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Attractor of a Shallow Water Equations Model

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Abstract : In this research, attractors in a shallow water equations model, STSWM, are investigated. The shallow water equations model is a simple model that describes fluid flow and is widely used in many fields including atmospheric and oceanographic studies. The shallow water equations model can sometimes behave in a chaotic manner. The time series, the power spectrum, Lyapunov exponent and reconstruction of the attractor are used to distinguish between chaotic and nonchaotic signal to determine the main frequencies of periodic oscillators.

1 Introduction

Fluid flow is a process that can be found at any place and at any time. Examples are the flows of water in rivers, lakes, and oceans. Another important example is atmospheric flow which causes winds. Shallow water equations model describes fluid flow and is widely used in many fields including atmospheric and oceanographic studies. For large scale flow such as circulations of the atmosphere and oceans, spherical shape of the earth must be taken into account when formulating the equations. This results in spherical shallow water equations, which is important for atmospheric and oceanic numerical model development and applications. For examples, it is a basic model for atmospheric prediction, climate change study and pollution dispersion in the atmosphere. Although simplified from the full set of equations, the shallow water equations still maintain an important property, nonlinear behavior. As a consequence, the shallow water equations can sometimes behave in a chaotic manner, that is unpredictable. In order to gain more understanding about chaos in the system, a main characteristic of chaos in a spherical shallow water equations, attractor, is investigated in this study. There are many measurements of chaotic behavior. In this research some measurements are discussed. Chaos is aperiodic long-term behavior in a deterministic system that exhibits sensitive dependence on initial condition. Aperiodic long-term behavior means that the trajectories do not converge to fixed points as $t \to \infty$. Deterministic means that the system at the present (initial state) completely determines the future (the forward orbit of the state). Sensitive dependence on initial condition means that a minor change in the initial state leads to dramatically different long term behavior.

2 Model

The experiments in this research are performed using a spectral transform shallow water equations model, STSWM (CHAMMP, 1992).

2.1 Governing equations and solution method

In vector form, the horizontal momentum and mass continuity equations are written as

$$\frac{d\mathbf{V}}{dt} = -f\mathbf{k} \times \mathbf{V} - \nabla\Phi \tag{2.1}$$

and

$$\frac{d\Phi}{dt} = -\Phi\nabla \cdot \mathbf{V} \tag{2.2}$$

where

$\mathbf{V} \equiv \mathbf{i}u + \mathbf{j}v$ is the horizontal (with respect to the surface of the sphere) vector velocity

$\Phi \equiv gh$ is the free surface geopotential

h is the free surface height

g is the acceleration of gravity

$f \equiv 2\Omega \sin\varphi$ is the Coriolis parameter

φ is latitude

Ω is the angular velocity of the earth.

The derivative is given by

$$\frac{d}{dt}() = \frac{\partial}{\partial t}() + (\mathbf{V} \cdot \nabla)() \tag{2.3}$$

and the ∇ operator is defined in spherical coordinates as

$$\nabla() \equiv \frac{\mathbf{i}}{a\cos\varphi}\frac{\partial}{\partial\varphi}() + (\mathbf{V} \cdot \nabla)() \tag{2.4}$$

where λ denotes longitude

a is the radius of the earth.

3 Measurement of chaos

There are many measurements of chaos. In this research, the following measurements are applied.

3.1 Time series

A time series is a sequence of data points, measured typically at successive times, spaced at (often uniform) time intervals. Time series analysis comprises methods that attempt to understand such time series, often either to understand the underlying theory of the data points, or to make forecasts

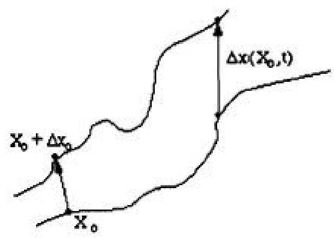


Figure 1: Lyapunov exponent

3.2 Power spectrum

Periodic signals give peaks at a fundamental frequency and its harmonics; quasiperiodic signals give peaks at linear combinations of two or more irrationally related frequencies; and chaotic dynamics give broad band components to the spectrum. The later may be used as a criterion for identifying the dynamics as chaotic. The power spectrum or power density

$$P(f) = \left| \int_{-\infty}^{\infty} x(t) e^{i} 2\pi f t dt \right|^2 \tag{3.1}$$

of the signal $x(t)$ as a function of frequency f can detect the presence of harmonics and sub harmonics as suitable peaks while a continuous broadband spectrum indicates the presence of chaotic motions.

3.3 Lyapunov exponent

The Lyapunov exponent is the rate of divergence or convergence of nearby trajectories in different directions in the phase space.

Consider two points in a space, X_0 and $X_0 + \Delta x_0$, each of which will generate an orbit in that space using some equation or system of equations. Because sensitive dependence can arise only in some portions of a system, this separation is also a function of the location of the initial value and has the form $\Delta x(X_0, t)$. For chaotic points, the function $\Delta x(X_0, t)$ will behave erratically. It is thus useful to study the mean exponential rate of divergence of two initially close orbits using the formula

$$\lambda = \lim_{t \to \infty} \frac{1}{t} \ln \left(\frac{|\Delta x\,(X_0, t)|}{|\Delta x_0|} \right) \tag{3.2}$$

This number,λ, called the Lyapunov exponent is useful for distinguishing among the various types of orbits.

The exponent can be used to identify whether the motion is periodic or chaotic by consider the largest nonzero Lyapunov exponents λ : if $\lambda < 0$, then the motion is

periodic and if $\lambda > 0$, then the motion is chaotic.
A positive Lyapunov exponent indicates a divergence between nearby trajectories, that is, a high sensitive to initial conditions.

3.4 Reconstruction of the attractor

Phase reconstruction of a chaos is an important step for the study of chaotic time series. From an observed time series $x(t)$, a data vector is generated as $Y(t) = F(x(t), x(t-T), , x(t-(D-1)T))$ where T is the time delay, the $Y(t)$ indicates one point of a D-dimensional reconstructed phase space $R^D : D$ is called the embedding dimension. When any attractor appears in the original dynamical systems, another attractor, which also retains the phase structure of the first attractor, will appear in the reconstructed state space. Attractor reconstruction refers to methods for inference of geometrical and topological about a dynamical attractor from observation.

4 The experiment

In this research, running the spectral transform shallow water model,STSWM, (CHAMMP, 1992) on Linux. The First test cases in the standard test set from Williamson et al. are implemented in the STSWM.
Williamson's First Case: Advection of Cosine Bell over the Pole. First case tests the advective component of the shallow water equations,

$$\frac{\partial}{\partial t}(h) + \frac{u}{a\cos\theta}\frac{\partial}{\partial\lambda}(h) + \frac{v}{a}\frac{\partial}{\partial\theta}(h) + \frac{h}{a\cos\theta}\left[\frac{\partial u}{\partial\lambda} + \frac{\partial(v\cos\theta)}{\partial\theta}\right] = 0 \tag{4.1}$$

The solid body rotation is given by

$$u = u_0(\cos\theta\cos\alpha + \sin\theta\cos\lambda\sin\alpha)$$

$$v = -u_0\sin\lambda\sin\alpha$$

where $u_0 \approx 40\ m/s$, is the advecting wind velocity, $a = 6.37122 \times 10^6 m$ (the mean radius of the earth) and α is the angle between the axis of solid body rotation and the polar axis of the spherical coordinate system, θ is latitude and λ is longitude. The initial cosine bell test pattern to be advected is given by

$$h(\lambda,\theta) = \begin{cases} (\frac{h_0}{2})(1+\cos\frac{\pi r}{R})) & ; \quad r < R \\ 0 & ; \quad r \geq R \end{cases} \tag{4.2}$$

where $h_0 = 1000\ m$, $R = \frac{a}{3}$, r is the great circle distance between (λ,θ) and the center, $(\lambda_c,\theta_c) = \left(\frac{3\pi}{2}, 0\right)$.

Experiments in this study are summarized in Table1.

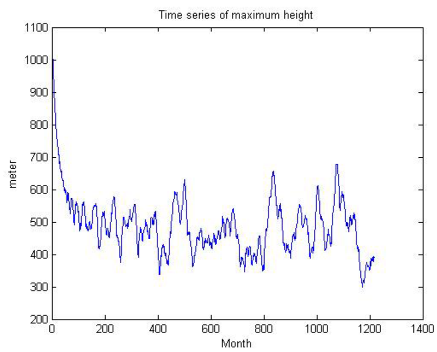


Figure2. Time series for Ex1.

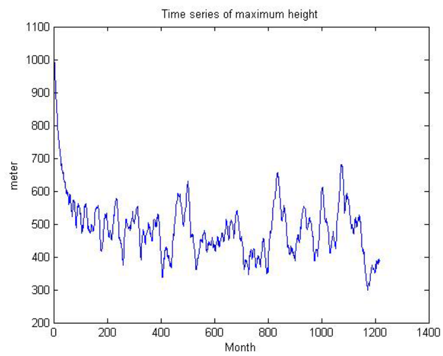


Figure3. Time series for Ex2.

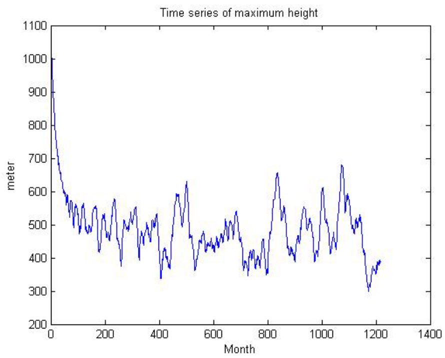


Figure4. Time series for Ex3.

Table1. Parameters for the experiments

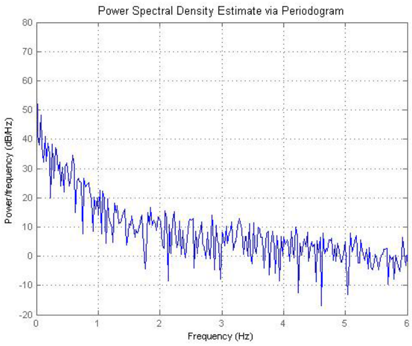
Experiment	α	Simulation length (hrs)	Output interval(hrs)
Ex1	0.0	875520	720
Ex2	0.5	875520	720
Ex3	$\frac{\pi}{2} - 0.5$	875520	720

5 Result

From the experiments, the maximum height of the model is discussed.

5.1 Time series

Figures 2-4 . show time series for Ex1-Ex3. There are no period in the time series. It indicates that the solutions are aperiodic. According to Moon (2004), this implies chaotic behavior.



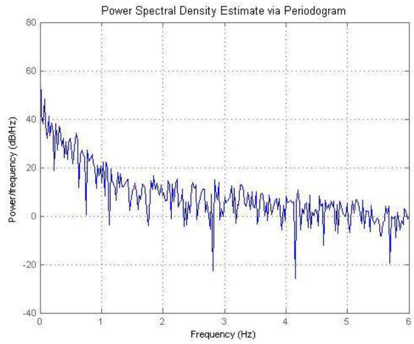


Figure5.Power spectrum density for Ex1. Figure6.Power spectrum density for Ex2.

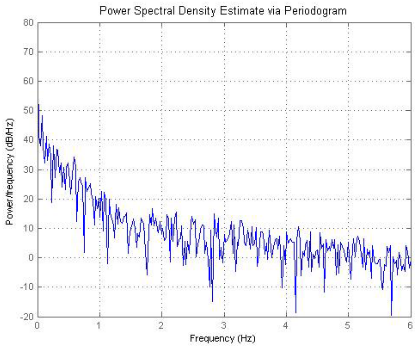


Figure7.Power spectrum density for Ex3.

5.2 Power spectrum

Figures 5-7 show the power spectrum density for Ex1-Ex3. The pattern of graph are a continuous broadband spectrum. This indicates the presence of chaotic motion (Hassouba,2006).

5.3 Lyapunov exponent

Figures 8-10 shows the Lyapunov exponent for Ex1-Ex3. The Lyapunov exponents are positive values which indicate divergence between nearby trajectories, that is, a high sensitive to initial conditions (Zeng,1991).

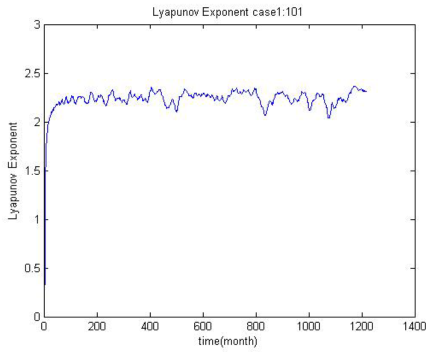


Figure8.Lyapunov exponent for Ex1.

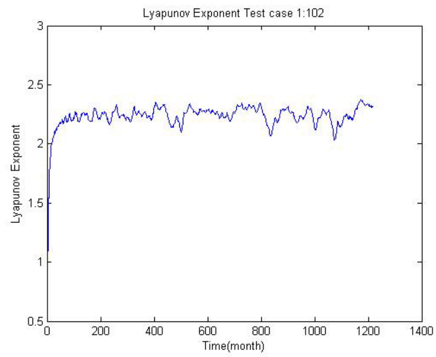


Figure9.Lyapunov exponent for Ex2.

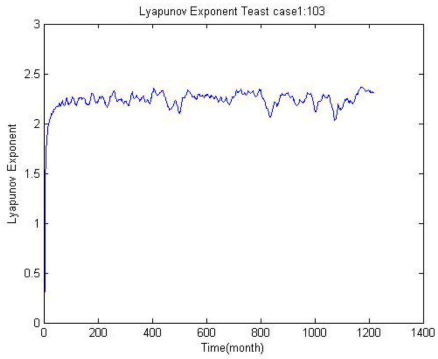


Figure10.Lyapunov exponent for Ex3.

5.4 Reconstruction of the attractor

Figures 11-13(a) show the attractor reconstruction in three dimensional state space. The maximum height decrease during the first 300 time-step, after that it shows no prefer direction with finite boundary between 400 to 600. It indicates that the first 300 time-steps is stable while the motion after 300 time-step is chaotic. To consider chaotic behavior, the first 300 times-steps which is stable is neglected in the plot as shown in Figures11-13 (b) (Silva, 2002). Figure11(a)Attractor reconstruction in three dimensional state space for Ex1. 11(b)Attractor reconstruction in three dimensional state space for Ex1 after 300 time steps. Figure12(a)Attractor reconstruction in three dimensional state space for Ex2. 12(b)Attractor reconstruction in three dimensional state space for Ex2 after 300 time steps. Figure13(a)Attractor reconstruction in three dimensional state space for Ex2. 13(b)Attractor reconstruction in three dimensional state space for Ex2 after 300 time steps.

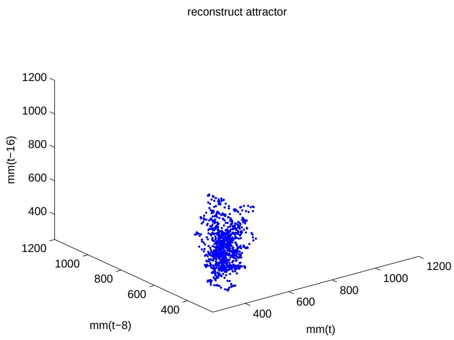


Figure11(a)

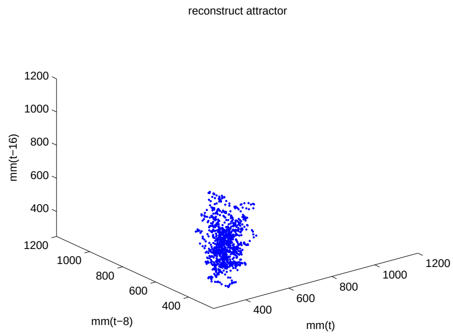


Figure11(a)

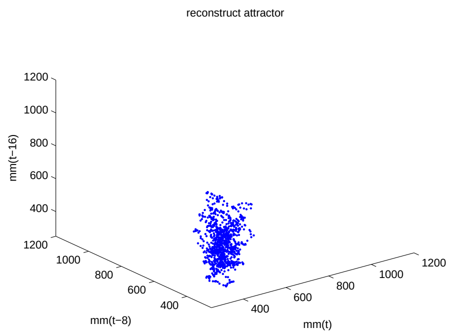


Figure12(a)

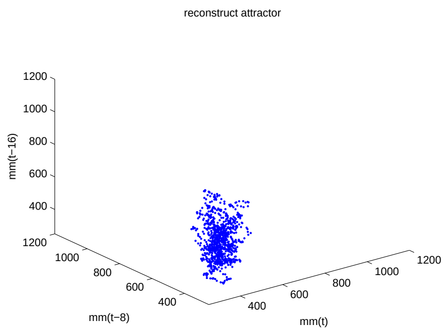


Figure12(a)

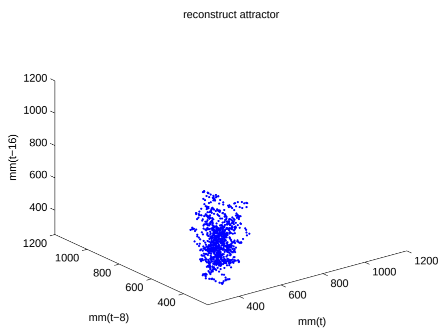


Figure13(a)

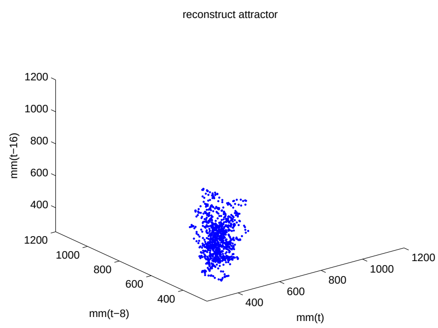


Figure13(a)

6 Conclusion

In this research, numerical experiments of the shallow water equation model STSWM are performed and some of measurements of chaotic behavior are investigated. From the experiments, each measurement of chaotic indicates that the shallow water equations model is chaotic. The reconstruction of the attractor method can be used to show geometry of attractors of the model.

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