Thai Journal of Mathematics : 61-80
Special Issue : Annual Meeting in Mathematics 2019
http://thaijmath.in.cmu.ac.th
Online ISSN 1686-0209

Jensen's Type Inequalities Involving Tracy-Singh Products, Khatri-Rao Products, Tracy-Singh Sums and Khatri-Rao Sums

Arnon Ploymukda and Pattrawut Chansangiam[1]

Department of Mathematics, Faculty of Science, King Mongkut's Institute of Technology Ladkrabang, Bangkok 10520, Thailand.
e-mail : arnon.p.math@gmail.com (A. Ploymukda)
pattrawut.ch@kmitl.ac.th (P. Chansangiam)

Abstract : In this paper, we establish a number of Jensen's type inequalities for Hilbert space operators involving convex/concave functions, unital positive linear maps, and certain operator products and sums. The products and sums considered here include the Tracy-Singh product, the Khatri-Rao products, the Tracy-Singh sum, and the Khatri-Rao sum. Moreover, we generalize Jensen's type inequalities in term of functional calculus of two-variable functions. In particular, we obtain Kantorovich-type operator inequalities involving the products and sums.

Keywords : Jensen's inequality; Tracy-Singh product (sum); Khatri-Rao product (sum); Hilbert space operator; operator convexity.
2010 Mathematics Subject Classification : 47A63; 47A80.

1 Introduction

Jensen's inequality is an inequality involving convexity of a function. It has many applications in mathematics and statistics. This inequality states that for any real-valued convex function f defined on an interval J,

$$f\Big(\sum_{i=1}^{k} w_i x_i\Big) \leqslant \sum_{i=1}^{k} w_i f(x_i) \tag{1.1}$$

[1]Corresponding author.

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where $x_i \in J$ and $w_i \geqslant 0$ for all $i = 1, \ldots, k$ with $\sum_{i=1}^{k} w_i = 1$.

Mond and Pečarić [1] gave matrix versions, with matrix weights, of converse inequalities for (1.1). Some matrix inequalities involving Hadamard products were also presented in [1]. In [2], Mićić established Jensen's type inequality and its converses involving Khatri-Rao products of unital positive linear maps on positive definite matrices.

Throughout this paper, let $\mathcal{H}$ and $\mathcal{K}$ be complex Hilbert spaces. When $\mathcal{X}$ and $\mathcal{Y}$ are Hilbert spaces, the symbol $\mathbb{B}(\mathcal{X}, \mathcal{Y})$ stands for the algebra of bounded linear operators from $\mathcal{X}$ into $\mathcal{Y}$, and we write $\mathbb{B}(\mathcal{X})$ instead of $\mathbb{B}(\mathcal{X}, \mathcal{X})$. The set of all self-adjoint operators on $\mathcal{H}$ is denoted by $\mathbb{B}(\mathcal{H})^{sa}$. For operators $A, B \in \mathbb{B}(\mathcal{H})^{sa}$, the situation $A \geqslant B$ means that $A - B$ is a positive operator. Denote the set of all positive invertible operators on $\mathcal{H}$ by $\mathbb{B}(\mathcal{H})^{+}$. Denote the spectrum of an operator A by $\mathrm{Sp}(A)$. The identity operator is denoted by I. A linear map $\phi : \mathbb{B}(\mathcal{H}) \to \mathbb{B}(\mathcal{K})$ is said to be positive if $\phi(A) \geqslant 0$ whenever $A \geqslant 0$. It is said to be unital if $\phi(I) = I$.

Jensen's inequality (1.1) can be extended to various operator inequalities. For any convex function $f : J \to \mathbb{R}$ and $A \in \mathbb{B}(\mathcal{H})^{sa}$ with $\mathrm{Sp}(A) \subseteq J$, we have [3]:

$$f(\langle Ax, x\rangle) \ \leqslant \ \langle f[A]x, x\rangle \tag{1.2}$$

holds for every unit vector $x \in \mathcal{H}$. Moreover, complementary inequalities of (1.2) were also established in [3]. Seo et.al. [4] presented Hadamard product versions of complementary inequalities of (1.2).

In [5], Mond and Pečarić gave an another operator version of (1.1) for unital positive linear maps associated with convex functions. Let $A_i \in \mathbb{B}(\mathcal{H})^{sa}$ be such that $\mathrm{Sp}(A_i) \subseteq J$, $\phi_i : \mathbb{B}(\mathcal{H}) \to \mathbb{B}(\mathcal{K})$ a unital positive linear map and $w_i \geqslant 0$ with sum one. Then for every operator convex function $f : J \to \mathbb{R}$,

$$f[\sum_{i=1}^{k} w_i \phi_i(A_i)] \ \leqslant \ \sum_{i=1}^{k} w_i \phi_i(f[A_i]). \tag{1.3}$$

Some bounds of (1.3) were obtained in [6]. In [7], the authors gave several complementary inequalities of (1.3) in the case $k = 1$. Hansen et al.[8] obtained a generalization of (1.3) for unital fields of positive linear maps and obtained converse inequalities in the new formulation.

In this paper, we establish Jensen type inequalities for bounded linear operators on a Hilbert space, convex/concave functions and unital positive linear maps involving certain operator products and sums. The products and the sums concerned here are the Tracy-Singh product, the Khatri-Rao product, the Tracy-Singh sum, and the Khatri-Rao sum. We apply Mond-Pačarić method to certain operator-convex functions to get Jensen's type operator inequalities. Moreover, we derive some generalizations of Jensen's type operator inequalities concerning functional calculus of two-variable functions. Our results include Kantorovich-type operator inequalities for the products and the sums.

This paper is organized as follows. In Section 2, we provide some preliminaries about two kinds of operator products and sums, and Mond-Pečarić method for convex functions. These facts will be used in Sections 3 and 5. Section 3, we derive several inequalities of Jensen's type involving Tracy-Singh products and Khatri-Rao products. Jensen's type inequalities concerning Tracy-Singh products and Khatri-Rao products in terms of functional calculus of two-variable functions are presented in Section 4. In Section 5, we establish Jensen's type inequalities for Tracy-Singh sums and Khatri-Rao sums.

2 Preliminaries

2.1 Operator Products and Operator Sums

From the projection theorem for Hilbert spaces, we can make the following decompositions:

$$\mathcal{H} = \bigoplus_{i=1}^{n} \mathcal{H}_i, \quad \mathcal{K} = \bigoplus_{j=1}^{m} \mathcal{K}_j.$$

where all $\mathcal{H}_i$ and $\mathcal{K}_j$ are Hilbert spaces. For each i and j, let $E_i : \mathcal{H}_i \to \mathcal{H}$ and $F_j : \mathcal{K}_j \to \mathcal{K}$ be the canonical embeddings, and $P_i : \mathcal{H} \to \mathcal{H}_i$ and $Q_j : \mathcal{K} \to \mathcal{K}_j$ be the orthogonal projections. Each $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$ thus can be expressed uniquely as operator matrices

$$A = [A_{ij}]_{i,j=1}^{n,n} \quad \text{and} \quad B = [B_{kl}]_{k,l=1}^{m,m}$$

where $A_{ij} = P_i A E_i \in \mathbb{B}(\mathcal{H}_j, \mathcal{H}_i)$ and $B_{kl} = Q_k B F_l \in \mathbb{B}(\mathcal{K}_l, \mathcal{K}_k)$ for each i, j, k, l.

Recall that the tensor product of $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$ is a unique bounded linear operator from $\mathcal{H} \otimes \mathcal{K}$ into itself such that for all $x \in \mathcal{H}$ and $y \in \mathcal{K}$,

$$(A \otimes B)(x \otimes y) = Ax \otimes By.$$

Definition 2.1. Let $A = [A_{ij}]_{i,j=1}^{n,n} \in \mathbb{B}(\mathcal{H})$ and $B = [B_{kl}]_{k,l=1}^{m,m} \in \mathbb{B}(\mathcal{K})$. We define the Tracy-Singh product of A and B to be the operator matrix

$$A \boxtimes B = \left[[A_{ij} \otimes B_{kl}]_{kl} \right]_{ij} \tag{2.1}$$

which is a bounded linear operator from $\bigoplus_{i=1}^{n} \bigoplus_{j=1}^{m} \mathcal{H}_i \otimes \mathcal{K}_j$ into itself. When $m = n$, we define the Khatri-Rao product of A and B to be the bounded linear operator

$$A \boxdot B = [A_{ij} \otimes B_{ij}]_{ij} \tag{2.2}$$

which is a bounded linear operator from $\bigoplus_{i=1}^{n} \mathcal{H}_i \otimes \mathcal{K}_i$ into itself.

Lemma 2.2 ([9, 10]). *Let A, B, C, D be compatible operators. Then*

1. *The map $(A, B) \mapsto A \boxtimes B$ is bilinear.*
2. *$(A \boxtimes B)(C \boxtimes D) = (AC) \boxtimes (BD)$.*
3. *If A and B are positive and invertible, then $(A \boxtimes B)^p = A^p \boxtimes B^p$ for any $p \in \mathbb{R}$.*
4. *If $A \geqslant C \geqslant 0$ and $B \geqslant D \geqslant 0$, then $A \boxtimes B \geqslant C \boxtimes D \geqslant 0$.*

For each $i = 1, \ldots, k$, let $\mathcal{H}_i$ be a Hilbert space and decompose $\mathcal{H}_i = \bigoplus_{r=1}^{n_i} \mathcal{H}_{i,r}$ where all $\mathcal{H}_{i,r}$ are Hilbert spaces. For a finite number of operators $A_i \in \mathbb{B}(\mathcal{H}_i)$ for $i = 1, \ldots, k$, we use the following notations

$$\begin{aligned} \mathop{\boxtimes}_{i=1}^{k} A_i &= ((A_1 \boxtimes A_2) \boxtimes \cdots \boxtimes A_{k-1}) \boxtimes A_k, \\ \mathop{\boxdot}_{i=1}^{k} A_i &= ((A_1 \boxdot A_2) \boxdot \cdots \boxdot A_{k-1}) \boxdot A_k. \end{aligned}$$

Lemma 2.3 ([11]). *There is a unital positive linear map ψ such that*

$$\psi\Big(\mathop{\boxtimes}_{i=1}^{k} A_i \Big) = \mathop{\boxdot}_{i=1}^{k} A_i \tag{2.3}$$

for any $A_i \in \mathbb{B}(\mathcal{H}_i)$, $i = 1, \ldots, k$.

Definition 2.4. Let $A = [A_{ij}]_{i,j=1}^{n,n} \in \mathbb{B}(\mathcal{H})$ and $B = [B_{kl}]_{k,l=1}^{m,m} \in \mathbb{B}(\mathcal{K})$. We define the Tracy-Singh sum of A and B to be

$$A \boxplus B = A \boxtimes I_{\mathcal{K}} + I_{\mathcal{H}} \boxtimes B \tag{2.4}$$

which is a bounded linear operator from $\bigoplus_{i,j=1}^{n,m} \mathcal{H}_i \otimes \mathcal{K}_j$ into itself. When $m = n$, we define the *Khatri-Rao sum* of A and B to be

$$A \circledast B = A \boxdot I_{\mathcal{K}} + I_{\mathcal{H}} \boxdot B \tag{2.5}$$

which is a bounded linear operator from $\bigoplus_{i=1}^{n} \mathcal{H}_i \otimes \mathcal{K}_i$ into itself.

Applying the unital positive linear map in Lemma 2.3, we obtain a relation between Tracy-Singh and Khatri-Rao sums as in [12].

Lemma 2.5 ([12]). *There is a unital positive linear map ψ such that $\psi(A \boxplus B) = A \circledast B$ for any $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$.*

2.2 Bounds for Jensen's Inequality for Operators

In [6], Mond and Pečarić established the following results which give rise to the reverse of operator Jensen's inequality (1.3).

Lemma 2.6 ([6])**.** *Let $A_i \in \mathbb{B}(\mathcal{H})^{sa}$ be such that* $\mathrm{Sp}(A_i) \subseteq [m, M]$*,* $\phi_i : \mathbb{B}(\mathcal{H}) \to \mathbb{B}(\mathcal{K})$ *be a unital positive linear map and w_i a positive real number for $i = 1, \dots, k$ with $\sum_{i=1}^{k} w_i = 1$. Let $f : [m, M] \to \mathbb{R}$ be a strictly-convex twice-differentiable function. Suppose that either (i) $f(x) > 0$ for all $x \in [m, M]$, or (ii) $f(x) < 0$ for all $x \in [m, M]$. Then*

$$\sum_{i=1}^{k} w_i \phi_i(f[A_i]) \leqslant \lambda f[\sum_{i=1}^{k} w_i \phi_i(A_i)] \tag{2.6}$$

holds for some $\lambda > 1$ in case (i), or $\lambda \in (0, 1)$ in case (ii).

Lemma 2.7 ([6])**.** *Let A_i, ϕ_i and w_i be as Lemma 2.6. Let $f : [m, M] \to \mathbb{R}$ be a differentiable convex function such that f' is strictly increasing on $[m, M]$. Then*

$$\sum_{i=1}^{k} w_i \phi_i(f[A_i]) \leqslant \kappa I + f[\sum_{i=1}^{k} w_i \phi_i(A_i)] \tag{2.7}$$

holds for some κ satisfying $0 < \kappa < (M - m)(\mu - f'(m))$ where $\mu = \frac{f(M)-f(m)}{M-m}$.

Lemmas 2.6 and 2.7 were known as Mond-Pečarić method.

3 Jensen's Type Inequalities Involving Tracy-Singh Products and Khatri-Rao Products via Mond-Pečarić Method

In this section, we apply the Mond-Pečarić method to derive certain inequalities of Jensen's type for Tracy-Singh products and Khatri-Rao products of operators.

Let us start with recalling some terminologies. Let J be an interval. A function $f : J \to \mathbb{R}$ if said to be convex if

$$f((1 - \alpha)s + \alpha t) \leqslant (1 - \alpha)f(s) + \alpha f(t) \tag{3.1}$$

for any $s, t \in J$ and $\alpha \in [0, 1]$. We say that f is concave if $-f$ is convex. More generally, f is said to be operator convex if

$$f[(1 - \alpha)A + \alpha B] \leqslant (1 - \alpha)f[A] + \alpha f[B] \tag{3.2}$$

for any $\alpha \in [0, 1]$ and $A, B \in \mathbb{B}(\mathcal{H})^{sa}$ whose spectra are in J. We say that f is operator concave if $-f$ is operator convex.

Definition 3.1. A function $f : J \to \mathbb{R}$ is said to be submultiplicative (resp. supermultiplicative) with respect to the Tracy-Singh product if

$$f[A \boxtimes B] \leqslant f[A] \boxtimes f[B] \quad (\text{resp. } f[A \boxtimes B] \leqslant f[A] \boxtimes f[B])$$

for all $A \in \mathbb{B}(\mathcal{H})^{sa}$ and $B \in \mathbb{B}(\mathcal{K})^{sa}$ whose spectra of A, B and $A \boxtimes B$ are contained in J.

Proposition 3.2. *Let $A_i \in \mathbb{B}(\mathcal{H})^{sa}, B_i \in \mathbb{B}(\mathcal{K})^{sa}$ be such that $\mathrm{Sp}(A_i \boxtimes B_i) \subseteq [m, M]$ and $w_i > 0$ for $i = 1, \dots, k$ with $\sum_{i=1}^{k} w_i = 1$. Let $f : [m, M] \to \mathbb{R}$ be a function.*

1. *If f is operator-convex and submultiplicative with respect to the Tracy-Singh product, then*

$$f[\sum_{i=1}^{k} w_i A_i \boxtimes B_i] \leqslant \sum_{i=1}^{k} w_i f[A_i] \boxtimes f[B_i], \tag{3.3}$$

$$f[\sum_{i=1}^{k} w_i A_i \boxdot B_i] \leqslant \sum_{i=1}^{k} w_i f[A_i] \boxdot f[B_i]. \tag{3.4}$$

2. *If f is strictly-convex twice-differentiable and supermultiplicative with respect to the Tracy-Singh products, and either (i) $f(x) > 0$ for all $x \in [m, M]$, or (ii) $f(x) < 0$ for all $x \in [m, M]$, then*

$$\sum_{i=1}^{k} w_i f[A_i] \boxtimes f[B_i] \leqslant \lambda f[\sum_{i=1}^{k} w_i A_i \boxtimes B_i], \tag{3.5}$$

$$\sum_{i=1}^{k} w_i f[A_i] \boxdot f[B_i] \leqslant \lambda f[\sum_{i=1}^{k} w_i A_i \boxdot B_i], \tag{3.6}$$

where λ is given in Lemma 2.6.

3. *If f is operator-concave and supermultiplicative with respect to the Tracy-Singh product, then the opposite inequalities hold in* (3.3) *and* (3.4)*.*

4. *If f is strictly-concave twice-differentiable and submultiplicative with respect to the Tracy-Singh product, and either (i) $f(x) > 0$ for all $x \in [m, M]$, or (ii) $f(x) < 0$ for all $x \in [m, M]$, then the opposite inequalities hold in* (3.5) *and* (3.6)*.*

Proof. 1. Using the submultiplicativity of f with respect to the Tracy-Singh product and the classical Jensen inequality (1.3), we have

$$f[\sum_{i=1}^{k} w_i A_i \boxtimes B_i] \leqslant \sum_{i=1}^{k} w_i f[A_i \boxtimes B_i] \leqslant \sum_{i=1}^{k} w_i f[A_i] \boxtimes f[B_i].$$

Applying Lemma 2.3 and Inequality (1.3), we get

$$\begin{aligned} f[\sum_{i=1}^{k} w_i A_i \boxdot B_i] &= f[\sum_{i=1}^{k} w_i \psi(A_i \boxtimes B_i)] \leqslant \sum_{i=1}^{k} w_i \psi(f[A_i \boxtimes B_i]) \\ &\leqslant \sum_{i=1}^{k} w_i \psi(f[A_i] \boxtimes f[B_i]) = \sum_{i=1}^{k} w_i f[A_i] \boxdot f[B_i]. \end{aligned}$$

2. Using the supermultiplicativity of f with respect to the Tracy-Singh product and Lemma 2.6, we have

$$\sum_{i=1}^{k} w_i f[A_i] \boxtimes f[B_i] \leqslant \sum_{i=1}^{k} w_i f[A_i \boxtimes B_i] \leqslant \lambda f[\sum_{i=1}^{k} w_i A_i \boxtimes B_i].$$

Applying Lemmas 2.3 and 2.6, we get

$$\begin{aligned} \sum_{i=1}^{k} w_i f[A_i] \boxdot f[B_i] &= \sum_{i=1}^{k} w_i \psi(f[A_i] \boxtimes f[B_i]) \leqslant \sum_{i=1}^{k} w_i \psi(f[A_i \boxtimes B_i]) \\ &\leqslant \lambda f[\sum_{i=1}^{k} w_i \psi(A_i \boxtimes B_i)] = \lambda f[\sum_{i=1}^{k} w_i A_i \boxdot B_i]. \end{aligned}$$

The proof of Cases 3 and 4 are similar to those for Cases 1 and 2, respectively. □

We now consider the special case when $f(t) = t^p,\ x > 0$. It is well-known that f is convex if either $p < 0$ or $p \geqslant 1$, while it is concave if $0 < p \leqslant 1$. It is clearly from Lemma 2.2 that $(A \boxtimes B)^p = A^p \boxtimes B^p$ for any $p \in \mathbb{R}$. For this reason, in the next corollary, we focus only on inequalities concerning Khatri-Rao products.

Following [13], the generalized Kantorovich constant $K(m, M, p)$ and the constant $C(m, M, p)$ are defined as follows:

$$\begin{aligned} K(m, M, p) &= \frac{mM^p - Mm^p}{(p-1)(M-m)} \left[\frac{(p-1)(M^p - m^p)}{p(mM^p - Mm^p)} \right]^p, \\ C(m, M, p) &= \frac{Mm^p - mM^p}{M - m} + (p-1) \left(\frac{M^p - m^p}{p(M-m)} \right)^{\frac{p}{p-1}}. \end{aligned}$$

We denote $K(m, M) = K(m, M, -1) = K(m, M, 2)$ the original Kantorovich constant.

Corollary 3.3. *Let $A \in \mathbb{B}(\mathcal{H})^+$ and $B \in \mathbb{B}(\mathcal{K})^+$ be such that $\mathrm{Sp}(A \boxtimes B) \subseteq [m, M]$. For any $p > 1$, we have*

$$(A \boxdot B)^p \leqslant A^p \boxdot B^p \leqslant K(m, M, p)(A \boxdot B)^p. \qquad (3.7)$$

While, for $0 < p < 1$, we have the reverse inequality in (3.7). If $A \boxdot B$ is invertible, then (3.7) holds for any $p < 0$ or $p > 1$.

Corollary 3.3 includes the following Kantorovich type inequalities for operators:

$$(A \boxdot B)^2 \;\leqslant\; A^2 \boxdot B^2 \;\leqslant\; K(m,M)(A \boxdot B)^2, \tag{3.8}$$

$$(A \boxdot B)^{-1} \;\leqslant\; A^{-1} \boxdot B^{-1} \;\leqslant\; K(m,M)(A \boxdot B)^{-1}. \tag{3.9}$$

Both inequalities were originally proved in [14].

Proposition 3.4. *Let A_i, B_i and w_i be as in Proposition 3.2. Let $f : [m,M] \to \mathbb{R}$ be a differentiable function.*

1. *If f is convex and supermultiplicative with respect to the Tracy-Singh product, and f' is strictly-increasing on $[m,M]$, then*

$$\sum_{i=1}^{k} w_i f[A_i] \boxtimes f[B_i] - f[\sum_{i=1}^{k} w_i A_i \boxtimes B_i] \;\leqslant\; \kappa I, \tag{3.10}$$

$$\sum_{i=1}^{k} w_i f[A_i] \boxdot f[B_i] - f[\sum_{i=1}^{k} w_i A_i \boxdot B_i] \;\leqslant\; \kappa I, \tag{3.11}$$

where κ is given in Lemma 2.7.

2. *If f is concave and submultiplicative with respect to the Tracy-Singh product, and f' is strictly-decreasing on $[m,M]$, then the opposite inequalities hold in* (3.10) *and* (3.11).

Proof. We only prove the Case 1. Applying Lemma 2.6, we get

$$\sum_{i=1}^{k} w_i f[A_i] \boxtimes f[B_i] \;\leqslant\; \sum_{i=1}^{k} w_i f[A_i \boxtimes B_i] \;\leqslant\; \kappa I + f[\sum_{i=1}^{k} w_i A_i \boxtimes B_i].$$

We have by Lemmas 2.3 and 2.6 that

$$\begin{aligned}\sum_{i=1}^{k} w_i f[A_i] \boxdot f[B_i] \;&=\; \sum_{i=1}^{k} w_i \psi(f[A_i] \boxtimes f[B_i]) \;\leqslant\; \sum_{i=1}^{k} w_i \psi(f[A_i \boxtimes B_i]) \\ &\leqslant\; \kappa I + f[\sum_{i=1}^{k} w_i \psi(A_i \boxtimes B_i)] \;=\; \kappa I + f[\sum_{i=1}^{k} w_i A_i \boxdot B_i]. \;\square\end{aligned}$$

Corollary 3.5. *Let $A \in \mathbb{B}(\mathcal{H})^+$ and $B \in \mathbb{B}(\mathcal{K})^+$ be such that* $\mathrm{Sp}(A \boxtimes B) \subseteq [m,M]$. *For any $p > 1$, we have*

$$A^p \boxdot B^p - (A \boxdot B)^p \;\leqslant\; C(m,M,p)I, \tag{3.12}$$

While, for $0 < p < 1$, we have the reverse inequality in (3.12). *Moreover, if $A \boxdot B$ is invertible, then* (3.12) *holds for $p < 0$ or $p > 1$.*

Corollary 3.5 includes the following operator Kantorovich type inequalities:

$$A^2 \boxdot B^2 - (A \boxdot B)^2 \leqslant \frac{(M-m)^2}{4Mm} I,$$
$$A^{-1} \boxdot B^{-1} - (A \boxdot B)^{-1} \leqslant \frac{(\sqrt{M}-\sqrt{m})^2}{Mm} I.$$

Both inequalities were proved already in [14, Proposition 3].

4 Generalizations of Jensen's Type Operator Inequalities in Terms of Two-Variable Functions

In this section, we generalize Jensen's type operator inequalities involving Tracy-Singh products and Khatri-Rao products in terms of functional calculus of two-variable functions. Let us introduce some hypotheses and notations used in this section.

Hypothesis 4.1. *For each $i = 1, \ldots, k$, let $A_i \in \mathbb{B}(\mathcal{H}_i)^{sa}$ be such that* $\mathrm{Sp}(A_i) \subseteq [m, M]$ *with $m < M$.*

Hypothesis 4.2. $\mathrm{Sp}\left(\boxtimes_{i=1}^k A_i\right) \subseteq [m_A, M_A]$ *with $m_A < M_A$.*

Hypothesis 4.3. *For each $i = 1, \ldots, k$, let $\phi_i : \mathbb{B}(\mathcal{H}_i) \to \mathbb{B}(\mathcal{K}_i)$ be a unital positive linear map.*

Hypothesis 4.4. $\mathrm{Sp}\left(\boxtimes_{i=1}^k \phi_i(A_i)\right) \subseteq [m_\phi, M_\phi]$ *with $m_\phi < M_\phi$.*

For a function $f : [m, M] \to \mathbb{R}$, we denote the slope and the intercept of a linear function through $(m, f(m))$ and $(M, f(M))$ by μ_f and ν_f, respectively, i.e.

$$\mu_f = \frac{f(M) - f(m)}{M - m}, \quad \nu_f = \frac{Mf(m) - mf(M)}{M - m}.$$

For a function $f : [m, M] \cup [m_A, M_A] \to \mathbb{R}$, we denote $\tilde{\mu}_f$ and $\tilde{\nu}_f$ for the slope and the intercept of a linear function through $(m_A, f(m_A))$ and $(M_A, f(M_A))$, respectively.

In order to defined $F[A, B]$ where F a real-valued function of two variables, we apply the functional calculus on the tensor products (see e.g. [15, 16]). In particular, if $F(u, v) = v^{-1/2} u v^{-1/2}$, then $F[A, B] = B^{-1/2} A B^{-1/2}$.

Theorem 4.5. *Assume Hypotheses 4.1 and 4.2. Let $f : [m, M] \cup [M_A, M_A] \to \mathbb{R}$, $g : [m_A, M_A] \to \mathbb{R}$ and $F : U \times V \to \mathbb{R}$ be functions such that*

$$\mathrm{Sp}\left(\boxtimes_{i=1}^{k} f[A_i]\right) \cup \mathrm{Sp}\left(\tilde{\mu}_f \boxtimes_{i=1}^{k} A_i + \tilde{\nu}_f I\right) \subseteq U, \quad g([m_A, M_A]) \subseteq V.$$

Suppose that F is bounded and operator-monotone in the first variable.

1. *If f is convex and supermultiplicative with respect to the Tracy-Singh product, then*

$$F\Big[\boxtimes_{i=1}^{k} f[A_i], g[\boxtimes_{i=1}^{k} A_i]\Big] \leqslant \Big\{\sup_{m_A\leqslant t\leqslant M_A} F(\tilde{\mu}_f t+\tilde{\nu}_f, g(t))\Big\} I, \tag{4.1}$$

$$F\Big[\boxdot_{i=1}^{k} f[A_i], g[\boxdot_{i=1}^{k} A_i]\Big] \leqslant \Big\{\sup_{m_A\leqslant t\leqslant M_A} F(\tilde{\mu}_f t+\tilde{\nu}_f, g(t))\Big\} I. \tag{4.2}$$

2. *If f is concave and submultiplicative with respect to the Tracy-Singh products, then the opposite inequalities hold in* (4.1) *and* (4.2) *with* inf *instead of* sup.

Proof. We only prove Case 1. It follows from the convexity of f that $f(t) \leqslant \tilde{\mu}_f t+\tilde{\nu}_f$ for every $t \in [m_A, M_A]$. Using functional calculus, we have

$$f[\boxtimes_{i=1}^{k} A_i] \leqslant \tilde{\mu}_f \boxtimes_{i=1}^{k} A_i + \tilde{\nu}_f I.$$

Using the supermultiplicativity of f with respect to the Tracy-Singh product, we get

$$\boxtimes_{i=1}^{k} f[A_i] \leqslant \tilde{\mu}_f \boxtimes_{i=1}^{k} A_i + \tilde{\nu}_f I. \tag{4.3}$$

Since F is operator-monotone in the first variable, we obtain

$$\begin{aligned} F\Big[\boxtimes_{i=1}^{k} f[A_i], g[\boxtimes_{i=1}^{k} A_i]\Big] &\leqslant F\Big[\tilde{\mu}_f \boxtimes_{i=1}^{k} A_i + \tilde{\nu}_f I, g[\boxtimes_{i=1}^{k} A_i]\Big] \\ &\leqslant \Big\{\sup_{m_A\leqslant t\leqslant M_A} F(\tilde{\mu}_f t+\tilde{\nu}_f, g(t))\Big\} I. \end{aligned}$$

Applying Lemma 2.3 with (4.3) and using the monotonicity of $F(\cdot, v)$, we obtain (4.2). □

Notice that Theorem 4.5, when $k = 2$, can be viewed as Tracy-Singh/Khatri-Rao products versions of [4, Theorem 2].

Mićić et.al [17] showed that if $f : [m, M] \to \mathbb{R}$ is a continuous convex function, $g : [m, M] \to \mathbb{R}$ is a continuous function, and $A \in \mathbb{B}(\mathcal{H})^{sa}$ with $\mathrm{Sp}(A) \subseteq [m, M]$, then for a given $\alpha \in \mathbb{R}$, there exists the suitable constant β such that

$$\langle f[A]x, x\rangle \leqslant \alpha g(\langle Ax, x\rangle) + \beta \tag{4.4}$$

holds for every unit vector $x \in \mathcal{H}$. Now, we will derive Tracy-Singh/Khatri-Rao products versions of (4.4) by applying Theorem 4.5 for $F(u, v) = u - \alpha v$.

Corollary 4.6. *Assume Hypotheses 4.1 and 4.2. Let* $f : [m, M] \cup [m_A, M_A] \to \mathbb{R}$, $g : [m_A, M_A] \to \mathbb{R}$ *and* $\alpha > 0$.

1. *If f is convex and supermultiplicative with respect to the Tracy-Singh product, and g is strictly-convex and differentiable, then*

$$\mathop{\boxtimes}_{i=1}^{k} f[A_i] \leqslant \alpha g[\mathop{\boxtimes}_{i=1}^{k} A_i] + \beta I, \tag{4.5}$$

$$\mathop{\boxdot}_{i=1}^{k} f[A_i] \leqslant \alpha g[\mathop{\boxdot}_{i=1}^{k} A_i] + \beta I, \tag{4.6}$$

hold for $\beta = \tilde{\mu}_f t_0 + \tilde{\nu}_f - \alpha g(t_0)$, *where* μ_f, ν_f *are given in Theorem 4.5 and*

$$t_0 = \begin{cases} \text{the unique solution of } \alpha g'(t) = \tilde{\mu}_f & \text{if } \alpha g'(m_A) \leqslant \tilde{\mu}_f \leqslant \alpha g'(M_A), \\ m_A & \text{if } \tilde{\mu}_f \leqslant \alpha g'(m_A), \\ M_A & \text{if } \tilde{\mu}_f \geqslant \alpha g'(M_A). \end{cases}$$

2. *If f is convex and supermultiplicative with respect to the Tracy-Singh product, and g is convex and continuous, then* (4.5) *and* (4.6) *hold for*

$$t_0 = \begin{cases} m_A & \text{if } \tilde{\mu}_f \leqslant \alpha \tilde{\mu}_g, \\ M_A & \text{if } \tilde{\mu}_f \geqslant \alpha \tilde{\mu}_g. \end{cases}$$

3. *If f is concave and submultiplicative with respect to the Tracy-Singh product, and g is strictly-concave and differentiable, then the opposite inequalities hold in* (4.5) *and* (4.6) *with the same t_0 in Case 1 but the opposite condition while determining t_0.*

4. *If f is concave and submultiplicative with respect to the Tracy-Singh product, and g is convex and continuous, then the opposite inequalities hold in* (4.5) *and* (4.6) *with the same t_0 in Case 2 but the opposite condition while determining t_0.*

Notice that when $k = 2$, Corollary 4.6 can be viewed generalization of of [4, Theorem 5].

Corollary 4.7. *Assume Hypotheses 4.1 and 4.2. Let* $f : [m, M] \cup [m_A, M_A] \to \mathbb{R}$ *be a continuously twice-differentiable function,* $g : [m_A, M_A] \to \mathbb{R}$ *be a continuous function and* $\alpha > 0$.

1. *If f is convex and supermultiplicative with respect to the Tracy-Singh product, and g is strictly positive, then*

$$\mathop{\boxtimes}_{i=1}^{k} f[A_i] \leqslant \max_{m_A \leqslant t \leqslant M_A} \left\{ \frac{\tilde{\mu}_f t + \tilde{\nu}_f}{g(t)} \right\} g[\mathop{\boxtimes}_{i=1}^{k} A_i], \tag{4.7}$$

$$\mathop{\boxdot}_{i=1}^{k} f[A_i] \leqslant \max_{m_A \leqslant t \leqslant M_A} \left\{ \frac{\tilde{\mu}_f t + \tilde{\nu}_f}{g(t)} \right\} g[\mathop{\boxdot}_{i=1}^{k} A_i]. \tag{4.8}$$

2. *If f is convex and supermultiplicative with respect to the Tracy-Singh product, and g is strictly negative, then the inequalities* (4.7) *and* (4.8) *hold with* min *instead of* max.

3. *If f is concave and submultiplicative with respect to the Tracy-Singh product, and g is strictly positive, then the opposite inequalities hold in* (4.7) *and* (4.8) *with* min *instead of* max.

4. *If f is concave and submultiplicative with respect to the Tracy-Singh product, and g is strictly negative, then the opposite inequalities hold in* (4.7) *and* (4.8).

Proof. Applying Theorem 4.5 for the function $F(u,v) = v^{-1/2}uv^{-1/2}$. Since $h(t) = \frac{\tilde{\mu}_f t+\tilde{\nu}_f}{g(t)}$ is continuous on $[m_A, M_A]$, it has the global extreme points on $[m_A, M_A]$. □

In the next theorem, we give Jensen's type inequalities concerning Tracy-Singh and Khatri-Rao products.

Theorem 4.8. *Assume Hypotheses 4.1, 4.3 and 4.4. Let $f : [m, M] \cup [m_\phi, M_\phi] \to \mathbb{R}$ be a function.*

1. *If f is operator-convex and submultiplicative with respect to the Tracy-Singh product, then*

$$f[\boxtimes_{i=1}^{k} \phi_i(A_i)] \leqslant \boxtimes_{i=1}^{k} \phi_i(f[A_i]), \tag{4.9}$$

$$f[\boxdot_{i=1}^{k} \phi_i(A_i)] \leqslant \boxdot_{i=1}^{k} \phi_i(f[A_i]). \tag{4.10}$$

2. *If f is operator-concave and supermultiplicative with respect to the Tracy-Singh product, then the opposite inequalities of* (4.9) *and* (4.10) *hold.*

Proof. We only prove Case 1. The inequality (4.9) follows from the submultiplicativity of f respect to the Tracy-Singh product, and Jensen's inequality (1.3). Applying Lemmas 2.2 and 2.3, and inequalities (1.3) and (4.9), we have

$$\begin{aligned} f[\boxdot_{i=1}^{k} \phi_i(A_i)] &= f[\psi(\boxtimes_{i=1}^{k} \phi_i(A_i))] \leqslant \psi(f[\boxtimes_{i=1}^{k} \phi_i(A_i)]) \\ &\leqslant \psi(\boxtimes_{i=1}^{k} f[\phi_i(A_i)]) \leqslant \boxdot_{i=1}^{k} \phi_i(f[A_i]). \end{aligned}$$ □

We mention that (4.10) is an operator extension of [2, Theorem 2.1].

Corollary 4.9. *Assume Hypotheses 4.1, 4.3 and 4.4. Let* $f : [m, M] \cup [m_\phi, M_\phi] \to \mathbb{R}$, $g : [m_\phi, M_\phi] \to \mathbb{R}$ *and* $F : U \times V \to \mathbb{R}$ *be functions such that*

$$\mathrm{Sp}\,\big(\boxtimes_{i=1}^{k} \phi_i(f[A_i])\big) \cup f([m_\phi, M_\phi]) \;\subseteq\; U, \quad g([m_\phi, M_\phi]) \subseteq V.$$

Suppose that F *is bounded and operator-monotone in the first variable.*

1. *If* f *is operator-convex and submultiplicative with respect to the Tracy-Singh product, then*

$$F\big[\boxtimes_{i=1}^{k} \phi_i(f[A_i]), g[\boxtimes_{i=1}^{k} \phi_i(A_i)]\big] \;\geqslant\; \Big\{\inf_{m_\phi \leqslant t \leqslant M_\phi} F(f(t), g(t))\Big\} I, \tag{4.11}$$

$$F\big[\boxdot_{i=1}^{k} \phi_i(f[A_i]), g[\boxdot_{i=1}^{k} \phi_i(A_i)]\big] \;\geqslant\; \Big\{\inf_{m_\phi \leqslant t \leqslant M_\phi} F(f(t), g(t))\Big\} I. \tag{4.12}$$

2. *If* f *is operator-concave and supermultiplicative with respect to the Tracy-Singh product, then the opposite inequalities hold in* (4.11) *and* (4.12) *with* inf *instead of* sup.

Proof. It follows from the monotonicity of $F(\cdot, v)$ and Theorem 4.8. □

Theorem 4.10. *Assume Hypotheses 4.1, 4.3 and 4.4. Let* $f : [m, M] \to \mathbb{R}$, $g : [m_\phi, M_\phi] \to \mathbb{R}$ *and* $F : U \times V \to \mathbb{R}$ *be functions such that*

$$\mathrm{Sp}\,\big(\boxtimes_{i=1}^{k} \phi_i(f[A_i])\big) \cup \mathrm{Sp}\,\big(\mu_f^k \boxtimes_{i=1}^{k} \phi_i(A_i) + \nu_f^k I\big) \;\subseteq\; U, \quad g([m_\phi, M_\phi]) \subseteq V.$$

Suppose that F *is bounded and operator-monotone in the first variable.*

1. *If* f *convex, then*

$$F\big[\boxtimes_{i=1}^{k} \phi_i(f[A_i]), g[\boxtimes_{i=1}^{k} \phi_i(A_i)]\big] \;\leqslant\; \Big\{\sup_{m_\phi \leqslant t \leqslant M_\phi} F(\mu_f^k t + \nu_f^k, g(t))\Big\} I, \tag{4.13}$$

$$F\big[\boxdot_{i=1}^{k} \phi_i(f[A_i]), g[\boxdot_{i=1}^{k} \phi_i(A_i)]\big] \;\leqslant\; \Big\{\sup_{m_\phi \leqslant t \leqslant M_\phi} F(\mu_f^k t + \nu_f^k, g(t))\Big\} I. \tag{4.14}$$

2. *If* f *is concave, then the opposite inequalities hold in* (4.13) *and* (4.14) *with* inf *instead of* sup.

Proof. For the first assertion, assume that f is convex. Then $f(t) \leqslant \mu_f t + \nu_f$ for every $t \in [m, M]$. Using functional calculus, we get

$$f[A_i] \leqslant \mu_f A_i + \nu_f I$$

for all $i = 1, \dots, k$. Since ϕ_i is a unital positive linear map, we get

$$\phi_i(f[A_i]) \leqslant \mu_f \phi_i(A_i) + \nu_f I.$$

We have by the monotonicity of the Tracy-Singh product that

$$\boxtimes_{i=1}^{k} \phi_i(f[A_i]) \leqslant \boxtimes_{i=1}^{k} \mu_f \phi_i(A_i) + \boxtimes_{i=1}^{k} \nu_f I = \mu_f^k \boxtimes_{i=1}^{k} \phi_i(A_i) + \nu_f^k I. \tag{4.15}$$

Since F is operator-monotone in the first variable, we obtain

$$\begin{aligned} F\Big[\boxtimes_{i=1}^{k} \phi_i(f[A_i]), g[\boxtimes_{i=1}^{k} \phi_i(A_i)]\Big] &\leqslant F\Big[\mu_f^k \boxtimes_{i=1}^{k} \phi_i(A_i) + \nu_f^k I, g[\boxtimes_{i=1}^{k} \phi_i(A_i)]\Big] \\ &\leqslant \left\{ \sup_{m_\phi \leqslant t \leqslant M_\phi} F(\mu_f^k t + \nu_f^k, g(t)) \right\} I. \end{aligned}$$

We get (4.14) by applying (4.15) with Lemma 2.3 and using monotonicity of $F(\cdot, v)$. The proof for the second assertion is similar to the previous case. $\square$

Applying Theorem 4.10 for the function $F(u, v) = u - \alpha v$, we obtain the following corollary.

Corollary 4.11. *Assume Hypotheses 4.1, 4.3 and 4.4. Let $f : [m, M] \to \mathbb{R}$, $g : [m_\phi, M_\phi] \to \mathbb{R}$ and $\alpha > 0$.*

1. *If f is convex and g is strictly-convex differentiable, then*

$$\boxtimes_{i=1}^{k} \phi_i(f[A_i]) \leqslant \alpha g[\boxtimes_{i=1}^{k} \phi_i(A_i)] + \beta I, \tag{4.16}$$

$$\boxdot_{i=1}^{k} \phi_i(f[A_i]) \leqslant \alpha g[\boxdot_{i=1}^{k} \phi_i(A_i)] + \beta I, \tag{4.17}$$

hold for $\beta = \mu_f^k t_0 + \nu_f^k - \alpha g(t_0)$, where

$$t_0 = \begin{cases} \text{the unique solution of } \alpha g'(t) = \mu_f^k & \text{if } \alpha g'(m_\phi) \leqslant \mu_f^k \leqslant \alpha g'(M_\phi), \\ m_\phi & \text{if } \mu_f^k \leqslant \alpha g'(m_\phi), \\ M_\phi & \text{if } \mu_f^k \geqslant \alpha g'(M_\phi). \end{cases}$$

2. *If f is convex and g is concave, then (4.16) and (4.17) hold for*

$$t_0 = \begin{cases} m_\phi & \text{if } \mu_f^k \leqslant \alpha \mu_g, \\ M_\phi & \text{if } \mu_f^k \geqslant \alpha \mu_g. \end{cases}$$

3. *If f is concave and g is convex, the opposite inequalities hold in* (4.16) *and* (4.17) *with the same t_0 in the case 1 but the opposite condition while determining t_0.*

4. *If f is concave and g is strictly-concave differentiable, the opposite inequalities hold in* (4.16) *and* (4.17) *with the same t_0 in the case 2 but the opposite condition while determining t_0.*

Notice that the cases 1 and 3, when $k = 1$, in Corollary 4.11 conclude the results in [7, Theorem 2.1 and 2.2].

If $f : \mathbb{R} \to \mathbb{R}$ is a convex function, then a subdifferential function of f on $[m, M]$ is any function $s : [m, M] \to \mathbb{R}$ such that

$$s(t) \in [f'_{-}(t), f'_{+}(t)], \quad t \in (m, M),$$

where f'_{-} and f'_{+} are the one-sided derivatives of f and $s(m) = f'_{+}(m)$ and $s(M) = f'_{-}(M)$. Subdifferential function for concave functions is defined in analogous way. In [8], Hansen et.al. gave some inequalities of Jensen's type for unital fields if linear maps involving subdifferential functions. In the next theorem, we give related inequalities of Theorem 4.10 by using subdifferentials.

Theorem 4.12. *Assume Hypotheses 4.1, 4.3 and 4.4. Let $f : \mathbb{R} \to \mathbb{R}$, $g : [m_\phi, M_\phi] \to \mathbb{R}$ and $F : U \times V \to \mathbb{R}$ be functions such that $g([m_\phi, M_\phi]) \subseteq V$ and*

$$\mathrm{Sp}\,\Big(\boxtimes_{i=1}^{k} \phi_i(f[A_i])\Big) \cup \{f(x)^k + s(x)^k(y - x^k) : x, y \in [m, M]\} \;\subseteq\; U.$$

Suppose that F is bounded and operator-monotone in the first variable.

1. *If f is convex on $[m, M]$, then for every $x \in [m, M]$,*

$$F\Big[\boxtimes_{i=1}^{k} \phi_i(f[A_i]), g[\boxtimes_{i=1}^{k} \phi_i(A_i)]\Big] \geqslant \left\{ \inf_{m_\phi \leqslant t \leqslant M_\phi} F(f(x)^k + s(x)^k(t - x^k), g(t)) \right\} I, \tag{4.18}$$

$$F\Big[\boxdot_{i=1}^{k} \phi_i(f[A_i]), g[\boxdot_{i=1}^{k} \phi_i(A_i)]\Big] \geqslant \left\{ \inf_{m_\phi \leqslant t \leqslant M_\phi} F(f(x)^k + s(x)^k(t - x^k), g(t)) \right\} I. \tag{4.19}$$

2. *If f is concave on $[m, M]$, the opposite inequalities hold in* (4.18) *and* (4.19) *with* sup *instead of* inf.

Proof. We only prove the case f is convex. Let $x \in [m, M]$. Since f is convex, we have $f(y) \geqslant f(x) + s(x)(y - x)$ for $y \in [m, M]$. Using functional calculus, we get $f[A_i] \geqslant f(x)I + s(x)(A_i - xI)$ for all $i = 1, \ldots, k$. Since ϕ_i is a unital positive linear map, we have

$$\phi_i(f[A_i]) \geqslant f(x)I - s(x)(\phi_i(A_i) - xI).$$

We have by the monotonicity of the Tracy-Singh product and Lemma 2.2 that

$$\begin{aligned}
\mathop{\boxtimes}_{i=1}^{k} \phi_i(f[A_i]) &\geqslant \mathop{\boxtimes}_{i=1}^{k} f(x)I - \mathop{\boxtimes}_{i=1}^{k} s(x)(\phi_i(A_i) - xI) \\
&= f(x)^k \mathop{\boxtimes}_{i=1}^{k} I - s(x)^k \Big(\mathop{\boxtimes}_{i=1}^{k} \phi_i(A_i) - \mathop{\boxtimes}_{i=1}^{k} xI \Big) \\
&= f(x)^k I - s(x)^k \big(\mathop{\boxtimes}_{i=1}^{k} \phi_i(A_i) - x^k I \big).
\end{aligned}$$

Using Lemma 2.3, we get

$$\mathop{\boxdot}_{i=1}^{k} \phi_i(f[A_i]) \geqslant f(x)^k I - s(x)^k \big(\mathop{\boxdot}_{i=1}^{k} \phi_i(A_i) - x^k I \big).$$

Applying the monotonicity of $F(\cdot, v)$, we obtain (4.18) and (4.19). □

Applying Theorem 4.12 for the function $F(u, v) = u - \alpha v$, we obtain the following corollary.

Corollary 4.13. *Assume Hypotheses 4.1, 4.3 and 4.4. Let $f : \mathbb{R} \to \mathbb{R}$, $g : [m, M] \to \mathbb{R}$ be functions, $\alpha > 0$ and $c \in [m, M]$.*

1. *If f is convex and g is strictly-concave differentiable, then*

$$\mathop{\boxtimes}_{i=1}^{k} \phi_i(f[A_i]) \geqslant \alpha g[\mathop{\boxtimes}_{i=1}^{k} \phi_i(A_i)] + \beta I, \tag{4.20}$$

$$\mathop{\boxdot}_{i=1}^{k} \phi_i(f[A_i]) \geqslant \alpha g[\mathop{\boxdot}_{i=1}^{k} \phi_i(A_i)] + \beta I, \tag{4.21}$$

hold for $\beta = f(c)^k + s(c)^k(t_0 - c^k) - \alpha g(t_0)$, where

$$t_0 = \begin{cases} \text{the unique solution of } \alpha g'(t) = s(c)^k & \text{if } \alpha g'(M_\phi) \leqslant s(c)^k \leqslant \alpha g'(m_\phi), \\ m_\phi & \text{if } s(c)^k \geqslant \alpha g'(m_\phi), \\ M_\phi & \text{if } s(c)^k \leqslant \alpha g'(M_\phi). \end{cases}$$

2. *If f is convex and g is continuous and convex, then the inequalities hold in* (4.20) *and* (4.21) *for*

$$t_0 = \begin{cases} m_\phi & \text{if } s(c)^k \geqslant \alpha\mu_g, \\ M_\phi & \text{if } s(c)^k \leqslant \alpha\mu_g. \end{cases}$$

3. *If f is concave and g is strictly-convex differentiable, the opposite inequalities hold in* (4.20) *and* (4.21) *with the same t_0 in the case 1 but the opposite condition while determining t_0.*

4. *If f is concave and g is continuous and concave, the opposite inequalities hold in* (4.20) *and* (4.21) *with the same t_0 in the case 2 but the opposite condition while determining t_0.*

5 Jensen's Type Inequalities Involving Tracy-Singh Sums and Khatri-Rao Sums

This section deals with operator inequalities for Tracy-Singh sums and Khatri-Rao sums. recall the following result.

Lemma 5.1 ([18])**.** *Let $A \in \mathbb{B}(\mathcal{H})^+$ and $B \in \mathbb{B}(\mathcal{K})^+$. For any $p \in \mathbb{N}$, we have*

$$A^p \boxplus B^p \leqslant (A \boxplus B)^p.$$

In [12, Corollary 26], this inequality holds for the Khatri-Rao sum of operators when A and B are the direct sums $A = A_1 \oplus \cdots \oplus A_n$ and $B = B_1 \oplus \cdots \oplus B_n$ when $A_i \in \mathbb{B}(\mathcal{H}_i)$ and $B \in \mathbb{B}(\mathcal{K}_i)$ for each $i = 1, \ldots, n$. Now, we will give an upper bound for $A^p \circledast B^p$ without these conditions.

Proposition 5.2. *Let $A \in \mathbb{B}(\mathcal{H})^+$ and $B \in \mathbb{B}(\mathcal{K})^+$ be such that* $\mathrm{Sp}(A \boxplus B) \subseteq [m, M]$*. For any $p \in \mathbb{N}$, we have*

$$A^p \circledast B^p \leqslant K(m, M, p)(A \circledast B)^p, \tag{5.1}$$

$$A^p \circledast B^p \leqslant (A \circledast B)^p + C(m, M, p)I. \tag{5.2}$$

Proof. Using Lemmas 2.5, 5.1 and applying Lemma 2.6 for the function $f(t) = t^p$ with $k = 1$, we get

$$\begin{aligned} A^p \circledast B^p &= \psi(A^p \boxplus B^p) \leqslant \psi((A \boxplus B)^p) \\ &\leqslant K(m, M, p)\psi(A \boxplus B)^p = K(m, M, p)(A \circledast B)^p. \end{aligned}$$

Using Lemmas 2.5, 5.1 and applying Lemma 2.7 for the function $f(t) = t^p$ with $k = 1$, we get

$$\begin{aligned} A^p \circledast B^p - (A \circledast B)^p &= \psi(A^p \boxplus B^p) - [\psi(A \boxplus B)]^p \\ &\leqslant \psi((A \boxplus B)^p) - \psi(A \boxplus B)^p \leqslant C(m, M, p)I. \qquad \square \end{aligned}$$

Matrix versions of inequalities in Proposition 5.2 were given in [19, Theorem 3.11].

Corollary 5.3. *Let $A \in \mathbb{B}(\mathcal{H})^+$ and $B \in \mathbb{B}(\mathcal{K})^+$ be such that* $\mathrm{Sp}(A \boxplus B) \subseteq [m, M]$. *Then*

$$A^2 \circledast B^2 \leqslant K(m, M)(A \circledast B)^2, \tag{5.3}$$

$$A^2 \circledast B^2 \leqslant (A \circledast B)^2 + \frac{(M-m)^2}{4} I. \tag{5.4}$$

We now consider special cases of Lemmas 2.6 and 2.7. For any $A, B \in \mathbb{B}(\mathcal{H})^+$ and $p \in (-\infty, 0) \cup [1, \infty)$, we have

$$A^p + B^p \leqslant \frac{K(m, M, p)}{2^{p-1}}(A + B)^p, \tag{5.5}$$

$$A^p + B^p \leqslant \frac{1}{2^{p-1}}(A + B)^p + 2C(m, M, p)I. \tag{5.6}$$

From above inequalities, we will consider the Khatri-Rao sum and Tracy-Singh sum as the "sum". The following theorem gives upper bounds for $(A \boxplus B)^p$ and $(A \circledast B)^p$ when $p \in \mathbb{R}$.

Theorem 5.4. *Let $A \in \mathbb{B}(\mathcal{H})^+$ and $B \in \mathbb{B}(\mathcal{K})^+$ be such that* $\mathrm{Sp}(A) \subseteq [m, M]$ *and* $\mathrm{Sp}(B) \subseteq [m, M]$. *For any $p \geqslant 1$, we have*

$$A^p \circledast B^p \leqslant \frac{K(m, M, p)}{2^{p-1}}(A \circledast B)^p, \tag{5.7}$$

$$A^p \circledast B^p \leqslant \frac{1}{2^{p-1}}(A \circledast B)^p + 2C(m, M, p)I. \tag{5.8}$$

While, for $0 < p < 1$, we have the reverse inequalities in (5.7) *and* (5.8). *If $A \boxdot B$ is invertible, then* (5.7) *and* (5.8) *hold for $p < 0$ and $p > 1$. We can replace the Khatri-Rao sum $\circledast$ in* (5.7) *and* (5.8) *by the Tracy-Singh sum $\boxplus$.*

Proof. From Lemma 2.6, setting $k = 2$, $w_1 = w_2 = \frac{1}{2}$ and $f(t) = t^p$ ($p < 0$ or $p > 1$), we get

$$\frac{1}{2}\left[\phi_1(A^p) + \phi_2(B^p)\right] \leqslant \frac{K(m, M, p)}{2^p}[\phi_1(A) + \phi_2(B)]^p. \tag{5.9}$$

Using Lemmas 2.2, 2.3 and (5.9), we have

$$\begin{aligned} A^p \circledast B^p &= A^p \boxdot I + I \boxdot B^p \\ &= \psi((A \boxtimes I)^p) + \psi((I \boxtimes B)^p) \\ &\leqslant \frac{K(m, M, p)}{2^{p-1}}\left[\psi(A \boxtimes I) + \psi(I \boxtimes B)\right]^p \\ &= \frac{K(m, M, p)}{2^{p-1}}(A \circledast B)^p. \end{aligned}$$

Similarly, we get (5.8) by applying Lemmas 2.6, 2.2 and 2.3. □

Corollary 5.5. *Let* $A \in \mathbb{B}(\mathcal{H})^+$ *and* $B \in \mathbb{B}(\mathcal{K})^+$ *be such that* $\mathrm{Sp}(A) \subseteq [m, M]$ *and* $\mathrm{Sp}(B) \subseteq [m, M]$*. Then*

$$A^2 \circledast B^2 \;\leqslant\; \frac{K(m,M)}{2}(A \circledast B)^2, \tag{5.10}$$

$$A^2 \circledast B^2 \;\leqslant\; \frac{1}{2}(A \circledast B)^2 + \frac{(M-m)^2}{2} I. \tag{5.11}$$

If $A \circledast B$ *is invertible, then*

$$A^{-1} \circledast B^{-1} \;\leqslant\; 4K(m,M)(A \circledast B)^{-1}, \tag{5.12}$$

$$A^{-1} \circledast B^{-1} \;\leqslant\; 4(A \circledast B)^{-1} + \frac{2(\sqrt{M}-\sqrt{m})^2}{Mm} I. \tag{5.13}$$

All inequalities in Corollary 5.5 were proved in [20, Theorems 15 and 17] under the condition $mI \leqslant (A \boxtimes I) \oplus (I \boxtimes B) \leqslant MI$.

Acknowledgement : The first author expresses his gratitude towards Thailand Research Fund for providing the Royal Golden Jubilee Ph.D. Scholarship, grant no. PHD60K0225 to support his Ph.D. study.

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(Received 27 May 2019)
(Accepted 24 December 2019)