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On Δ-Convergence Theorems in b-CAT(0) Spaces

Cholatis Suanoom

Program of Mathematics, Faculty of Science and Technology,
Kamphaengphet Rajabhat University, Kamphaengphet 62000,Thailand.
e-mail : cholatis.suanoom@gmail.com

Abstract : In this work, we extend and improve CAT(0) spaces to b-CAT(0) spaces by using the concept of b-metric spaces. Second, we establish new spaces, that is b-CAT(0) spaces and prove the results of a fixed point for non-expansive mappings on b-CAT(0) spaces. Moreover, we obtain Δ-convergence theorems for non-expansive mappings on b-CAT(0) spaces and present some properties.

Keywords : b-CAT(0); $b-CN$-inequality; $b-CN_p$-inequality; Δ-convergence; non-expansive.
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1 Introduction and Preliminaries

A metric space X is a CAT(0) space if it is geodesically connected, and if every geodesic triangle in X is at least as "thin" as its comparison triangle in the Euclidean plane, introduced by Gromov [1]. It is well-known that any complete, simply connected Riemannian manifold having non positive sectional curvature is a CAT(0) space. Other examples include the classical hyperbolic spaces, Euclidean buildings (see [2]), the complex Hilbert ball with a hyperbolic metric (see [3]), and many others. The Δ-convergence in a general metric space setting is introduced by Lim [4] in 1976. In 2008, Dhompongsa and Panyanak [5] proved Δ-convergence theorems in CAT(0) spaces by using the concept of Δ-convergence introduced by Lim [4], and gived the CAT(0) space analogs of results on weak convergence of the Picard, Mann and Ishikawa iterates proved in uniformly convex Banach spaces by Opial [6], Ishikawa [7] and Tan and Xu [8]. In 2017, Khamsi and Shukri [9] extended the Gromov geometric definition of CAT(0) spaces to the case where the comparison triangles are not in the Euclidean plane but belong to a general Banach space. In particular, many other authors studied the case where the Banach space is l_p, for $p > 2$.

Next, Bakhtin [10] and Czerwik [11] developed the notion of b-metric spaces and established some fixed point theorems in b-metric spaces in 1989. Subsequently, several results appeared in this direction ([14]-[21]) as follows:

Definition 1.1 ([11])**.** A b-metric on a set X is a mapping $d : X \times X \to [0, +\infty)$ satisfying the following conditions: for any $x, y, z \in X$,
(b_1) $d(x, y) = 0$ if and only if $x = y$;
(b_2) $d(x, y) = d(y, x)$;
(b_3) there exists $s \geq 1$ such that $d(x, y) \leq s(d(x, z) + d(z, y))$.
Then (X, d) is known as a b-metric space with coefficient s.

Note that every metric space is a b-metric space with $s = 1$. Some examples of b-metric space are given below: Let $\mathbb{R}$ be a vecter space. Define a mapping $d : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ by

$$d(x, y) = |x - y|^p$$

for all $x, y \in X,\ p = 2, 3, ...$. Then $(\mathbb{R}, |\cdot|)$ is a b-metric space with coefficient $s = 2^{p-1}$.

After that, Al-Saphory, Al-Janabi and Al-Delfi [22] introduced a quasi-Banach space as follows:

Definition 1.2. Let X be a real linear space. A quasi-norm is a real-valued function on X satisfying the following:
(qb_1) $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$,
(qb_2) $\|\lambda x\| = |\lambda| \, \|x\|$ for all $\lambda \in \mathbb{R}$ and all $x \in X$,
(qb_3) There is a constant $K \geq 1$ such that $\|x + y\| \leq K(\|x\| + \|y\|)$ for all $x, y \in X$.
The pair $(X, \|\cdot\|)$ is called a quasi-normed space if $\|\cdot\|$ is a quasi-norm on X.
A quasi-normed $\|\cdot\|$ is called a p-norm $(0 < p \leq 1)$ if
$\|x + y\|^p \leq \|x\|^p + \|y\|^p$ for all $x, y \in X$. In this case, a quasi-normed spaces (quasi-Banach space) is called a p-quasi-normed spaces (p-quasi-Banach space).
Note that every a Banach space is a quasi-Banach space with $K = 1$ and every a quasi-Banach space is a b-metric space with $d(x, y) := \|x - y\|$ and coefficient $s = K$.

In this work, we extend and improve CAT(0) spaces to b-CAT(0) spaces by using the concept of b-metric spaces. Second, we establish new spaces, that is b-CAT(0) spaces and prove the results of a fixed point for non-expansive mappings on b-CAT(0) spaces. Moreover, we obtain Δ-convergence theorems for non-expansive mappings on b-CAT(0) spaces and present some properties.

2 Main Results

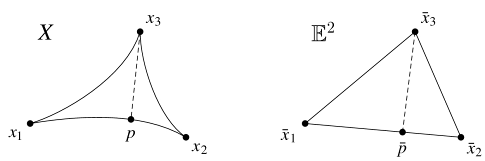
Let (X,d) be a b-metric space with $s \geq 1$. A b-geodesic joining $x \in X$ to $y \in X$ is a continuous mapping $\gamma : [0,d(x,y)] \to X$ such that

- $\gamma(0) = x$,
- $\gamma(d(x,y)) = y$,
- $d(\gamma(t_1), \gamma(t_2)) = |t_1 - t_2|$ for any $t_1, t_2 \in [0, d(x,y)]$.

We will say that (X,d) is a (uniquely) b-geodesic metric space if any two points are connected by a (unique) b-geodesic. In this case, we denote such geodesic by $[x,y]$. Note that in general such b-geodesic is not uniquely determined by its endpoints. For a point $z \in [x,y]$, we will use the notation $z = (1-t)x \oplus ty$, where $t = \frac{d(x,z)}{d(x,y)}$, $1 - t = \frac{d(y,z)}{d(x,y)}$ assuming $x \neq y$. Let (X,d) be a b-geodesic metric space with $s \geq 1$. A b-geodesic triangle consists of three point $p,q,r \in X$ and three geodesics $[p,q],[q,r],[r,p]$. Denote $\Delta([p,q],[q,r],[r,p])$. For such a triangle, there is a comparison triangle $\overline{\Delta}(\overline{p},\overline{q},\overline{r}) \to \mathbb{E}^2 :$ $d(p,q) = d(\overline{p},\overline{q})$, $d(q,r) = d(\overline{q},\overline{r})$, $d(r,p) = d(\overline{r},\overline{p})$.

Definition 2.1. A b-geodesic space is said to be a b-CAT(0) space if all b-geodesic triangles of appropriate size satisfy the following comparison axiom.
b-CAT(0): Let Δ be a b-geodesic triangle in b-metric space X and let $\bar{\Delta} \in \mathbb{E}^2$ be a comparison triangle for Δ. Then Δ is said to satisfy the b-CAT(0) inequality if for all $x,y \in \Delta$ and all comparison points $\overline{x},\overline{y} \in \overline{\Delta}$ such that

$$d(x,y) \leq ||\overline{x} - \overline{y}||.$$



We call $b - CAT_p(0)$ metric spaces, for p-quasi-normed spaces $(\mathbb{E}^2, \|\cdot\|)$.

Example 2.2. (I). Let $X := l_p(\mathbb{R})$ with $0 < p < 1$ where $l_p(\mathbb{R}) := \{\{x_n\} \subset \mathbb{R} : \sum_{i=1}^{\infty} |x_i|^p < \infty\}$. Define $d : X \times X \to [0,\infty)$ as:

$$||x|| = (\sum_{i=1}^{\infty} |x_i|^p)^{\frac{1}{p}}$$

where $x = \{x_n\}$. Then d is a b-metric space with coefficient $s = 2^{p-1}$, see([23] - [25]). And, defined a continuous mapping $\gamma : [0,d(x,y)] \to X$ by $\gamma(z) = (1-t)x + ty$ for all $t \in [0,d(x,y)]$. and all $z \in X$. Then (X,d) is a b-CAT(0) space.

(II). Let $X := L_p[0,1]$ be the space of all real functions $x(t),\ t \in [0,1]$ such that $\int_0^1 |x(t)|^p dt < \infty$ with $0 < p < 1$. Define $d : X \times X \to [0,\infty)$ as:

$$||x|| = (\int_0^1 |x(t)|^p dt)^{\frac{1}{p}}$$

with $0 < p < 1$ where $x = x(t)$. Then d is a b-metric space with coefficient $s = 2^{p-1}$, see([23] -[25]). And, defined a continuous mapping $\gamma : [0, d(x,y)] \to X$ by $\gamma(z) = (1-t)x + ty$ for all $t \in [0, d(x,y)]$. and all $z \in X$. Then (X,d) is a b-CAT(0) space.

Now, we establish lemma about $(b - CN)$ inequality and $(b - CN_p)$ inequality.

Lemma 2.3. *Let (X,d) be a b-CAT(0) metric space. Then for any x, y_1, y_2 in X, we have*

$$d(x, \frac{y_1 \oplus y_2}{2}) \leq Kd(x,y_1) + Kd(x,y_2) - \frac{1}{2}d(y_1,y_2)$$

which we will call the (b-CN) inequality.

Proof. Let x, y_1, y_2 be in X and Δ be the associated geodesic triangle in X. Since X is a b-CAT(0) space, there exists a comparison geodesic triangle Δ. The associated comparison points in $\mathbb{E}^2$ will be denoted by $\overline{x}, \overline{y_1}, \overline{y_2}$. The comparison axiom implies:

$$d(x, \frac{y_1 \oplus y_2}{2}) \leq ||\overline{x} - \frac{\overline{y_1} + \overline{y_2}}{2}||$$

By the inequality of b-CAT(0), we get $||a+b|| + ||a-b|| \leq 2K(||a|| + ||b||)$ for any $a, b \in \mathbb{E}^2$. Applying this inequality for $a = \frac{\overline{x} - \overline{y_1}}{2}$ and $b = \frac{\overline{x} - \overline{y_2}}{2}$, yields:

$$||\frac{\overline{x} - \overline{y_1}}{2} + \frac{\overline{x} - \overline{y_2}}{2}|| + ||\frac{\overline{x} - \overline{y_1}}{2} - \frac{\overline{x} - \overline{y_2}}{2}|| \leq 2K(||\frac{\overline{x} - \overline{y_1}}{2}|| + ||\frac{\overline{x} - \overline{y_2}}{2}||).$$

So,

$$||\frac{\overline{x} - \overline{y_1}}{2} + \frac{\overline{x} - \overline{y_2}}{2}|| \leq 2K(||\frac{\overline{x} - \overline{y_1}}{2}|| + ||\frac{\overline{x} - \overline{y_2}}{2}||) - ||\frac{\overline{x} - \overline{y_1}}{2} - \frac{\overline{x} - \overline{y_2}}{2}||,$$

or,

$$||\overline{x} - \frac{\overline{y_1} + \overline{y_2}}{2}|| \leq K(||\overline{x} - \overline{y_1}|| + ||\overline{x} - \overline{y_2}||) - \frac{1}{2}||\overline{y_1} - \overline{y_2}||.$$

Since $||\overline{y_i} - \overline{y_j}|| = d(y_i.y_j)$, for $i, j \in \{1,2\}$, we get

$$d(x, \frac{y_1 \oplus y_2}{2}) \leq Kd(x,y_1) + Kd(x,y_2) - \frac{1}{2}d(y_1,y_2).$$

□

Note that the $(b - CN)$ inequality coincides with the classical (CN) inequality if $K = 1$. One of the implications of the (CN) inequality is the uniform convexity of the distance of a CAT(0) space.

Next we discuss the $(b - CN_p)$ inequality of the b-CAT$_p(0)$ metric spaces.

Lemma 2.4. *Let (X, d) be a b-$CAT_p(0)$ metric space. Then for any x, y_1, y_2 in X, we have*

$$d^p(x, \frac{y_1 \oplus y_2}{2}) \leq \frac{1}{2^{p-1}} d^p(x, y_1) + \frac{1}{2^{p-1}} d^p(x, y_2) - \frac{1}{2^p} d^p(y_1, y_2),$$

where $0 < p \leq 1$, which we will call the $(b - CN_p)$ inequality.

Proof. Let x, y_1, y_2 be in X and Δ be the associated geodesic triangle in X. Since X is a $\mathrm{CAT}_p(0)$ space, there exists a comparison geodesic triangle Δ. The associated comparison points in $\mathbb{R}$ will be denoted by $\overline{x}, \overline{y_1}, \overline{y_2}$. The comparison axiom implies:

$$d(x, \frac{y_1 \oplus y_2}{2}) \leq ||\overline{x} - \frac{\overline{y_1} + \overline{y_2}}{2}||,$$

which implies

$$d(x, \frac{y_1 \oplus y_2}{2})^p \leq ||\overline{x} - \frac{\overline{y_1} + \overline{y_2}}{2}||^p.$$

By the inequality of b-$\mathrm{CAT}_p(0)$, we get $||a + b||^p + ||a - b||^p \leq 2(||a||^p + ||b||^p)$ for any $a, b \in \mathbb{E}^2$. App blying this inequality for $a = \frac{\overline{x} - \overline{y_1}}{2}$ and $b = \frac{\overline{x} - \overline{y_2}}{2}$, yields:

$$||\frac{\overline{x} - \overline{y_1}}{2} + \frac{\overline{x} - \overline{y_2}}{2}||^p + ||\frac{\overline{x} - \overline{y_1}}{2} - \frac{\overline{x} - \overline{y_2}}{2}||^p \leq 2(||\frac{\overline{x} - \overline{y_1}}{2}||^p + ||\frac{\overline{x} - \overline{y_2}}{2}||^p).$$

So,

$$||\frac{\overline{x} - \overline{y_1}}{2} + \frac{\overline{x} - \overline{y_2}}{2}||^p \leq 2(||\frac{\overline{x} - \overline{y_1}}{2}||^p + ||\frac{\overline{x} - \overline{y_2}}{2}|||^p) - ||\frac{\overline{x} - \overline{y_1}}{2} - \frac{\overline{x} - \overline{y_2}}{2}||^p,$$

or,

$$||\overline{x} - \frac{\overline{y_1} + \overline{y_2}}{2}||^p \leq \frac{1}{2^{p-1}}(||\overline{x} - \overline{y_1}||^p + ||\overline{x} - \overline{y_2}||^p) - \frac{1}{2^p}||\overline{y_1} - \overline{y_2}||^p.$$

Since $||\overline{y_i} - \overline{y_j}|| = d(y_i.y_j)$, for $i, j \in \{1, 2\}$, we get

$$d^p(x, \frac{y_1 \oplus y_2}{2}) \leq \frac{1}{2^{p-1}} d^p(x, y_1) + \frac{1}{2^{p-1}} d^p(x, y_2) - \frac{1}{2^p} d^p(y_1, y_2),$$

where $0 < p \leq 1$. □

We now give the definition and collect some basic properties of the Δ-convergence:

Definition 2.5. Let $\{x_n\}$ be a bounded sequence in a b-metric space X. For $x \in X$, we set

$$r(x, \{x_n\}) = \limsup_{n \to \infty} d(x, x_n).$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\},$$

and the asymptotic center A $(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}$$

A sequence $\{x_n\}$ in X is said to Δ-converge to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case we write $\Delta - \lim_n x_n = x$ and call x the Δ-limit of $\{x_n\}$.

Remark 2.6. *Let $\{x_n\}$ be a bounded sequence in a b-metric space X.*

(i) Every bounded sequence in X has a Δ-convergent subsequence

(ii) If C is a closed convex subset of X and if $\{x_n\}$ is a bounded sequence in C, then the asymptotic center of $\{x_n\}$ is in C .

(iii) If C is a closed convex subset of X and if $f : C \to X$ i a nonexpansive mapping, then the conditions,$\{x_n\}$ Δ-convergestox and $d(x_n, f(x_n)) \to 0$, imply $x \in C$and $f(x) = x$

(iv) If $\{x_n\}$ is a bounded sequence in X with $A(\{x_n\}) = \{x\}$ and $\{u_n\}$ is a subsequence of $\{x\}$ with $A(\{u_n\}) = \{u\}$ and the sequence $\{d(x_n, u)\}$ converges, then $x = u$.

The following lemma is crucial in the study my theorem and it can prove follow as of the proof of Dhompongsa and Panyanak [5]

Lemma 2.7. *Let C be a closed convex subset of a b-CAT(0) space X, followand let $T : C \to X$ be a nonexpansive mapping. Suppose $\{x_n\}$ is a bounded sequence in C such that limn $d(x_n, Tx_n) = 0$ and $d(x_n, v)$ converges for all $v \to F(T)$, then $\bigcup A(\{u_n\}) \subset F(T)$. Here $\bigcup A(\{u_n\})$ where the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$. Moreover, $\bigcup A(\{u_n\})$ consists of exactly one point.*

We recall the definition a nonexpansive mapping:

Definition 2.8. From now on, X is a b-metric space, C is a nonempty convex subset of X and $T : C \to C$ is a mapping. A mappimg T is called nonexpansive if for each $x, y \in C$,

$$d(Tx, Ty) \leq d(x, y).$$

A point $x \in C$ is called a fixed point of T if $x = Tx$. We shall denote with $F(T)$ the set of fixed points of T.

Now, we proof main results:

Theorem 2.9. *Let C be a bounded closed convex subset of b-CAT(0) spaces X, and $F(T) \neq \varnothing$. Suppose that $T : C \to C$ a nonexpansive mapping. Then for any initial point x_0 in C, the iterate sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n,\ n = 0, 1, 2, ...,$ and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, then the Picard iterate sequence Δ-converges to a fixed point of T.*

Proof. Since T is nonexpansive, $\{d(x_n, p)\}$ is decreasing for each $p \in F(T)$, so it is convergent. By Lemma 2.7, $\bigcup A(\{u_n\})$ consists of exactly one point and is contained in $F(T)$. This shows that $\{x_n\}$ Δ-converges to an element of $F(T)$. $\square$

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