &= \sum_{k=1}^{n} \sum_{j\in\mathbb{Z}} \int_{-\tan^{-1}\frac{1}{n\epsilon}}^{-\frac{\pi}{2}} \frac{-\cot^2(y) f_k\,(j\pi + y)}{n^2} dy + \sum_{k=1}^{n} \sum_{j\in\mathbb{Z}} \int_{\frac{\pi}{2}}^{\tan^{-1}\frac{1}{n\epsilon}} \frac{-\cot^2(y) f_k\,(j\pi + y)}{n^2} dy. \end{aligned} \tag{3.1}$$

If we can show that

$$\lim_{\epsilon\to 0^+} \lim_{n\to\infty} \sum_{k=1}^{n} \sum_{j\in\mathbb{Z}} \int_{-\tan^{-1}\frac{1}{n\epsilon}}^{-\frac{\pi}{2}} \frac{-\cot^2(y) f_k\,(j\pi + y)}{n^2} dy \le 0$$

and

$$\lim_{\epsilon\to 0^+}\lim_{n\to\infty}\sum_{k=1}^{n}\sum_{j\in\mathbb{Z}}\int_{\frac{\pi}{2}}^{\tan^{-1}\frac{1}{n\epsilon}}\frac{-\cot^2(y)f_k\left(j\pi+y\right)}{n^2}dy\leq 0,$$

then $\lim\limits_{\epsilon\to 0^+}\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\left\{\int_{|x|<\epsilon}x^2\ dF_{nk}(x)-\left(\int_{|x|<\epsilon}x\ dF_{nk}(x)\right)^2\right\}=0.$

By assumption (i), there exists $0<\delta<\frac{\pi}{2}$ such that $f_k\left(j\pi+y\right)<f_k\left(j\pi\right)+\frac{1}{|j^p|+1}$ for all $|y|<\delta$, $k\in\mathbb{N}$ and $j\in\mathbb{Z}$. Then

$$\begin{aligned}
&\sum_{j\in\mathbb{Z}}\int_{-\tan^{-1}\frac{1}{n\epsilon}}^{-\frac{\pi}{2}}-\cot^2(y)f_k\left(j\pi+y\right)\ dy\\
&\leq\sum_{j\in\mathbb{Z}}\int_{-\frac{\pi}{2}}^{-\delta}\cot^2(y)f_k\left(j\pi+y\right)\ dy+\sum_{j\in\mathbb{Z}}\int_{-\delta}^{-\tan^{-1}\frac{1}{n\epsilon}}\cot^2(y)\left(f_k\left(j\pi\right)+\frac{1}{|j^p|+1}\right)\ dy\\
&\leq\cot^2(\delta)+(n\epsilon-\cot(\delta)+\tan^{-1}\frac{1}{n\epsilon}-\delta)\left(\sum_{j\in\mathbb{Z}}(f_k\left(j\pi\right)+\frac{1}{|j^p|+1})\right). \qquad (3.2)
\end{aligned}$$

It implies that $\lim\limits_{\epsilon\to 0^+}\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\sum\limits_{j\in\mathbb{Z}}\int_{-\tan^{-1}\frac{1}{n\epsilon}}^{-\frac{\pi}{2}}\frac{-\cot^2(y)f_k\left(j\pi+y\right)}{n^2}dy\leq 0$ by using assumption (ii).

To show that $\lim\limits_{\epsilon\to 0^+}\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\sum\limits_{j\in\mathbb{Z}}\int_{\frac{\pi}{2}}^{\tan^{-1}\frac{1}{n\epsilon}}\frac{-\cot^2(y)f_k\left(j\pi+y\right)}{n^2}dy\leq 0$, we may follow the same way of the term $\int_{-\tan^{-1}\frac{1}{n\epsilon}}^{-\frac{\pi}{2}}\frac{-\cot^2(y)f_k\left(j\pi+y\right)}{n^2}dy$. Hence we have the result in 1. as desired.

2. With the same method of the inequality (3.1), we have

$$\begin{aligned}
&\sum_{k=1}^{n}\left\{\int_{|x|<\epsilon}x^2\ dF_{nk}^r(x)-\left(\int_{|x|<\epsilon}x\ dF_{nk}^r(x)\right)^2\right\}\\
&\leq\sum_{k=1}^{n}\sum_{j\in\mathbb{Z}}\int_{\tan^{-1}(\frac{1}{n\epsilon^{\frac{1}{r}}})}^{\frac{\pi}{2}}\frac{f_k(j\pi-y)}{n^{2r}\tan^{2r}(y)}dy+\sum_{k=1}^{n}\sum_{j\in\mathbb{Z}}\int_{\tan^{-1}(\frac{1}{n\epsilon^{\frac{1}{r}}})}^{\frac{\pi}{2}}\frac{f_k(j\pi+y)}{n^{2r}\tan^{2r}(y)}dy.
\end{aligned}$$

By assumption (ii), we have

$$
\begin{aligned}
&\lim_{\epsilon\to 0^+}\lim_{n\to\infty}\sum_{k=1}^{n}\sum_{j\in\mathbb{Z}}\int_{\tan^{-1}(\frac{1}{n\epsilon^{\frac{1}{r}}})}^{\frac{\pi}{2}}\frac{f_k(j\pi-y)}{n^{2r}\tan^{2r}y}\,dy\\
&\leq \lim_{\epsilon\to 0^+}\lim_{n\to\infty}\sum_{k=1}^{n}\sum_{j\in\mathbb{Z}}\frac{1}{n^{2r}}\int_{\tan^{-1}(\frac{1}{n\epsilon^{\frac{1}{r}}})}^{\delta}\left(f_k(j\pi)+\frac{1}{|j^p|+1}\right)\frac{1}{\tan^{2r}y}\,dy\\
&+\lim_{\epsilon\to 0^+}\lim_{n\to\infty}\sum_{k=1}^{n}\sum_{j\in\mathbb{Z}}\frac{1}{n^{2r}}\int_{\delta}^{\frac{\pi}{2}}\frac{f_k(j\pi-y)}{\tan^{2r}y}\,dy\\
&\leq \lim_{\epsilon\to 0^+}\left(\frac{\tan^{1-2r}\delta}{1-2r}\right)\lim_{n\to\infty}\left[\frac{1}{n^{2r-1}}\left(\frac{1}{n}\sum_{k=1}^{n}\sum_{j\in\mathbb{Z}}f_k(j\pi)\right)+\frac{1}{n^{2r-1}}\sum_{j\in\mathbb{Z}}\frac{1}{|j^p|+1}\right]\\
&+\lim_{\epsilon\to 0^+}\left(\frac{\epsilon^{\frac{2r-1}{r}}}{2r-1}\right)\lim_{n\to\infty}\left[\frac{1}{n}\sum_{k=1}^{n}\sum_{j\in\mathbb{Z}}f_k(j\pi)+\sum_{j\in\mathbb{Z}}\frac{1}{|j^p|+1}\right]+\lim_{\epsilon\to 0^+}\lim_{n\to\infty}\frac{1}{n^{2r}\tan^{2r}\delta}\\
&=0.
\end{aligned}
$$

We note that $\lim\limits_{n\to\infty}\dfrac{1}{n^{2r-1}}$ and $\lim\limits_{\epsilon\to 0^+}\epsilon^{\frac{2r-1}{r}}$ have to be zero so this is the reason that we assume $r>\frac{1}{2}$. Furthermore, it is the same details of the inequlity (3.2) to show that

$$\lim_{\epsilon\to 0^+}\lim_{n\to\infty}\frac{1}{n^r}\sum_{k=1}^{n}\sum_{j\in\mathbb{Z}}\int_{\tan^{-1}\frac{1}{n\epsilon^{\frac{1}{r}}}}^{\frac{\pi}{2}}f_k(j\pi+y)\,\frac{1}{n^r\tan^{2r}y}\,dy\leq 0.$$

Hence we have the lemma as desired. □

3.1 Proof of Theorem 2.1

In order to guarantee the existence of (A_n) such that the sequence of the distribution function of $\sum\limits_{k=1}^{n}X_{nk}-A_n$ converges, we have to examine the conditions of Theorem A. It is clear that the sequence (X_{nk}) is infinitesimal from Lemma 3.3. The non-decreasing functions M and N in condition A1 and A2 follow from Lemma 3.4(1) and Lemma 3.5(1), respectively and the value of σ in the condition A3 of Theorem A is zero. Thus all conditions of Theorem A are satisfied and it implies that there exists a sequence of real numbers (A_n) such that the distribution function of $\sum\limits_{k=1}^{n}X_{nk}-A_n$ converges. Moreover, we can determine its limit distribution, say F, by using Levy's representation, as details in [11, p. 93]. If $L=0$, we know that $F=1_{[0,\infty)}$ and if $L>0$, the distribution function F is Cuachy distribution function with parameter $L\pi$.

The proof of Theorem 2.1(2) is the same as the footprints of Theorem 2.1(1) and we also get that $F^r = 1_{[0,\infty)}$ if $L = 0$ and F^r is a stable distribution function with characteristic exponent $\frac{1}{r}$ if $L > 0$ where F^r is the limit distribution function of $\sum_{k=1}^{n} X_{nk}^r - A_n(r)$ for each $r > \frac{1}{2}$, see the table in [11, p. 93].

To prove Theorem 2.2 (case $r = \frac{1}{2}$), we use the Theorem B as an important tool.

3.2 Proof of Theorem 2.2.

First, we shall show that the condition B1 of Theorem B is satisfied. Let $c_n = \sqrt{n\sqrt{\ln n}}$ for $n = 2, 3, ...$ and $c_1 = 1$ and $Y_n = \frac{1}{\sqrt{|\tan(X_n)|}}$. From Lemma 3.2, the distribution function of Y_n is represented as follows:

$$G_n(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ 1 + \sum_{j\in\Lambda} \left[F_n\left(j\pi - \arctan(\frac{1}{x^2})\right) - F_n\left(j\pi + \arctan(\frac{1}{x^2})\right)\right] & \text{if } x > 0. \end{cases}$$

By condition (i) and the fact that $\lim_{n\to\infty} c_n = \infty$, there exist $\delta > 0$ and $N \in \mathbb{N}$ such that for every $k \in \mathbb{N}$, $j \in \Lambda$ and $x \in \mathbb{R}$ if $|x - j\pi| < \delta$ then $|f_k(x) - f_k(j\pi)| < \frac{C}{|j^p|+1}$ where $C = (\frac{L}{2})\left(\sum_{j\in\Lambda} \frac{1}{|j^p|+1}\right)$ and $|\arctan(\frac{1}{c_n^2})| < \delta$ for any $n \geq N$. Thus we have $0 \leq \sum_{k=1}^{n} \int_{|x|\geq c_n} dG_k(x) = \sum_{k=1}^{n}\sum_{j\in\Lambda} \int_{-\arctan(\frac{1}{c_n^2})}^{\arctan(\frac{1}{c_n^2})} f_k(j\pi+y)dy \leq$ $(2\arctan\frac{1}{c_n^2})\sum_{k=1}^{n}\sum_{j\in\Lambda}(f_k(j\pi)+\frac{1}{|j^p|+1})$ for any $n \geq N$. So $\lim_{n\to\infty}\sum_{k=1}^{n}\int_{|x|\geq c_n} dG_k(x) \leq$ $0 \cdot L + 0 = 0$ because of the condition (ii) and $\lim_{n\to\infty} n \cdot \arctan(\frac{1}{c_n^2}) = 0$. Then the condition B1 of Theorem B holds. Next, we will show that the condition B2 of Theorem B is also satisfied. For $n \geq N$, and $k \in \{1, 2, ..., n\}$, we obtain that

$$\begin{aligned} 0 &\leq \int_{|x|<c_n} x dG_k(x) \\ &\leq \sum_{j\in\Lambda}\int_0^{c_n} x d\left(\left[F_k\left(j\pi + \arctan\left(-\frac{1}{x^2}\right)\right) - F_k\left(j\pi + \arctan\left(\frac{1}{x^2}\right)\right)\right]\right) \end{aligned}$$

$$\leq \sum_{j\in\Lambda}\left[\frac{1}{\sqrt{\tan(\delta)}}\int_{j\pi-\frac{\pi}{2}}^{j\pi-\delta} f_k(y)dy + \left(f_k(j\pi)+\frac{C}{|j^p|+1}\right)\int_{j\pi-\delta}^{j\pi+\frac{1}{n\sqrt{\ln n}}}\frac{1}{\sqrt{\tan(j\pi-y)}}dy\right.$$

$$\left. + \left(f_k(j\pi)+\frac{C}{|j^p|+1}\right)\int_{j\pi+\arctan(\frac{1}{n\ln n})}^{j\pi+\delta}\frac{1}{\sqrt{\tan(y-j\pi)}}dy + \frac{1}{\sqrt{\tan(\delta)}}\int_{j\pi+\delta}^{j\pi+\frac{\pi}{2}} f_k(y)dy\right]$$

$$= \frac{1}{\sqrt{\tan(\delta)}}\int_{-\infty}^{\infty} f_k(y)dy + \sum_{j\in\Lambda}\left(f_k(j\pi)+\frac{C}{|j^p|+1}\right)2\left[\sqrt{\tan(\delta)}-\frac{1}{c_n}\right]$$

$$+ \sum_{j\in\Lambda}\left(f_k(j\pi)+\frac{C}{|j^p|+1}\right)2\left[\sqrt{\tan\delta}-\frac{1}{c_n}\right] + \frac{1}{\sqrt{\tan(\delta)}}\int_{-\infty}^{\infty} f_k(y)dy$$

$$\leq (\sum_{j\in\Lambda} f_k(j\pi))K_n + L_n \tag{3.3}$$

where $K_n = 4(\sqrt{\tan(\delta)} - \frac{1}{c_n})$, $L_n = \left(\sum_{j\in\Lambda}\frac{1}{|j^p|+1}\right)A_n + \frac{2}{\sqrt{\tan(\delta)}}$ and we also have

$$\begin{aligned}\int_{|x|<c_n} x^2 dG_k(x) &\geq \sum_{j\in\Lambda}\int_{j\pi+b_n}^{j\pi+\frac{\pi}{2}}\frac{f_k(y)}{\tan(y-j\pi)}dy\\ &\geq \sum_{j\in\Lambda}\int_{j\pi+b_n}^{j\pi+\delta}\frac{f_k(y)}{\tan(y-j\pi)}dy\\ &\geq \sum_{j\in\Lambda}\left(f_k(j\pi)-\frac{1}{|j^p|+1}\right)\int_{\frac{1}{c_n^2}}^{\tan(\delta)}\frac{1}{x(1+x^2)}dx\\ &\geq \sum_{j\in\Lambda}\left(f_k(j\pi)-\frac{1}{|j^p|+1}\right)\frac{1}{\sqrt{1+(\tan(\delta))^2}}\ln(c_n^2\tan(\delta)). \quad (3.4)\end{aligned}$$

From (3.3) and (3.4) we conclude that

$$\frac{1}{c_n^2}\sum_{k=1}^{n}\left\{\int_{|x|<c_n} x^2 dG_k(x) - (\int_{|x|<c_n} x dG_k(x))^2\right\}$$

$$\geq \frac{1}{c_n^2}\sum_{k=1}^{n}\left\{\sum_{j\in\Lambda}\left(f_k(j\pi)-\frac{C}{|j^p|+1}\right)\frac{1}{\sqrt{1+(\tan(\delta))^2}}\ln(c_n^2\tan(\delta)) - [(\sum_{j\in\Lambda} f_k(j\pi))K_n + L_n]^2\right\}$$

$$\geq (\frac{1}{\sqrt{1+(\tan(\delta))^2}})\left[\frac{1}{n}\sum_{k=1}^{n}\sum_{j\in\Lambda} f_k(j\pi) - \frac{L}{2}\right]\left(\frac{n\ln(c_n^2\tan\delta)}{c_n^2}\right)$$

$$- K_n^2\left(\frac{1}{n}\sum_{k=1}^{n}\sum_{j\in\Lambda} f_k(j\pi)\right) - (2K_nL_n)\left(\frac{n}{c_n^2}\right)\left(\frac{1}{n}\sum_{k=1}^{n}\sum_{j\in\Lambda} f_k(j\pi)\right) - \left(\frac{nL_n^2}{c_n^2}\right).$$

Since $\lim_{n\to\infty} \frac{n\ln(c_n^2\tan\delta)}{c_n^2} = \infty$, $\lim_{n\to\infty} \frac{n}{c_n^2} = 0$, and both of $\lim_{n\to\infty} K_n$ and $\lim_{n\to\infty} L_n$ are constants, the condition B2 of Theorem B holds. Hence we have the Theorem 2.2 as desired.

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