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Some Remarks on Paramedial Semigroups and Medial Semigroups

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Abstract : Let S be a semigroup. We say that S is a medial if $abcd = acbd$ for all $a, b, c, d \in S$ and S is a paramedial if $abcd = dbca$ for all $a, b, c, d \in S$. In this paper, we investigate some properties of the regularity and Green's relations. Moreover, we describe compatibility with the natural partial order on paramedial semigroups and medial semigroups.

Keywords : regular; left regular; right regular; intra-regular; Green's relations; natural partial order.
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1 Introduction

Semigroups satisfying some type of generalized commutativity were considered in quite a number of papers. Lajos [3], Nagy [4] and Yamada [5] dealed with semigroups satisfying the identity $abc = cba$ for all $a, b, c \in S$. These semigroups are called *externally commutative semigroups*. The class of externally commutative semigroups appears as a natural generalization of the class of a commutative semigroup.

A *medial semigroup* [6] is a semigroup satisfying the medial law:

$$abcd = acbd \text{ for all } a, b, c, d \in S.$$

Hence the class of externally commutative semigroups is a subclass of the class of medial semigroups. Chrislock [6] has investigated medial Archimedean semigroups.

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Protić [7] introduced the concept of paramedial semigroups as a generalization of externally commutative semigroups. A *paramedial semigroup* is a semigroup satisfying the paramedial law:

$$abcd = dbca \text{ for all } a, b, c, d \in S.$$

He investigated that some general properties of paramedial semigroups and semilattice decomposition of paramedial semigroups are described.

In this paper, we investigate some properties of the regularity, Green's relations and natural partial order on paramedial semigroups and medial semigroups.

In this section, we present a number of definitions and notations most of which will be indispensable for our research.

Definition 1.1. Let S be a semigroup and $x \in S$. Then

- (i) x is a *regular element* if $x = xyx$ for some $y \in S$.
- (ii) x is an *intra-regular element* if $x = yx^2z$ for some $y, z \in S$.
- (iii) x is a *left regular element* if $x = yx^2$ for some $y \in S$.
- (iv) x is a *right regular element* if $x = x^2y$ for some $y \in S$.
- (v) x is a *completely regular element* if $x = xyx$ and $xy = yx$ for some $y \in S$.

The sets of all regular, intra-regular, left regular, right regular and completely regular elements of a semigroup S are called the *regular, intra-regular, left regular, right regular* and *completely regular part* of S, and are denoted by $Reg(S)$, $IReg(S)$, $LReg(S)$, $RReg(S)$ and $CReg(S)$, respectively.

A semigroup S is a *regular* (*intra-regular, left regular, right regular, completely regular*) *semigroup* if $Reg(S) = S$ ($IReg(S) = S, LReg(S) = S, RReg(S) = S, CReg(S) = S$).

Lemma 1.1 ([11])**.** $CReg(S) = LReg(S) \cap RReg(S)$.

Definition 1.2 ([9])**.** Let S be a semigroup and $a, b \in S$. We say that

1. $(a, b) \in \mathcal{L}$ if $S^1a = S^1b$,
2. $(a, b) \in \mathcal{R}$ if $aS^1 = bS^1$ and
3. $(a, b) \in \mathcal{J}$ if $S^1aS^1 = S^1bS^1$,

where S^1 is denoted a semigroup with identity obtained from S by adjoining an identity if necessary. We then have $\mathcal{L}, \mathcal{R}$ and $\mathcal{J}$ are equivalence relations on S. Let $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$ and $\mathcal{D} = \mathcal{L} \circ \mathcal{R}$. Since $\mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$, we have $\mathcal{D}$ and $\mathcal{H}$ are also equivalence relations on S. Five equivalence relations on S are called *Green's relations.*

Definition 1.3. Let x and y be elements in a semigroup S, then y is an *inverse* of x if $xyx = x$ and $yxy = y$. A semigroup S is an *inverse semigroup* if every element of S has a unique inverse.

In 1952, Vagner [12] defined the *natural partial order* for any inverse semigroup S by defining $\leq$ on S as follows:

$$a \leq b \text{ if and only if } a = be \text{ for some } e \in E(S). \tag{1.1}$$

where $E(S)$ is the set of all idempotent elements of S. Later, Nambooripad [10] extended this partial order $\leq$ on a regular semigroup S. The partial order is defined by

$$a \leq b \text{ if and only if } a = eb = bf \text{ for some } e, f \in E(S). \tag{1.2}$$

For an inverse semigroup S this relation is just the natural partial order (1.1).

In 1986, Mitsch [8] extended the above partial order to any semigroup S by defining $\leq$ on S as follows:

$$a \leq b \text{ if and only if } a = xb = by \text{ and } a = ay \text{ for some } x, y \in S^1. \tag{1.3}$$

This natural partial order coincides with the relation (1.2) if the semigroup S is regular.

2 Paramedial semigroups

In this section, we consider the properties of the paramedial semigroups.

Definition 2.1 ([7])**.** A semigroup S is a *paramedial semigroup* if the paramedial law

$$(\forall a, b, c, d \in S) \quad abcd = dbca$$

holds in S.

The following example depicts the existence of a paramedial semigroup.

Example 2.2. Consider the semigroup $S = \{a, b, c\}$ in the following Cayley table:

	a	b	c
a	a	a	a
b	a	a	a
c	a	b	c

It is not hard to check the paramedial law in the given Cayley table. Hence S is a paramedial semigroup but not commutative.

A semigroup S is a *semilattice* if for all $x, y \in S, x^2 = x$ and $xy = yx$.

Lemma 2.3 ([7])**.** *Let S be a paramedial semigroup. If $E(S) \neq \emptyset$, then $E(S)$ is a semilattice.*

Lemma 2.4 ([7])**.** *Let S be a paramedial semigroup. Then S^3 is a commutative semigroup.*

The next corollaries are a consequence of Lemma 2.4.

Corollary 2.1. *Every regular paramedial semigroup is commutative.*

Corollary 2.2. *Let S be a paramedial semigroup and $a, b \in S$. Then aSb is a commutative semigroup.*

Next, we describe the relationship of the regularity on paramedial semigroups.

Theorem 2.3. *Let S be a paramedial semigroup and $a \in S$. Then the following statements are equivalent.*

(i) *a is regular.*

(ii) *a is left regular.*

(iii) *a is right regular.*

(iv) *a is completely regular.*

(v) *a is intra-regular.*

Proof. $(i) \Rightarrow (ii)$ Assume that a is regular. Then $a = axa$ for some $x \in S$. Thus $a = axa = axaxa = xxaaa = (xxa)a^2$. Hence a is left regular.

$(ii) \Rightarrow (iii)$ Suppose that a is left regular. Then $a = xa^2$ for some $x \in S$. Hence $a = xa^2 = xaa = xa(xa^2) = a^2axx$ which implies that a is right regular.

$(iii) \Rightarrow (iv)$ Similarly the proof $(ii) \Rightarrow (iii)$, we can prove that if a is right regular, then a is left regular. Hence a is completely regular by Lemma 1.1.

$(iv) \Rightarrow (v)$ Assume that a is completely regular. Then $a = axa$ and $ax = xa$ for some $x \in S$. This implies that $a = axa = axaxa = xaaxa = xa^2xa$. Thus a is intra-regular.

$(v) \Rightarrow (i)$ If a is intra-regular, then $a = xa^2y$ for some $x, y \in S$. Thus $a = xaay = x(xaay)ay = ax(axy(ay)) = ax((ay)xya)$. Hence a is regular. □

Lemma 2.4. *If x is a regular element of a paramedial semigroup S, then x has a unique inverse.*

Proof. Suppose that x is a regular element of a paramedial semigroup S. Then $x = xax$ for some $a \in S$. Thus axa is an inverse of x. Let $y \in S$ be such that $x = xyx$ and $y = yxy$. Since $ax, xa, xy, yx \in E(S)$ and by Lemma 2.3, we have that $axa = axyxa = yxaxa = yxa = yxyxa = yxaxy = yxy = y$. Hence x has a unique inverse. □

A regular semigroup S is an *orthodox semigroup* if $E(S)$ is a subsemigroup of S.

As an immediate consequence of Theorem 2.3 and Lemmas 2.3 and 2.4, we have the following.

Corollary 2.5. *Let S be a paramedial semigroup. Then the following statements are equivalent.*

(i) *S is a regular semigroup.*

(ii) *S is a left regular semigroup.*

(iii) *S is a right regular semigroup.*

(iv) *S is a completely regular semigroup.*

(v) *S is an intra-regular semigroup.*

(vi) *S is an inverse semigroup.*

(vii) *S is an orthodox semigroup.*

Theorem 2.6. *Let S be a paramedial semigroup. If $Reg(S) \neq \emptyset$, then $Reg(S)$ is a subsemigroup of S.*

Proof. Let $a, b \in Reg(S)$. Then $a = axa$ and $b = byb$ for some $x, y \in S$. Since $xa, by \in E(S)$ and by Lemma 2.3, we have

$$ab = axabyb = abyxab.$$

Hence $ab \in Reg(S)$. We conclude that $Reg(S)$ is a subsemigroup of S. □

Now, we investigate the Green's relations on paramedial semigroups.

Theorem 2.7. *In a paramedial semigroup S, the Green's relations $\mathcal{R}, \mathcal{L}, \mathcal{H}, \mathcal{D}$ and $\mathcal{J}$ are coincided.*

Proof. Since $\mathcal{R}, \mathcal{L}, \mathcal{H}, \mathcal{D} \subseteq \mathcal{J}$, it suffices to show that $\mathcal{J} \subseteq \mathcal{R}$ and $\mathcal{J} \subseteq \mathcal{L}$. Let $a, b \in S$ be such that $(a, b) \in \mathcal{J}$. Then $a = u_1bu_2$ and $b = v_1av_2$ for some $u_1, u_2, v_1, v_2 \in S^1$. Thus

$$a = u_1bu_2 = u_1v_1av_2u_2 = u_1v_1u_1bu_2v_2u_2 = bv_1u_1u_1u_2v_2u_2 = u_2v_1u_1u_1u_2v_2b$$

and

$$b = v_1av_2 = v_1u_1bu_2v_2 = v_1u_1v_1av_2u_2v_2 = au_1v_1v_1v_2u_2v_2 = v_2u_1v_1v_1v_2u_2a.$$

Hence $(a, b) \in \mathcal{R}$ and $(a, b) \in \mathcal{L}$. This shows that $\mathcal{R} = \mathcal{L} = \mathcal{H} = \mathcal{D} = \mathcal{J}$, as required. □

Let ρ be a partial order on a semigroup S. An element $c \in S$ is said to be *left compatible* with ρ if $(ca, cb) \in \rho$ for all $(a, b) \in \rho$. *Right compatibility* with ρ is defined dually. If c is both left and right compatible, then c is *compatible* with ρ.

Theorem 2.8. *Every element of a paramedial semigroup is compatible with $\leq$.*

Proof. Let S be a paramedial semigroup and let $a, b \in S$ with $a \leq b$. Then $a = xb = by = ay$ for some $x, y \in S^1$. Let $c \in S$, then we have $ac = xbc = ayc = ayyc = ayyyc = yyyac = acyyy$ and $bcyyy = byyyc = ayyc = ayc = ac$. This implies that $ac \leq bc$. Hence c is a right compatible. Similarly, we can show that $ca = yyycb = cby = cay$. This shows that $ca \leq cb$. Hence c is left compatible. We conclude that c is compatible with $\leq$. □

3 Medial semigroups

Now it is time to consider some less familiar structure properties of medial semigroups. Some results, as we shall see arise from paramedial semigroups. But many medial semigroups have no such direct connection with paramedial semigroups.

Definition 3.1 ([6]). A semigroup S is a *medial semigroup* if the medial law

$$(\forall a, b, c, d \in S) \quad abcd = acbd$$

holds in S. Such a semigroup S satisfies $(ab)^n = a^n b^n$ and $(SaS)^n = S^n a^n S^n$ for all $a, b \in S$ and $n \in \mathbb{N}$.

A semigroup S is a *left (right) zero semigroup* if $ab = a$ $(ab = b)$ for all $a, b \in S$.

Example 3.2. Every left (right) zero semigroup is a medial semigroup. If S is a left (right) zero semigroup with $|S| \geq 2$, then S is not commutative.

Proposition 3.1. *Let S be a medial semigroup and $a, b \in S$. Then aSb is a commutative semigroups.*

Proof. Since $aSbaSb \subseteq aSb$, aSb is a subsemigroup of S. Let $x, y \in aSb$. Then $x = aub$ and $y = avb$ for some $u, v \in S$. Consequently,

$$xy = aubavb = aubvab = auvbab = avubab = avbuab = avbaub = yx.$$

□

Example 3.3. Let S be the left zero semigroup such that $|S| \geq 2$ and let $a, b \in S$ with $a \neq b$. Then $abb, bba \in S^3$ and $(abb)(bba) = a \neq b = (bba)(abb)$. Thus S^3 is not a commutative semigroups.

Theorem 3.2. *Let S be a medial semigroup and $a \in S$. Then the following statements are equivalent.*

(*i*) *a is regular.*

(*ii*) *a is left regular.*

(*iii*) *a is right regular.*

(*iv*) *a is completely regular.*

(*v*) *a is intra-regular.*

Proof. $(i) \Rightarrow (ii)$ Suppose that a is regular. Then $a = axa$ for some $x \in S$. Thus $a = axa = axaxa = axxaa$. Hence a is a left regular element.

$(ii) \Rightarrow (iii)$ Suppose that a is left regular. Then $a = xa^2$ for some $x \in S$. and so $a = xa^2 = xaa = xa(xa^2) = xaxa(xa^2) = xaaxxaa = axxaa = aaxxa = a^2(xxa)$ which implies that a is right regular.

$(iii) \Rightarrow (iv)$ Similarly the proof $(ii) \Rightarrow (iii)$, we can prove that if a is right regular, then a is left regular. By Lemma 1.1, we have a is completely regular.

$(iv) \Rightarrow (v)$ Assume that a is completely regular. Then $a = axa$ and $ax = xa$ for some $x \in S$. This implies that $a = axa = axaxa = xaaxa = xa^2xa$. Thus a is intra-regular.

$(v) \Rightarrow (i)$ If a is intra-regular, then $a = xa^2y$ for some $x, y \in S$. Thus $a = xaay = x(xa^2y)(xa^2y)y = x(a^2yx)(xa^2y)y = x(a^2yx)(yxa^2)y = (xa^2y)xy(xa^2y) = axya$. Hence a is a regular element. □

Lemma 3.3. *Let S be a medial semigroup. If $E(S) \neq \emptyset$, then $E(S)$ is a subsemigroup of S.*

Proof. Let $e, f \in E(S)$. Then

$$(ef)^2 = efef = eeff = ef.$$

Hence $ef \in E(S)$. □

Example 3.4 ([2]). Let a semigroup S be given by the Cayley table

	a	b	c
a	b	b	b
b	b	b	b
c	c	c	c

The semigroups given by the above table is a medial semigroup [2]. Since $b, c \in E(S)$ and $bc \neq cb$, it follows that $E(S)$ is not commutative. Hence $E(S)$ is not a semilattice.

By Theorem 3.2 and Lemma 3.3, we have the following results.

Corollary 3.4. *Let S be a medial semigroup. Then the following statements are equivalent.*

(i) *S is a regular semigroup.*

(ii) *S is a left regular semigroup.*

(iii) *S is a right regular semigroup.*

(iv) *S is a completely regular semigroup.*

(v) *S is an intra-regular semigroup.*

(vi) *S is an orthodox semigroup.*

Example 3.5. Let S be the left zero semigroup such that $|S| \geq 2$ and let $a, b \in S$ with $a \neq b$. Then S is a regular semigroup. Since $a = aaa = aba$ and $b = bab$, we have a and b are inverses of a and $a \neq b$. Hence S is not an inverse semigroup.

Theorem 3.5. *Let S be a medial semigroup. If $Reg(S) \neq \emptyset$, then $Reg(S)$ is a subsemigroup of S.*

Proof. Let $a, b \in Reg(S)$. Then $a = axa$ and $b = byb$ for some $x, y \in S$. Therefore

$$ab = axabyb = abyxab.$$

Hence $ab \in Reg(S)$. □

By Theorem 2.7, these five Green's relations in paramedial semigroups are coincided but not true in medial semigroups as the following example.

Example 3.6. Let S be defined as in Example 3.4. Consider

$$\begin{aligned} S^1a &= \{a,b,c\}, \\ S^1b &= \{b,c\} = S^1c, \\ aS^1 &= \{a,b\}, \\ bS^1 &= \{b\}, \\ cS^1 &= \{c\}, \\ S^1aS^1 &= \{a,b,c\} \text{ and} \\ S^1bS^1 &= \{b,c\} = S^1cS^1. \end{aligned}$$

Therefore $\mathcal{L} = \{(a,a),(b,b),(b,c),(c,b),(c,c)\} = \mathcal{J}$ and $\mathcal{R} = \{(a,a),(b,b),(c,c)\}$. Hence $\mathcal{L} = \mathcal{J} = \mathcal{D}$ and $\mathcal{R} = \mathcal{H}$.

An element e of a semigroup S is a *left (right) identity* if $ex = x(xe = x)$ for all $x \in S$.

Theorem 3.6. *Let S be a medial semigroup. The following statements hold.*

(i) *If S has a left identity, then $\mathcal{R} = \mathcal{D} = \mathcal{J}$ and $\mathcal{L} = \mathcal{H}$.*

(ii) *If S has a right identity, then $\mathcal{L} = \mathcal{D} = \mathcal{J}$ and $\mathcal{R} = \mathcal{H}$.*

(iii) *If S has the identity, then the Green's relations $\mathcal{R}, \mathcal{L}, \mathcal{H}, \mathcal{D}$ and $\mathcal{J}$ on S are coincided.*

Proof. (i) Let e be a left identity of S. If $(a,b) \in \mathcal{J}$, then $a = ubv$ and $b = xay$ for some $u,v,x,y \in S$. Since e is a left identity, $a = eubv = ebuv = buv$ and $b = exay = eaxy = axy$. This implies that $(a,b) \in \mathcal{R}$. Hence $\mathcal{J} = \mathcal{R} = \mathcal{D}$ and $\mathcal{L} = \mathcal{H}$.

(ii) Similarly, if S has a right identity, then $\mathcal{L} = \mathcal{D} = \mathcal{J}$ and $\mathcal{R} = \mathcal{H}$.

(iii) It follows from (i) and (ii). □

Finally, we describe the natural partial order on medial semigroups.

Theorem 3.7. *Let S be a medial semigroup. Every idempotent of S is compatible with $\leq$ on S.*

Proof. Let $c \in E(S)$. If $a \leq b$, then there exist $x, y \in S^1$ such that $a = xb = by = ay$. Thus $ac = xbc$. Since $ac = byc = bycc = bcyc$ and $acyc = aycc = ac$, we deduce that $ac \leq bc$. Note that $ca = cby = cay$ and $ca = cxb = ccxb = cxcb$. This implies that $ca \leq cb$. Hence c is compatible with $\leq$. □

Theorem 3.8. *Every element of a regular medial semigroup is compatible with $\leq$ on S.*

Proof. Let $a, b \in S$ be such that $a \leq b$ and $x \in S$. Then $a = be = fb$ and $x = xyx$ where $e, f \in E(S)$ and $y \in S$. Then $xa = xbe$ and $xa = xfb = xyxfb = xyfxb$. Since $xy, f \in E(S)$ by Lemma 3.3, $xyf \in E(S)$. Hence $xa \leq xb$. Note that $eyx \in E(S)$ by Lemma 3.3. Since $ax = fbx$ and $ax = bex = bexyx = bxeyx$, it follows that $ax \leq bx$. Hence x is compatible with $\leq$ on S. □

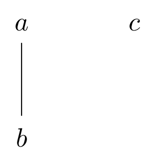
Theorem 3.9. *Let S be a medial semigroup. If S has a left (right) identity, then every element of S is left (right) compatible with $\leq$ on S.*

Proof. Let e be a left identity on S and let $a, b \in S$ be such that $a \leq b$. Then $a = xb = by = ay$ for some $x, y \in S^1$. For each $c \in S$, we have $ca = cxb = ecxb = excb = xcb$ and $ca = cby = cay$ which implies that $ca \leq cb$. Hence $\leq$ is left compatible.

Similarly, if S has a right identity, then the natural partial order on a regular medial semigroup is right compatible. □

The following example shows that the converses of the Theorems 3.8 and 3.9, are not true.

Example 3.7. Let S be defined as in Example 3.4. The natural partial order on S is as follows:



By the Cayley table, it is easy to see that S has no left and right identities. Since $a \neq axa$ for all $x \in S$, a is not regular. Hence S is not regular. But the natural partial order on a regular medial semigroup is compatible.

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