Thai Journal of Mathematics : 133-145
Special Issue : Annual Meeting in Mathematics 2019
http://thaijmath.in.cmu.ac.th
Online ISSN 1686-0209

Some Forbidden Rectangular Chessboards with Generalized Knight's Moves

Sirirat Singhun[†], **Krit Karudilok**[†,1], **and Ratinan Boonklurb**[‡]

[†]Department of Mathematics, Faculty of Science,
Ramkhamhaeng University, Bangkok 10240, Thailand
e-mail : `sin_sirirat@ru.ac.th (A. Singhun)`
`Kritkarudilok@hotmail.com (K. Karudilok)`
[‡] Department of Mathematics and Computer Science,
Faculty of Science, Chulalongkorn University,
Bangkok 10330 Thailand
e-mail : `ratinan.b@chula.ac.th (R. Boonklurb)`

Abstract : The $m \times n$ chessboard is an array with squares arranged in m rows and n columns. An (a, b)-*knight's move* or *generalized knight's move* is a move from one square to another by moving a knight passing a squares vertically or a squares horizontally and then passing b squares at 90 degrees angle. A *closed* (a, b)-*knight's tour* is an (a, b)-knight's move such that the knight lands on every square on the $m \times n$ chessboard once and returns to its starting square. In this paper, we show that (i) the $(a + b) \times n$ chessboard admits no closed (a, b)-knight's tours if $n \in [2b + 1, 4b - 1]$ where $1 \leq a < b$ or if $n \in [4b + 1, 5b]$ where $1 \leq a < b < 2a$, and (ii) the $(2a + 1) \times n$ chessboard admits no closed $(a, a + 1)$-knight's tours if $n = 4a + 4$ where $a \geq 1$, or if $n = 6a + 6$ where $a > 3$, or if $n = 6a + 8$ where $a > 3$.

Keywords : Hamiltonian cycle; closed knight's tour; $m \times n$ chessboard; generalized knight's move; forbidden chessboard.

2010 Mathematics Subject Classification : 05C38; 05C45.

---

[1]Corresponding author.

1 Introduction and Preliminaries

The chessboard is an array arranged in eight rows and eight columns. The interested chess piece on the chessboard is the knight because of its move. The knight can move one square vertically or one square horizontally and then two squares move at 90 degrees angle. In 1959, Euler [1] found that on the chessboard the knight can move from a square to another square such that it lands on every square once and returns to its starting square. This knight's move is called a *closed knight's tour*. The extension of the chessboard is the $m \times n$ chessboard. It is an array arranged in m rows and n columns. In 1991, Schwenk [2] obtained the sufficient and necessary conditions for the $m \times n$ chessboard to admit a closed knight's tour.

Theorem 1.1 ([2]). *The $m \times n$ chessboard with $m \leq n$ admits a closed knight's tour unless one or more of the following conditions hold:*
(i) m and n are both odd; or
(ii) $m = 1$ or 2 or 4; or
(iii) $m = 3$ and $n = 4$ or 6 or 8.

An (a, b)*-knight's move* or a *generalized knight's move* is an extension of a knight's move. It is defined by Chia et al. [3]. They generalized in such a way that the knight can move a squares vertically or a squares horizontally and then b squares move at 90 degrees angle. Then, $(1, 2)$-knight's move is the ordinary knight's move. For the $m \times n$ chessboard, we label each square by (i, j) in the matrix fashion. If a knight stands at square (i, j), then it can move to at most eight squares: $(i \pm a, j \pm b)$ and $(i \pm b, j \pm a)$. A closed knight's tour using (a, b)-knight's move is also extended on the $m \times n$ chessboard. If the knight moves to all squares of the $m \times n$ chessboard with an (a, b)-knight's move and returns to the starting square, then this move is called a *closed* (a, b)*-knight's tour* or a *generalized closed knight's tour*.

We see that an (a, b)-knight's move and (b, a)-knight's move are the same. In the case that $a = b$, the knight can move from a square to another square with the same colour, black to black or white to white. Then, the $m \times n$ chessboard admits no closed (a, a)-knight's tours. Thus, we shall assume that $a < b$. The authors in [3] obtained some chessboard admit a closed $(2, 3)$-knight's tour as follows.

Proposition 1.2 ([3]). *Suppose $k \geq 3$ is an integer. Then, the $5k \times n$ chessboard admits a closed $(2, 3)$-knight's tour if and only if*
(i) $n \geq 10$ is even and $n \neq 12$ when k is odd, or
(ii) $n = 5, 9, 10, 11$ or $n \geq 13$ when k is even.

Moreover, Chia et al. [3] obtained the necessary conditions for the $m \times n$ chessboard admitted no closed (a, b)-knight's tours as follows.

Theorem 1.3 ([3]). *Suppose that the $m \times n$ chessboard admits a closed (a, b)-knight's tour, where $a < b$ and $m \leq n$. Then,*
(i) $a + b$ is odd; and

(ii) m *or* n *is even; and*
(iii) $m \geq a+b$*; and*
(iv) $n \geq 2b$*.*

By Theorems 1.1 and 1.3, we see that which $m \times n$ chessboard admits no closed (a, b)-knight's tours. Then, Chia et al. [3] also obtained some chessboards which admit no closed (a, b)-knight's tours.

Theorem 1.4 ([3])**.** *Suppose that* $m = a+b+2t+1$*, where* $0 \leq t \leq a-1$*. Then, the* $m \times n$ *chessboard admits no closed* (a, b)*-knight's tour.*

Theorem 1.5 ([3])**.** *Suppose that* $m = a(k+2l)$*, where* $1 \leq l \leq \frac{k}{2}$*. Then, the* $m \times n$ *chessboard admits no closed* (a, ak)*-knight's tour, where* a *is odd and* k *is even.*

Theorem 1.6 ([3])**.** *Suppose that* $m = 2(ak+l)$*, where* $1 \leq k \leq l \leq a$*. Then, the* $m \times n$ *chessboard admits no closed* $(a, a+1)$*-knight's tour.*

Theorem 1.7 ([3])**.** *Suppose that* $m = 2a+2t+1$*, where* $1 \leq t \leq a-1$*. Then, the* $m \times n$ *chessboard admits no closed* $(a, a+1)$*-knight's tour.*

By Theorems 1.4-1.7, certain chessboards that admit no closed (a, b)-knight's tours for any $a < b$ depend on a and b. In 2017, the authors in [4] obtained the result for any $m \times n$ chessboard as follows.

Theorem 1.8 ([4])**.** *Let* $gcd(a, b) \neq 1$*. Then, there is no closed* (a, b)*-knight's tour for any* $m \times n$ *chessboard.*

By Theorem 1.3, we see that the smallest m of the $m \times n$ chessboard which may contain a closed (a, b)-knight's tour is $a+b$. This motivates us to focus the $m \times n$ chessboard with $m = a+b$. In this paper, we show that (i) the $(a+b) \times n$ chessboard admits no closed (a, b)-knight's tours if $n \in [2b+1, 4b-1]$ where $1 \leq a < b$ (in Theorem 2.5) or if $n \in [4b+1, 5b]$ where $1 \leq a < b < 2a$ (in Theorem 2.6), and (ii) the $(2a+1) \times n$ chessboard admits no closed $(a, a+1)$-knight's tours if $n = 4a+4$ where $a \geq 1$ (in Theorem 3.1), or if $n = 6a+6$ where $a > 3$ (in Theorem 3.2), or if $n = 6a+8$ where $a > 3$ (in Theorem 3.3).

2 The $(a+b) \times n$ Chessboards where $n \in [2b, 4b-1] \cup [4b+1, 5b]$

We start this section with the well-known graphs. The first one is a *Hamiltonian graph*. It is a graph containing a cycle that passes through each vertex exactly once. This cycle is called a *Hamiltonian cycle*. The second one is a *bipartite graph*. It is a graph that the vertex set can be partition into two sets and two vertices are joined by an edge if they are in the different partition set.

The knight's tour problem on the $m \times n$ chessboard is converted to question about a certain graph. An $m \times n$ *knight's graph* is a graph G that represents all

(a, b)-knight's moves on the $m \times n$ chessboard. That is, G contains mn vertices where each square of the $m \times n$ chessboard is replaced by a vertex and two vertices of G are joined by an edge if the knight can move by (a, b)-knight's move between these squares or vertices. In other words, if v is a square of the $m \times n$ chessboard, then v is a vertex of the $m \times n$ knight's graph. If v is a vertex of G, then the *degree* of v, denoted by deg v, is the number of squares that the knight can move to. We may say that the degree of a vertex v of the $m \times n$ knight's graph is the degree of the corresponding square v on the $m \times n$ chessboard. A closed knight's tour is a Hamiltonian cycle in G and the knight's tour problem on the $m \times n$ chessboard in the sense of a graph is to determine a Hamiltonian cycle of the $m \times n$ knight's graph. For the $m \times n$ chessboard, we color square (i, j) with black if $i + j$ is even, and white if $i + j$ is odd. If the $m \times n$ chessboard admits a closed (a, b)-knight's tour, then, by Theorem 1.3, $a + b$ is odd. If the knight stands at a black square (i, j), then squares $(i \pm a, j \pm b)$ and $(i \pm b, j \pm a)$, if exist, are colored by white. Also, if the knight stands at a white square (i, j), then squares $(i \pm a, j \pm b)$ and $(i \pm b, j \pm a)$, if exist, are colored by black. Thus, the knight only moves from a black square to a white square or a white square to a black square. We can conclude this remark in the following lemma.

Lemma 2.1. *If the $m \times n$ chessboard admits a closed (a, b)-knight's tour, then the $m \times n$ knight's graph is a bipartite graph.*

Now, we consider the $m \times n$ chessboard where $m = a + b$. In the case that $n = 2b$, we posted the following result in [4].

Theorem 2.2 ([4]). *Suppose that $m = a + b$ and $n = 2b$. Then, the $m \times n$ chessboard admits no closed (a, b)-knight's tours.*

For the case that $m = a + b$ and $n \geq 2b + 1$, the $m \times n$ chessboard can be devided into 11 parts depending on the degree of each vertex of the $m \times n$ knight's graph.

Lemma 2.3. *Let $1 \leq a < b, n \geq 2b + 1$ and B denote the $(a + b) \times n$ chessboard. Suppose that the knight moves with an (a, b)-knight's move. Then, B can be partition into 11 parts,namely $B_1^2, B_2^2, B_3^2, B_4^2, B_5^2, B_6^2, B_1^3, B_2^3, B_3^3, B_4^3$ and B_1^4, such that (i) if v belongs to B_i^2 for some $i \in \{1, 2, 3, 4, 5, 6\}$, then deg $v = 2$,*
(ii) if v belongs to B_i^3 for some $i \in \{1, 2, 3, 4\}$, then deg $v = 3$, and
(iii) if v belongs to B_1^4, then deg $v = 4$.

The 11 parts of B are shown in Figure 1. Then, we obtain the following remark in order to know which part of the $m \times n$ chessboard contains square (x, y).

Remark 2.1. *Let $1 \leq a < b, n \geq 2b + 1$ and (x, y) be a square of the $(a + b) \times n$ chessboard. Then,*

1. *square (x, y) belongs to B_1^2, if $1 \leq x \leq a$ and $1 \leq y \leq a$,*
2. *square (x, y) belongs to B_2^2, if $b + 1 \leq x \leq a + b$ and $1 \leq y \leq a$,*
3. *square (x, y) belongs to B_3^2, if $1 \leq x \leq a$ and $n - a + 1 \leq y \leq n$,*

4. *square* (x, y) *belongs to* B_4^2*, if* $b+1 \le x \le a+b$ *and* $n-a+1 \le y \le n$*,*
5. *square* (x, y) *belongs to* B_5^2*, if* $a+1 \le x \le b$ *and* $1 \le y \le b$*,*
6. *square* (x, y) *belongs to* B_6^2*, if* $a+1 \le x \le b$ *and* $n-b+1 \le y \le n$*,*
7. *square* (x, y) *belongs to* B_1^3*, if* $1 \le x \le a$ *and* $a+1 \le y \le b$*,*
8. *square* (x, y) *belongs to* B_2^3*, if* $b+1 \le x \le a+b$ *and* $a+1 \le y \le b$*,*
9. *square* (x, y) *belongs to* B_3^3*, if* $1 \le x \le a$ *and* $n-b+1 \le y \le n-a$*,*
10. *square* (x, y) *belongs to* B_4^3*, if* $b+1 \le x \le a+b$ *and* $n-b+1 \le y \le n-a$*, and*
11. *square* (x, y) *belongs to* B_1^4*, if* $1 \le x \le a+b$ *and* $b+1 \le y \le n-b$*.*

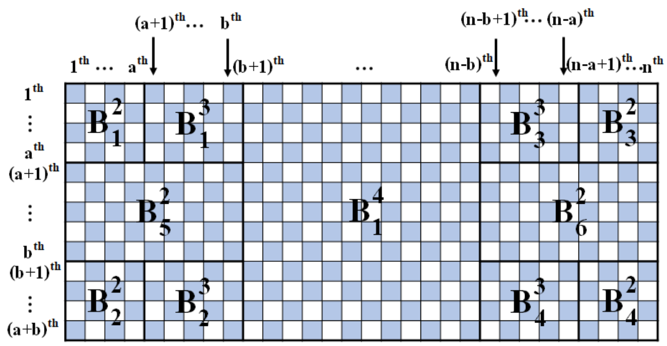

Figure 1: The 11 parts, $B_1^2, B_2^2, B_3^2, B_4^2, B_5^2, B_6^2, B_1^3, B_2^3, B_3^3, B_4^3$ and B_1^4, of the $(a+b) \times n$ chessboard in Lemma 2.3.

The following lemma is obtained directly from Theorem 1.3 and Lemma 2.1.

Lemma 2.4. *Let* a *and* b *be integers such that* $1 \le a < b$*. If* $a+b$ *is even or* n *is odd, then the* $(a+b) \times n$ *chessboard admits no closed* (a, b)*-knight's tours.*

Theorem 2.5. *Let* a *and* b *be integers such that* $1 \le a < b$*. Suppose that* $2b+1 \le n \le 4b-1$*. Then, the* $(a+b) \times n$ *chessboard admits no closed* (a, b)*-knight's tours.*

Proof. By Lemma 2.4, the $(a+b) \times n$ chessboard admits no closed (a, b)-knight's tours when (i) $a+b$ is even or (ii) n is odd. We assume that $a+b$ is odd and n is even where $2b+1 \le n \le 4b-1$.

Let G be an $(a+b) \times n$ knight's graph. We will show that there are two vertices of degree 2, u and v, such that they are adjacent to the same vertices, x and y.

Step 1: Since $a < b$, $a+b+1 \le 2b$ and $2a+2 \le a+b+1$. That is, $\frac{a+b+1}{2} \le b$ and $a+1 \le \frac{a+b+1}{2}$, respectively. Since $2b+1 \le n \le 4b-1$, $b + \frac{1}{2} \le \frac{n}{2} \le 2b - \frac{1}{2}$. That is, $1 \le \frac{n}{2} - b < b$. Then, the vertex $(\frac{a+b+1}{2}, \frac{n}{2} - b)$ is in B_5^2 of Lemma 2.3. Then, it is a degree-2 vertex adjacent to $(\frac{b-a+1}{2}, \frac{n}{2})$ and $(\frac{3a+b+1}{2}, \frac{n}{2})$.

Step 2: By Step 1, it suffices to show that $n - b < \frac{n}{2} + b \le n - 1$. Since $1 \le \frac{n}{2} - b < b$, $-b < -\frac{n}{2} + b \le -1$ Then, $n - b < \frac{n}{2} + b \le n - 1$. Thus, the vertex

$(\frac{a+b+1}{2}, \frac{n}{2}+b)$ is in B_6^2 of Lemma 2.3. Then, it is a degree-2 vertex adjacent to $(\frac{b-a+1}{2}, \frac{n}{2})$ and $(\frac{3a+b+1}{2}, \frac{n}{2})$.

If G contains a Hamiltonian cycle C, then two edges incident to $(\frac{a+b+1}{2}, \frac{n}{2}-b)$ and $(\frac{a+b+1}{2}, \frac{n}{2}+b)$ are in C. Thus, $(\frac{a+b+1}{2}, \frac{n}{2}-b)(\frac{b-a+1}{2}, \frac{n}{2})(\frac{a+b+1}{2}, \frac{n}{2}+b)(\frac{3a+b+1}{2}, \frac{n}{2})$ forms a 4-cycle (see Figure 2). Since $b > a \geq 1$ and $n \geq 2b+1$, the number of vertices of the $(a+b)\times n$ knight's graph is $(a+b)n \geq 3n \geq 3(2b+1) \geq 6b+1 \geq 13$. Then, such 4-cycle is not a Hamiltonian cycle. It is a contradiction. $\square$

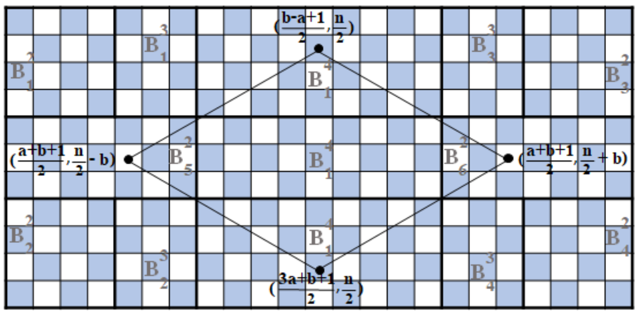

Figure 2: The 4-cycle in the proof of Theorem 2.5.

For the case that $4b+1 \leq n \leq 5b$ and $1 \leq a < b < 2a$, the $(a+b)\times n$ chessboard can be partition into 11 parts as in Lemma 2.3. We see that if the knight moves with an (a,b)-knight's move, then, by Remark 2.1,

1. square $(\frac{3a-b+1}{2}, b-a+1)$ belongs to B_1^2,
2. squares $(\frac{b-a+1}{2}, b+1), (\frac{3a+b+1}{2}, b+1)$ and $(\frac{a+b+1}{2}, 2b+1)$ belong to B_1^4,
3. square $(\frac{3b-a+1}{2}, b-a+1)$ belongs to B_2^2,
4. square $(\frac{a+b+1}{2}, 1)$ belong to B_5^2, and
5. square $(\frac{a+b+1}{2}, 4b+1)$ belongs to B_6^2.

Theorem 2.6. *Let a and b be integers such that $1 \leq a < b < 2a$. Suppose that $4b+1 \leq n \leq 5b$. Then, the $(a+b)\times n$ chessboard admits no closed (a,b)-knight's tours.*

Proof. By Lemma 2.4, the $(a+b)\times n$ chessboard admits no closed (a,b)-knight's tours when $a+b$ is even or n is odd. We assume that $a+b$ is odd and n is even where $4b+1 \leq n \leq 5b$.

Let G be an $(a+b)\times n$ knight's graph. We prove by contradiction. Suppose that G contains a Hamiltonian cycle C. If edges $(\frac{a+b+1}{2}, 2b+1)(\frac{b-a+1}{2}, 3b+1), (\frac{a+b+1}{2}, 2b+1)(\frac{3a+b+1}{2}, 3b+1), (\frac{a+b+1}{2}, 4b+1)(\frac{b-a+1}{2}, 3b+1)$ and $(\frac{a+b+1}{2}, 4b+1)(\frac{3a+b+1}{2}, 3b+1)$ are in C, then those edges form a 4-cycle. It is a contradiction.

Step 1: Since the square $(\frac{a+b+1}{2}, 1)$ belongs to B_5^2 of Lemma 2.3, it is a degree-2 vertex and edges $(\frac{a+b+1}{2}, 1)(\frac{b-a+1}{2}, b+1)$ and $(\frac{a+b+1}{2}, 1)(\frac{3a+b+1}{2}, b+1)$ must be included in C.
Step 2: Since the square $(\frac{3b-a+1}{2}, b-a+1)$ belongs to B_2^2 of Lemma 2.3, it is a degree-2 vertex and an edge $(\frac{3b-a+1}{2}, b-a+1)(\frac{b-a+1}{2}, b+1)$ must be included in C.
Step 3: Since $(\frac{b-a+1}{2}, b+1)$ belongs to B_1^4 of Lemma 2.3, it is a degree-4 vertex. By Steps 1 and 2, edges $(\frac{b-a+1}{2}, b+1)(\frac{a+b+1}{2}, 1)$ and $(\frac{b-a+1}{2}, b+1)(\frac{3b-a+1}{2}, b-a+1)$ belong to C. Then, $(\frac{b-a+1}{2}, b+1)(\frac{a+b+1}{2}, 2b+1)$ cannot be included in C.
Step 4: Since the square $(\frac{3a-b+1}{2}, b-a+1)$ belongs to B_1^2 of Lemma 2.3, it is a degree-2 vertex. Then, an edge $(\frac{3a-b+1}{2}, b-a+1)(\frac{3a+b+1}{2}, b+1)$ must be included in C.
Step 5: Since the square $(\frac{3a+b+1}{2}, b+1)$ belongs to B_1^4 of Lemma 2.3, it is a degree-4 vertex. By Steps 1 and 4, the edges $(\frac{a+b+1}{2}, 1)(\frac{3a+b+1}{2}, b+1)$ and $(\frac{3a-b+1}{2}, b-a+1)(\frac{3a+b+1}{2}, b+1)$ belong to C. Then, $(\frac{3a+b+1}{2}, b+1)(\frac{a+b+1}{2}, 2b+1)$ cannot be included in C.
Step 6: Since the square $(\frac{a+b+1}{2}, 2b+1)$ belongs to B_1^4 of Lemma 2.3, it is a degree-4 vertex. By Steps 3 and 5, the edges $(\frac{a+b+1}{2}, 2b+1)(\frac{b-a+1}{2}, b+1)$ and $(\frac{a+b+1}{2}, 2b+1)(\frac{3a+b+1}{2}, b+1)$ cannot be included in C. Then, edges $(\frac{a+b+1}{2}, 2b+1)(\frac{b-a+1}{2}, 3b+1)$ and $(\frac{a+b+1}{2}, 2b+1)(\frac{3a+b+1}{2}, 3b+1)$ must be included in C.
Step 7: Since the square $(\frac{a+b+1}{2}, 4b+1)$ belongs to B_6^2 of Lemma 2.3, it is a degree-2 vertex. Then, edges $(\frac{a+b+1}{2}, 4b+1)(\frac{b-a+1}{2}, 3b+1)$ and $(\frac{a+b+1}{2}, 4b+1)(\frac{3a+b+1}{2}, 3b+1)$ must be included in C.

By Steps 6 and 7, we see that edges $(\frac{a+b+1}{2}, 2b+1)(\frac{b-a+1}{2}, 3b+1)$, $(\frac{a+b+1}{2}, 2b+1)(\frac{3a+b+1}{2}, 3b+1)$, $(\frac{a+b+1}{2}, 4b+1)(\frac{b-a+1}{2}, 3b+1)$ and $(\frac{a+b+1}{2}, 4b+1)(\frac{3a+b+1}{2}, 3b+1)$ form a 4-cycle (see Figure 3), which is a contradiction. □

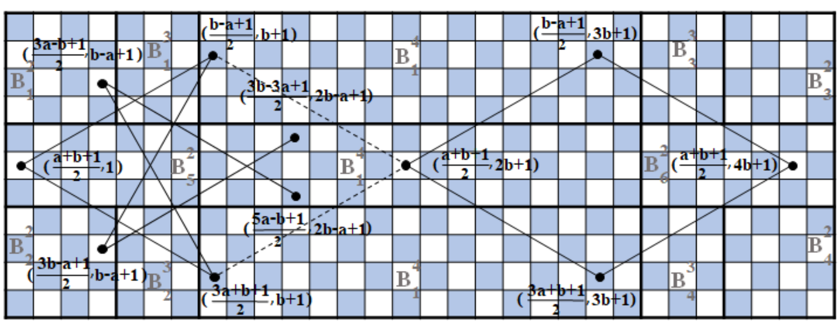

Figure 3: The 4-cycle in the proof of Theorem 2.6.

3 The $(2a+1)\times n$ Chessboards where $n=4a+4$, $6a+6$ or $6a+8$

In this section, an $(a, a+1)$-knight's move on the $(2a+1)\times n$ chessboard where $n = 4a+4, 6a+6$ or $6a+8$ is considered. Then, in Figure 1, B_1^3 and B_2^3 of B belong to the $(a+1)^{th}$ column and B_3^3 and B_4^3 of B belong to the $(n-a)^{th}$ column.

The certain $(2a+1)\times n$ chessboards admit no closed $(a, a+1)$-knight's tours where $n = 4a+4, n = 6a+6$ and $n = 6a+8$ which are obtained in Theorems 3.1, 3.2 and 3.3, respectively.

Theorem 3.1. *Suppose that* a *is a positive integer. Then, the* $(2a+1)\times(4a+4)$ *chessboard admits no closed* $(a, a+1)$*-knight's tours.*

Proof. If $a = 1$, then the 3×8 chessboard contains no closed $(1, 2)$-knight's tours by Theorem 1.1. If $a = 2$, then the 5×12 chessboard contains no closed $(2, 3)$-knight's tours by Proposition 1.2.

Let $a \geq 3$ and G be a $(2a+1)\times(4a+4)$ knight's graph. We prove by contradiction. Assume that G contains a Hamiltonian cycle C. Then, we will show that there is a vertex that has only one edge of C incident to it. Consider the following steps.

Step 1: Since vertex $(a+1, a+1)$ belongs to B_5^2 of Lemma 2.3, it is a degree-2 vertex adjacent to vertices $(1, 2a+2)$ and $(2a+1, 2a+2)$. Then, edge $(a+1, a+1)(1, 2a+2)$ must be included in C.

Step 2: Since vertices $(1, 2)$ and $(2, 1)$ belong to B_1^2 of Lemma 2.3, they are degree-2 vertices adjacent to vertex $(a+2, a+2)$. Then, edges $(2, 1)(a+2, a+2)$ and $(1, 2)(a+2, a+2)$ must be included in C. Since vertex $(a+2, a+2)$ belongs to B_1^4, edge $(a+2, a+2)(1, 2a+2)$ cannot be included in C.

Step 3: Since vertices $(1, 4a+2)$ and $(2, 4a+3)$ belong to B_3^2 of Lemma 2.3, they are degree-2 vertices adjacent to vertex $(a+2, 3a+2)$. Then, edges $(1, 4a+2)(a+2, 3a+2)$ and $(2, 4a+3)(a+2, 3a+2)$ must be included in C. Since vertex $(a+2, 3a+2)$ belongs to B_1^4, edge $(1, 2a+2)(a+2, 3a+2)$ cannot be included in C.

Step 4: Since vertices $(1, 4a+4)$ and $(2a+1, 4a+4)$ belong to B_3^2 and B_4^2 of Lemma 2.3, respectively, they are degree-2 vertices adjacent to vertex $(a+1, 3a+3)$. Then, edges $(1, 4a+4)(a+1, 3a+3)$ and $(2a+1, 4a+4)(a+1, 3a+3)$ must be included in C. Since vertex $(1, 2a+2)$ is adjacent to $(a+1, 3a+3)$, edge $(1, 2a+2)(a+1, 3a+3)$ cannot be included in C.

Since vertex $(1, 2a+2)$ belongs to B_1^4 of Lemma 2.3, it is a degree-4 vertex adjacent to vertices $(a+1, a+1), (a+2, a+2), (a+2, 3a+2)$ and $(a+1, 3a+3)$. By Steps 2, 3 and 4, edges $(1, 2a+2)(a+2, a+2), (1, 2a+2)(a+2, 3a+2)$ and $(1, 2a+2)(a+1, 3a+3)$ are not in C. By Step 1, only one edge $(a+1, a+1)(1, 2a+2)$ incident to vertex $(1, 2a+2)$ is in C (see Figure 4), which is a contradiction. □

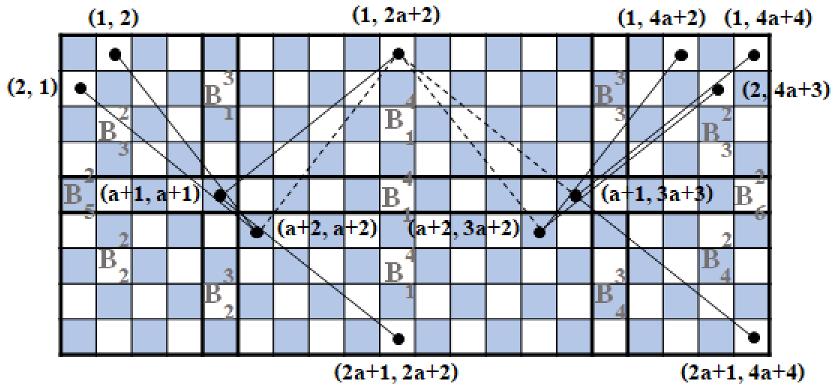

Figure 4: The only one edge incident to $(1, 2a+2)$ containing in C in the proof of Theorem 3.1.

Theorem 3.2. *Suppose that $a > 3$. Then, the $(2a+1) \times (6a+6)$ chessboard admits no closed $(a, a+1)$-knight's tours.*

Proof. Let G be a $(2a+1) \times (6a+6)$ knight's graph where $a > 3$. We prove by contradiction. Assume that G contains a Hamiltonian cycle C. Then, we will show that there are three edges of C incident to a vertex. Consider the following steps.

Step 1: Since vertex $(a+1, a+2)$ belongs to B_1^4 of Lemma 2.3, it is a degree-4 vertex adjacent to vertices $(1,1), (2a+1,1), (2a+1, 2a+3)$ and $(1, 2a+3)$. Since vertices $(1,1)$ and $(2a+1,1)$ belong to B_1^2 and B_2^2 of Lemma 2.3, respectively, such two vertices are degree-2 vertices. Then, the path $(1,1)(a+1,a+2)(2a+1,1)$ must be a part of C. Thus, edges $(2a+1,2a+3)(a+1,a+2)$ and $(1,2a+3)(a+1,a+2)$ cannot be included in C.

Step 2: Since vertex $(a, a+3)$ belongs to B_1^4 of Lemma 2.3, it is a degree-4 vertex adjacent to vertices $(2a,2), (2a+1,3), (2a,2a+4)$ and $(2a+1,2a+3)$. Since vertices $(2a,2)$ and $(2a+1,3)$ belong to B_2^2 of Lemma 2.3, they are degree-2 vertices. Then, the path $(2a,2)(a,a+3)(2a+1,3)$ must be a part of C. Thus, edges $(2a,2a+4)(a,a+3)$ and $(2a+1,2a+3)(a,a+3)$ cannot be included in C.

Step 3: Since vertex $(2a+1, 2a+3)$ belongs to B_1^4 of Lemma 2.3, it is a degree-4 vertex adjacent to vertices $(a+1,a+2), (a,a+3), (a+1,3a+4)$ and $(a,3a+3)$. By Steps 1 and 2, edges $(2a+1,2a+3)(a+1,a+2)$ and $(2a+1,2a+3)(a,a+3)$ cannot be included in C. Then, edges $(2a+1,2a+3)(a,3a+3)$ must be included in C. Also, edge $(2a+1,2a+3)(a+1,3a+4)$ must be included in C.

Step 4: Since vertex $(a, 5a+5)$ belongs to B_1^4 of Lemma 2.3, it is a degree-4 vertex adjacent to vertices $(2a,6a+6), (2a+1,6a+5), (2a,4a+4)$ and $(2a+1,4a+5)$. Since vertices $(2a,6a+6)$ and $(2a+1,6a+5)$ belong to B_4^2 of Lemma 2.3, they are vertices of degree 2. Then, the path $(2a,6a+6)(a,5a+5)(2a+1,6a+5)$ must

be a part of C. Thus, edges $(2a, 4a+4)(a, 5a+5)$ and $(2a+1, 4a+5)(a, 5a+5)$ cannot be included in C.
Step 5: Since vertex $(a-1, 5a+4)$ belongs to B_1^4 of Lemma 2.3, it is a degree-4 vertex adjacent to vertices $(2a-1, 6a+5), (2a, 6a+4), (2a-1, 4a+3)$ and $(2a, 4a+4)$. Since vertices $(2a-1, 6a+5)$ and $(2a, 6a+4)$ belong to B_4^2 of Lemma 2.3, they are vertices of degree 2. Then, the path $(2a-1, 6a+5)(a-1, 5a+4)(2a, 6a+4)$ must be a part of C. Thus, edges $(a-1, 5a+4)(2a-1, 4a+3)$ and $(a-1, 5a+4)(2a, 4a+4)$ cannot be included in C.
Step 6: Since vertex $(2a, 4a+4)$ belongs to B_1^4 of Lemma 2.3, it is a degree-4vertex adjacent to vertices $(a-1, 5a+4), (a, 5a+5), (a, 3a+3)$ and $(a-1, 3a+4)$. By Steps 4 and 5, edges $(2a, 4a+4)(a, 5a+5)$ and $(2a, 4a+4)(a-1, 5a+4)$ cannot be included in C. Thus, edge $(2a, 4a+4)(a, 3a+3)$ must be included in C. Also, edge $(2a, 4a+4)(a-1, 3a+4)$ must be included in C.
Step 7: Since vertex $(a+1, 5a+4)$ belongs to B_1^4 of Lemma 2.3, it is a degree-4 vertex adjacent to vertices $(2a+1, 6a+5), (1, 6a+5), (2a+1, 4a+3)$ and $(1, 4a+3)$. Since vertices $(2a+1, 6a+5)$ and $(1, 6a+5)$ belong to B_4^2 and B_3^2 of Lemma 2.3, respectively, they are vertices of degree 2. Then, the path $(2a+1, 6a+5)(a+1, 5a+4)(1, 6a+5)$ must be a part of C. Thus, edges $(a+1, 5a+4)(2a+1, 4a+3)$ and $(a+1, 5a+4)(1, 4a+3)$ cannot be included in C.
Step 8: Since vertex $(a, 5a+3)$ belongs to B_1^4 of Lemma 2.3, it is a degree-4 vertex adjacent to vertices $(2a, 6a+4), (2a+1, 6a+3), (2a, 4a+2)$ and $(2a+1, 4a+3)$. Since vertices $(2a, 6a+4)$ and $(2a+1, 6a+3)$ belong to B_4^2 of Lemma 2.3, they are vertices of degree 2. Then, the path $(2a, 6a+4)(a, 5a+3)(2a+1, 6a+3)$ must be a part of C. Thus, edges $(a, 5a+3)(2a, 4a+2)$ and $(a, 5a+3)(2a+1, 4a+3)$ cannot be included in C.
Step 9: Since vertex $(2a+1, 4a+3)$ belongs to B_1^4 of Lemma 2.3, it is a degree-4 vertex adjacent to vertices $(a, 5a+3), (a+1, 5a+4), (a, 3a+3)$ and $(a+1, 3a+2)$. By Steps 7 and 8, edges $(a+1, 5a+4)(2a+1, 4a+3)$ and $(a, 5a+3)(2a+1, 4a+3)$ cannot be included in C. Thus, edge $(2a+1, 4a+3)(a, 3a+3)$ must be included in C. Also, edge $(2a+1, 4a+3)(a+1, 3a+2)$ must be included in C.

By Steps 3, 6 and 9, respectively, edges $(2a+1, 2a+3)(a, 3a+3), (2a, 4a+4)(a, 3a+3)$ and $(2a+1, 4a+3)(a, 3a+3)$ must be in the Hamiltonian cycle C. Those three edges are incident to vertex $(a, 3a+3)$ (see Figure 5), which is a contradiction. □

Theorem 3.3. *Suppose that* $a > 3$. *Then, the* $(2a+1) \times (6a+8)$ *chessboard admits no closed* $(a, a+1)$*-knight's tours.*

Proof. Let G be a $(2a+1) \times (6a+8)$ knight's graph. We prove by contradiction. Suppose that G contains a Hamiltonian cycle C. Then, we will show that there are three edges of C incident to a vertex. Consider the following steps.
Step 1: Since vertex $(2a+1, 2)$ belongs to B_2^2 of Lemma 2.3, it is a degree-2 vertex adjacent to vertices $(a+1, a+3)$ and $(a, a+2)$. Thus, edge $(2a+1, 2)(a+1, a+3)$ must be included in C.
Step 2: Since vertex $(1, 2)$ belongs to B_1^2 of Lemma 2.3, it is a degree-2 vertex

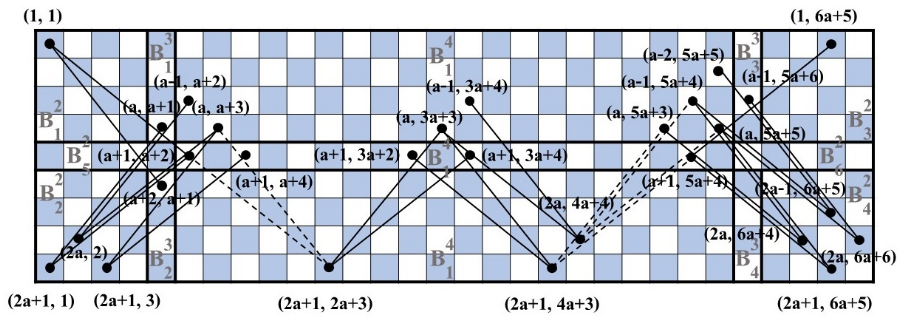

Figure 5: The three edges incident to $(a, 3a + 3)$ containing in C in the proof of Theorem 3.2.

adjacent to vertices $(a+1, a+3)$ and $(a+2, a+2)$. Thus, edge $(1, 2)(a+1, a+3)$ must be included in C.

Step 3: Since vertex $(a + 1, a + 3)$ belongs to B_1^4 of Lemma 2.3, it is a degree-4 vertex adjacent to vertices $(1, 2), (2a + 1, 2), (2a + 1, 2a + 4)$ and $(1, 2a + 4)$. By Steps 1 and 2, edges $(2a+1, 2)(a+1, a+3)$ and $(1, 2)(a+1, a+3)$ must be included in C. Thus, edge $(a + 1, a + 3)(2a + 1, 2a + 4)$ cannot be included in C.

Step 4: Since vertex $(2a, 3)$ belongs to B_2^2 of Lemma 2.3, it is a degree-2 vertex adjacent to vertices $(a, a + 4)$ and $(a - 1, a + 3)$. Thus, edge $(2a, 3)(a, a + 4)$ must be included in C.

Step 5: Since vertex $(2a+1, 4)$ belongs to B_2^2 of Lemma 2.3, it is a degree-2 vertex adjacent to vertices $(a, a + 4)$ and $(a + 1, a + 5)$. Thus, edge $(2a + 1, 4)(a, a + 4)$ must be included in C.

Step 6: Since vertex $(a, a+4)$ belongs to B_1^4 of Lemma 2.3, it is a degree-4 vertex adjacent to vertices $(2a, 3), (2a + 1, 4), (2a + 1, 2a + 4)$ and $(2a, 2a + 5)$. By Steps 4 and 5, edges $(2a, 3)(a, a + 4)$ and $(2a + 1, 4)(a, a + 4)$ must be included in C. Thus, edge $(a, a + 4)(2a + 1, 2a + 4)$ cannot be included in C.

Step 7: Since vertex $(2a + 1, 2a + 4)$ belongs to B_1^4 of Lemma 2.3, it is a degree-4 vertex adjacent to vertices $(a + 1, a + 3), (a, a + 4), (a, 3a + 4)$ and $(a + 1, 3a + 5)$. By Steps 3 and 6, edges $(a + 1, a + 3)(2a + 1, 2a + 4)$ and $(a, a + 4)(2a + 1, 2a + 4)$ cannot be included in C. Thus, edge $(2a+1, 2a+4)(a+1, 3a+5)$ must be included in C.

Step 8: Since vertex $(2a + 1, 6a + 8)$ belongs to B_4^2 of Lemma 2.3, it is a degree-2 vertex adjacent to vertices $(a+1, 5a+7)$ and $(a, 5a+8)$. Thus, edge $(2a+1, 6a+8)(a + 1, 5a + 7)$ must be included in C.

Step 9: Since vertex $(1, 6a + 8)$ belongs to B_3^2 of Lemma 2.3, it is a degree-2 vertex adjacent to vertices $(a + 1, 5a + 7)$ and $(a + 2, 5a + 8)$. Thus, edge $(1, 6a + 8)(a + 1, 5a + 7)$ must be included in C.

Step 10: Since vertex $(a + 1, 5a + 7)$ belongs to B_1^4 of Lemma 2.3, it is a degree-4 vertex adjacent to vertices $(1, 6a+8), (2a+1, 6a+8), (2a+1, 4a+6)$ and $(1, 4a+6)$.

By Steps 8 and 9, edges $(2a+1, 6a+8)(a+1, 5a+7)$ and $(1, 6a+8)(a+1, 5a+7)$ must be included in C. Thus, edges $(a+1, 5a+7)(2a+1, 4a+6)$ and $(a+1, 5a+7)(1, 4a+6)$ cannot be included in C.

Step 11: Since vertex $(2a+1, 6a+6)$ belongs to B_4^2 of Lemma 2.3, it is a degree-2 vertex adjacent to vertices $(a, 5a+6)$ and $(a+1, 5a+5)$. Thus, edge $(2a+1, 6a+6)(a, 5a+6))$ must be included in C.

Step 12: Since the vertex $(2a, 6a+7)$ belongs to B_4^2 of Lemma 2.3, it is a degree-2 vertex adjacent to vertices $(a, 5a+6)$ and $(a-1, 5a+7)$. Thus, edge $(2a, 6a+7)(a, 5a+6)$ must be included in C.

Step 13: Since vertex $(a, 5a+6)$ belongs to B_1^4 of Lemma 2.3, it is a degree-4 vertex adjacent to vertices $(2a+1, 6a+6), (2a, 6a+7), (2a+1, 4a+6)$ and $(2a, 4a+5)$. By Steps 11 and 12, edges $(2a+1, 6a+6)(a, 5a+6)$ and $(2a, 6a+7)(a, 5a+6)$ must be included in C. Thus, edge $(a, 5a+6)(2a+1, 4a+6)$ cannot be included in C.

Step 14: Since vertex $(2a+1, 4a+6)$ belongs to B_1^4 of Lemma 2.3, it is a degree-4 vertex adjacent to vertices $(a+1, 5a+7), (a, 5a+6), (a, 3a+6)$ and $(a+1, 3a+5)$. By Steps 10 and 13, edges $(a+1, 5a+7)(2a+1, 4a+6)$ and $(a, 5a+6)(2a+1, 4a+6)$ cannot be included in C. Thus, edge $(2a+1, 4a+6)(a+1, 3a+5)$ must be included in C.

Step 15: Since vertex $(1, 6a+6)$ belongs to B_3^2 of Lemma 2.3, it is a degree-2 vertex adjacent to vertices $(a+1, 5a+5)$ and $(a+2, 5a+6)$. Thus, edge $(1, 6a+6)(a+2, 5a+6)$ must be included in C.

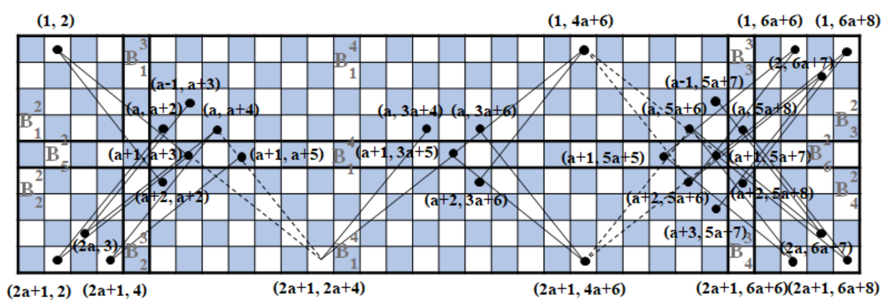

Figure 6: The three edges incident to $(a+1, 3a+5)$ containing in C in the proof of Theorem 3.3.

Step 16: Since vertex $(2, 6a+7)$ belongs to B_3^2 of Lemma 2.3, it is a degree-2 vertex adjacent to vertices $(a+3, 5a+7)$ and $(a+2, 5a+6)$. Thus, edge $(2, 6a+7)(a+2, 5a+6)$ must be included in C.

Step 17: Since vertex $(1, 4a+6)$ belongs to B_1^4 of Lemma 2.3, it is a degree-4 vertex adjacent to vertices $(a+1, 5a+7), (a+2, 5a+6), (a+2, 3a+6)$ and $(a+1, 3a+5)$. By Step 10, edge $(1, 4a+6)(a+1, 5a+7)$ cannot be included in C. By Steps 15 and 16, the edge $(1, 4a+6)(a+2, 5a+6)$ cannot be included in C. Thus, edge $(1, 4a+6)(a+1, 3a+5)$ must be included in C.

By Steps $7, 14$ and 17, we see that edges $(2a+1, 2a+4)(a+1, 3a+5)$, $(2a+1, 4a+6)(a+1, 3a+5)$ and $(1, 4a+6)(a+1, 3a+5)$ must be in C and those edges are incident to vertex $(a+1, 3a+5)$ (see Figure 6), which is a contradiction. $\square$

4 Conclusion

In this paper, we consider $m = a + b$ which is the smallest value of m for the $m \times n$ chessboard admitting a closed (a, b)-knight's tour. We show that (i) the $(a+b) \times n$ chessboard admits no closed (a, b)-knight's tours if $n \in [2b+1, 4b-1]$ where $1 \leq a < b$ in Theorem 2.5, or if $n \in [4b+1, 5b]$ where $1 \leq a < b < 2a$ in Theorem 2.6, and (ii) the $(2a+1) \times n$ chessboard admits no closed $(a, a+1)$-knight's tours if $n = 4a+4$ where $a \geq 1$ in Theorem 3.1, or if $n = 6a+6$ where $a > 3$ in Theorem 3.2, or if $n = 6a+8$ where $a > 3$ in Theorem 3.3. Moreover, we see that Theorem 3.1 is the special case of $n = 4b$ where $b = a+1$. Then, the remaining cases are still for other researchers to explore.

Acknowledgements : I would like to thank the referees for their comments and suggestions on the manuscript.

References

[1] J.J. Watkins, Across the boards: The mathematics of the chessboard problems, Princeton University, New Jersey, 2004.

[2] A.L. Schwenk, Which rectangular chessboards have a knight's tour, Math. Magazine 64 (1991), 325–332.

[3] G.L. Chia, Siew-Hui Ong, Generalized knight's tours on rectangular chessboards, Discrete Applied Math. 150 (2005), 80–89.

[4] K. Karudilok, S. Singhun, R. Boonklurb, Some forbidden rectangular chessboards with an (a, b)-Knight's move, The Proceeding of the 22^{nd} Annual Meeting in Mathematics (AMM2017), Chiang Mai University, Chiang Mai, Thailand (2017), GRA-01-1–GRA-01-6.

(Received 10 June 2019)
(Accepted 24 December 2019)