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On 2-absorbing Primary Ideals in Commutative Γ-semirings

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Abstract : The definition of 2-absorbing primary ideals which are generalizations of prime ideals and 2-absorbing ideals in commutative Γ-semirings is given. Then, various properties of 2-absorbing primary ideals are investigated. Moreover, characterizations of ideals to be 2-absorbing primary ideals are provided.

Keywords : Γ-semirings; 2-absorbing primary ideals; 2-absorbing ideals; prime ideals; primary ideals.
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1 Introduction

As a generalization of rings, Γ-rings were introduced by N. Nobusawa [1] in 1964. Also, as a generalization of semirings and Γ-rings, the notion of Γ-semirings was introduced by M. K. Rao [2] in 1995. Some properties of ideals and k-ideals in a Γ-semiring were also discussed by M. K. Rao [2] in 1995 and T. K. Dutta and S. K. Sardar [3] in 2000. T. K. Dutta and S. K. Sardar [4] in 2001 gave the definition of prime ideals in Γ-semirings and studied some of their properties. In 2017, M. K. Rao and B. Venkateswarlu [5] initiated the definition of primary ideals in Γ-semirings which is a generalization of prime ideals in Γ-semirings.

The concept of 2-absorbing ideals in commutative rings was introduced by A. Badawi [6] in 2007 which is a generalization of prime ideals in commutative rings. Recently, A. Badawi [7] in 2014 introduced the concept of 2- absorbing primary ideals in commutative rings and gave some characterizations related to it. The

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notion of 2-absorbing primary ideals in semirings was introduced by P. Kumar, M. K. Dubey and P. Sarohe [8] in 2016.

Our main goal is to provide the notion of 2-absorbing primary ideals in commutative Γ-semirings which are extended from those in Γ-rings. Also, we study these properties and present some of their characterizations.

2 Preliminaries

In this section, we recall some of fundamental concepts and definitions which are necessary for this paper.

Definition 2.1 ([2])**.** For any commutative semigroups $(R, +)$ and $(\Gamma, +)$, R is called a **Γ-semiring** if there exists a function $\cdot$ from $R \times \Gamma \times R$ into R, where $\cdot(x, \gamma, y)$ is denoted by $x\gamma y$ for all $x, y \in R$ and $\gamma \in \Gamma$, satisfying the following properties: for all $x, y, z \in R$ and $\gamma, \beta \in \Gamma$,
1. $x\gamma(y + z) = x\gamma y + x\gamma z$ and $(x + y)\gamma z = x\gamma z + y\gamma z$;
2. $x(\gamma + \beta)y = x\gamma y + x\beta y$; and
3. $(x\gamma y)\beta z = x\gamma(y\beta z)$.

Throughout this paper, let $\mathbb{Z}_0^+$ be the set of non-negative integers. Then $\mathbb{Z}_0^+$ is a semigroup under the usual addition. For a Γ-semiring R, $\emptyset \neq A, B \subseteq R$ and $\beta \in \Gamma$, let $A\Gamma B = \{ a\gamma b \mid a \in A, \gamma \in \Gamma \text{ and } b \in B \}$ and $A\beta B = \{ a\beta b \mid a \in A \text{ and } b \in B \}$.

Example 2.2. (1) Let R be the commutative semigroup containing all $m \times n$ matrices over $\mathbb{Z}_0^+$ under the usual addition and Γ be the commutative semigroup containing all $n \times m$ matrices over $\mathbb{Z}_0^+$ under the usual addition. Then R is a Γ-semiring where $a\gamma b$ is the usual matrix product for any $a, b \in R$ and $\gamma \in \Gamma$.
(2) For each $n \in \mathbb{N}$, recall that $n\mathbb{Z}_0^+ = \{ na \mid a \in \mathbb{Z}_0^+\}$ is a commutative semigroup under the usual addition of integers. Then $n\mathbb{Z}_0^+$ is an $m\mathbb{Z}_0^+$-semiring for all $m, n \in \mathbb{N}$ where $x\gamma y$ is the usual multiplication of integers for all $x, y \in n\mathbb{Z}_0^+$ and $\gamma \in m\mathbb{Z}_0^+$.

Definition 2.3 ([2])**.** A Γ-semiring R is said to have a **zero** element if there exists an element $0 \in R$ such that $x + 0 = x$ and $0\alpha x = x\alpha 0 = 0$ for all $x \in R$ and $\alpha \in \Gamma$.

Definition 2.4 ([9])**.** A Γ-semiring R is said to have a **unity** element if there exists an element $1 \in R$ such that for all $x \in R$, there exists $\alpha \in \Gamma$ such that $1\alpha x = x = x\alpha 1$.

Definition 2.5 ([2])**.** A Γ-semiring R is said to be **commutative** if $x\alpha y = y\alpha x$ for all $x, y \in R$ and $\alpha \in \Gamma$.

Definition 2.6 ([2])**.** Let R be a Γ-semiring and A be a subset of R. Then A is called a **Γ-subsemiring** of R if A is a subsemigroup of $(R, +)$ and $A\Gamma A \subseteq A$.

Proposition 2.7 ([2]). *Let R_i be a Γ_i-semiring for all $i \in \{1, 2, \ldots, n\}$. Then $R_1 \times R_2 \times \cdots \times R_n$ is a $(\Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_n)$-semiring where*

$$(x_1, x_1, \ldots, x_n) + (y_1, y_2, \ldots, y_n) = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n) \text{ and}$$
$$(x_1, x_1, \ldots, x_n)(\gamma_1, \gamma_2, \ldots, \gamma_n)(y_1, y_2, \ldots, y_n) = (x_1\gamma_1 y_1, x_2\gamma_2 y_2, \ldots, x_n\gamma_n y_n)$$

for all $x_i, y_i \in R_i$, $\gamma_i \in \Gamma_i$ and $i \in \{1, 2, \ldots, n\}$.

Moreover, if R_i is commutative for all $i \in \{1, 2, \ldots, n\}$, then the $(\Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_n)$-semiring $R_1 \times R_2 \times \cdots \times R_n$ is also commutative.

Definition 2.8 ([2]). A subset I of a Γ-semiring R is called an **ideal** in R if I is a subsemigroup of $(R, +)$, $I\Gamma R \subseteq I$ and $R\Gamma I \subseteq I$.

It is clear that a Γ-semiring R is an ideal in R. Moreover, if R is a Γ-semiring with zero 0, then $0 \in I$ for all ideal I in R.

Definition 2.9 ([2]). An ideal I in a Γ-semiring R is called a **k-ideal** in R if for all $x, y \in R, x + y \in I$ and $x \in I$ implies $y \in I$.

Example 2.10. From Example 2.2 (2), $\mathbb{Z}_0^+$ is a $5\mathbb{Z}_0^+$-semiring. Moreover, $3\mathbb{Z}_0^+$ is a k-ideal in $\mathbb{Z}_0^+$. However, $3\mathbb{Z}_0^+ - \{3\}$ is an ideal in $\mathbb{Z}_0^+$ but it is not a k-ideal in $\mathbb{Z}_0^+$ because $6 + 3 \in 3\mathbb{Z}_0^+ - \{3\}, 6 \in 3\mathbb{Z}_0^+ - \{3\}$ but $3 \notin 3\mathbb{Z}_0^+ - \{3\}$.

Proposition 2.11 ([4]). *Let R be a Γ-semiring with zero and $a \in R$. Define*

$$\langle a \rangle = \{ \; na + \sum_{j=1}^{p} a\eta_j t_j + \sum_{k=1}^{q} u_k\delta_k a + \sum_{l=1}^{s} v_l\mu_l a\lambda_l w_l \mid$$
$$n \in \mathbb{Z}_0^+, p, q, s \in \mathbb{Z}^+, \text{ all } t_j, u_k, v_l, w_l \in R \text{ and all } \eta_j, \delta_k, \mu_l, \lambda_l \in \Gamma \; \}.$$

Then $\langle a \rangle$ is an ideal in R containing a.

Definition 2.12 ([4]). Let ρ be an equivalence relation on a commutative Γ-semiring R. Then ρ is called a **Γ-congruence** on R if $x\rho x'$ and $y\rho y'$ implies $(x + y)\rho(x' + y')$ and $(x\gamma y)\rho(x'\gamma y')$ for all $x, y, x', y' \in R$ and $\gamma \in \Gamma$.

Definition 2.13 ([4]). Let I be a proper ideal in a commutative Γ-semiring R and ρ_I be a Γ-congruence on R. Then ρ_I is called **the Bourne Γ-congruence** on R if for all $x, y \in R$, $x\rho_I y$ if and only if $x + i_1 = y + i_2$ for some $i_1, i_2 \in I$.

The Bourne Γ-congruence class of an element r of R is denoted by r/ρ_I or simply by r/I and the set of all such Γ-congruence classes of the elements of R is denoted by R/ρ_I or simply by R/I.

For any proper ideal I of R, R/I is a commutative Γ-semiring where

$$r/I + r'/I = (r + r')/I \text{ and } (r/I)\alpha(r'/I) = (r\alpha r')/I \text{ for all } r, r' \in R \text{ and } \alpha \in \Gamma.$$

Proposition 2.14 ([10]). *If I and J are ideals in R and $I \subsetneq J$, then*

(i) I is also an ideal in the Γ-subsemiring J; and

(ii) J/I is an ideal in the Γ-semiring R/I.

Lemma 2.15. *Let I be a proper ideal in R and P be a k-ideal in R such that $I \subsetneq P$. Then, for all $a \in R$, $a/I \in P/I$ if and only if $a \in P$.*

Proof. If $a \in P$, then it is obvious that $a/I \in P/I$.

Next, let $a/I \in P/I$. Then, $a/I = p/I$ for some $p \in P$. Thus, there exist $i_1, i_2 \in I$ such that $a + i_1 = p + i_2$. Since $i_1, i_2 \in I \subseteq P$ and P is a k-ideal, we have $a \in P$. □

3 Radical Ideals

Throughout this section, properties of radical ideals in commutative Γ-semirings are investigated. However, we focus on those which are involving with 2-absorbing primary ideals which will be applied in the fourth section. In this section, let R be a commutative Γ-semiring.

Proposition 3.1 ([5]). *Let I be an ideal in R. Then*

$$\sqrt{I} := \{x \in R \mid \text{ there exists } n \in \mathbb{N} \text{ such that } (x\gamma)^{n-1}x \in I \text{ for all } \gamma \in \Gamma\}$$

*is an ideal in R containing I where $(x\gamma)^0 x = x$ and $(x\gamma)^n x = (x\gamma)^{n-1}x\gamma x$ for all $x \in R, \gamma \in \Gamma$ and $n \in \mathbb{N}$. The ideal $\sqrt{I}$ is called the **radical ideal** of I.*

Proposition 3.2. *Let I and J be ideals in R. If $I \subseteq J$, then $\sqrt{I} \subseteq \sqrt{J}$.*

Proof. The proof is straightforward. □

Proposition 3.3. *Let R_i be a commutative Γ_i-semiring for all $i \in \{1, 2\}$. If I_1 and I_2 are ideals in R_1 and R_2, respectively, then $\sqrt{I_1} \times \sqrt{I_2} = \sqrt{I_1 \times I_2}$.*

Proof. Let I_1 and I_2 be ideals in R_1 and R_2, respectively.

First, let $a \in \sqrt{I_1}$ and $b \in \sqrt{I_2}$. Then there exist $n, m \in \mathbb{N}$ such that $(a\alpha_1)^{n-1}a \in I_1$ and $(b\alpha_2)^{m-1}b \in I_2$ for all $\alpha_1 \in \Gamma_1$ and $\alpha_2 \in \Gamma_2$. So,

$$((a, b)(\alpha_1, \alpha_2))^{n+m-1}(a, b) \in I_1 \times I_2 \qquad \text{for all } \alpha_1 \in \Gamma_1 \text{ and } \alpha_2 \in \Gamma_2.$$

Thus, $(a, b) \in \sqrt{I_1 \times I_2}$. Hence, $\sqrt{I_1} \times \sqrt{I_2} \subseteq \sqrt{I_1 \times I_2}$.

Next, let $(p, q) \in \sqrt{I_1 \times I_2}$. Then, there exists $m \in \mathbb{N}$ such that

$$((p, q)(\alpha_1, \alpha_2))^{m-1}(p, q) \in I_1 \times I_2 \text{ for all } \alpha_1 \in \Gamma_1 \text{ and } \alpha_2 \in \Gamma_2.$$

Hence, $(p\alpha_1)^{m-1}p \in I_1$ and $(q\alpha_2)^{m-1}q \in I_2$ for all $\alpha_1 \in \Gamma_1$ and $\alpha_2 \in \Gamma_2$. So, $(p, q) \in \sqrt{I_1} \times \sqrt{I_2}$. Thus, $\sqrt{I_1 \times I_2} \subseteq \sqrt{I_1} \times \sqrt{I_2}$.

Therefore, $\sqrt{I_1} \times \sqrt{I_2} = \sqrt{I_1 \times I_2}$. □

Proposition 3.4. *Let I be a proper ideal in R and P be a k-ideal in R such that $I \subsetneq P$. Then $\sqrt{P}/I \subseteq \sqrt{P/I}$.*

Proof. Let $r \in \sqrt{P}$. Then there exists $n \in \mathbb{N}$ such that $(r\alpha)^{n-1}r \in P$ for any $\alpha \in \Gamma$. So, $((r/I)\alpha)^{n-1}(r/I) = ((r\alpha)^{n-1}r)/I \in P/I$ for any $\alpha \in \Gamma$. Thus, $r/I \in \sqrt{P/I}$.

Therefore, $\sqrt{P}/I \subseteq \sqrt{P/I}$. □

Definition 3.5 ([2])**.** Let R_1 and R_2 be Γ-semirings (not necessary commutative). Then $g : R_1 \to R_2$ is called a **homomorphism** if $g(x + y) = g(x) + g(y)$ and $g(x\gamma y) = g(x)\gamma g(y)$ for all $x, y \in R_1$ and $\gamma \in \Gamma$.

Definition 3.6. Let R_1 and R_2 be Γ-semirings (not necessary commutative) and $g : R_1 \to R_2$ be a homomorphism. Then g is called an **epimorphism** if g is surjective.

Example 3.7. Note that $3\mathbb{Z}_0^+$ and $\mathbb{Z}_0^+/7\mathbb{Z}_0^+$ are $5\mathbb{Z}_0^+$-semirings. Define $f : 3\mathbb{Z}_0^+ \to \mathbb{Z}_0^+/7\mathbb{Z}_0^+$ by $f(x) = x/7\mathbb{Z}_0^+$ for all $x \in 3\mathbb{Z}_0^+$. Then f is an epimorphism.

Proposition 3.8. *Let R_1 and R_2 be commutative Γ-semirings, $g : R_1 \to R_2$ be a homomorphism and I be an ideal in R_2. Then $g^{-1}(\sqrt{I}) = \sqrt{g^{-1}(I)}$.*

Proof. Clearly, $g^{-1}(I)$ is an ideal in R_1.

First, let $a \in g^{-1}(\sqrt{I})$.Then, $g(a) \in \sqrt{I}$. So, there exists $n \in \mathbb{N}$ such that $(g(a)\alpha)^{n-1}g(a) \in I$ for all $\alpha \in \Gamma$. Thus, $g((a\alpha)^{n-1}a) = (g(a)\alpha)^{n-1}g(a) \in I$ for all $\alpha \in \Gamma$. Then, $(a\alpha)^{n-1}a \in g^{-1}(I)$ for all $\alpha \in \Gamma$. So, $a \in \sqrt{g^{-1}(I)}$. Hence, $g^{-1}(\sqrt{I}) \subseteq \sqrt{g^{-1}(I)}$.

Next, let $a \in \sqrt{g^{-1}(I)}$. Then, there exists $n \in \mathbb{N}$ such that $(a\alpha)^{n-1}a \in g^{-1}(I)$ for all $\alpha \in \Gamma$. Thus, $(g(a)\alpha)^{n-1}g(a) = g((a\alpha)^{n-1}a) \in I$ for all $\alpha \in \Gamma$. So, $g(a) \in \sqrt{I}$, i.e., $a \in g^{-1}(\sqrt{I})$. Then, $\sqrt{g^{-1}(I)} \subseteq g^{-1}(\sqrt{I})$.

Therefore, $g^{-1}(\sqrt{I}) = \sqrt{g^{-1}(I)}$. □

Unlike the previous proposition, $g(\sqrt{I}) = \sqrt{g(I)}$ holds provided that g must also be surjective and I has to be a k-ideal.

Proposition 3.9. *Let R_1 and R_2 be commutative Γ-semirings, $g : R_1 \to R_2$ be an epimorphism and I be a k-ideal in R_1 such that $\{ x \in R_1 \mid \exists\, a, b \in R_1$ such that $x = a + b$ and $g(a) = g(b) \} \subseteq I$. Then $g(\sqrt{I}) = \sqrt{g(I)}$.*

Proof. Clearly, $g(I)$ is an ideal in R_2.

First, let $a \in g(\sqrt{I})$. Then, there exists $p \in \sqrt{I}$ such that $g(p) = a$. So, there exists $n \in \mathbb{N}$ such that $(p\alpha)^{n-1}p \in I$ for all $\alpha \in \Gamma$. Thus, $(a\alpha)^{n-1}a = (g(p)\alpha)^{n-1}g(p) = g((p\alpha)^{n-1}p)$ for all $\alpha \in \Gamma$. Since $(p\alpha)^{n-1}p \in I$, we have $(a\alpha)^{n-1}a \in g(I)$ for all $\alpha \in \Gamma$. So, $a \in \sqrt{g(I)}$. Hence, $g(\sqrt{I}) \subseteq \sqrt{g(I)}$.

Next, let $a \in \sqrt{g(I)}$. Then, there exists $n \in \mathbb{N}$ such that $(a\alpha)^{n-1}a \in g(I)$ for all $\alpha \in \Gamma$. Fix $\alpha \in \Gamma$. So, there exits $p \in I$ such that $g(p) = (a\alpha)^{n-1}a$. Since g is surjective, there exists $q \in R_1$ such that $g(q) = a$. Thus, $g(p) = (a\alpha)^{n-1}a =$

$(g(q)\alpha)^{n-1}g(q) = g((q\alpha)^{n-1}q)$. So, $p + (q\alpha)^{n-1}q \in \{ x \in R_1 \mid \exists\, b, c \in R_1$ such that $x = b + c$ and $g(b) = g(c) \} \subseteq I$. Since $p \in I$ and I is a k-ideal, $(q\alpha)^{n-1}q \in I$. Then, $q \in \sqrt{I}$. Thus, $a \in g(\sqrt{I})$. Hence, $\sqrt{g(I)} \subseteq g(\sqrt{I})$.

Therefore, $g(\sqrt{I}) = \sqrt{g(I)}$. □

4 2-absorbing Primary Ideals

In this section, we introduce the concept of 2-absorbing primary ideals in a commutative Γ-semiring and investigate some results related to it. Throughout this section, let R be a commutative Γ-semiring.

Definition 4.1 ([4]). A proper ideal P in a commutative Γ-semiring R is called a **prime ideal** in R if whenever $a, b \in R$, $a\Gamma b \subseteq P$ implies $a \in P$ or $b \in P$.

Example 4.2. From Example 2.2 (2), $\mathbb{Z}_0^+$ is a commutative $5\mathbb{Z}_0^+$-semiring. Then the ideal $2\mathbb{Z}_0^+$ is a prime ideal in $\mathbb{Z}_0^+$.

Proof. Let $x, y \in \mathbb{Z}_0^+$ be such that $x\Gamma y \subseteq 2\mathbb{Z}_0^+$. So, $2 \mid x \cdot 5 \cdot y$. Hence, $2 \mid x$ or $2 \mid y$. Thus, $x \in 2\mathbb{Z}_0^+$ or $y \in 2\mathbb{Z}_0^+$. Therefore, $2\mathbb{Z}_0^+$ is a prime ideal in $\mathbb{Z}_0^+$. □

Definition 4.3 ([5]). A proper ideal I in a commutative Γ-semiring R is called a **primary ideal** in R if whenever $a, b \in R$, $a\Gamma b \subseteq I$ implies $a \in I$ or $b \in \sqrt{I}$.

The following is the immediate result obtained from the definitions.

Remark 4.4. *Every prime ideal in R is a primary ideal in R.*

The following definitions that are given in the context of Γ-semirings were inspired by [11].

Definition 4.5. A proper ideal I in a commutative Γ-semiring R is called a **2-absorbing ideal** in R if whenever $x, y, z \in R$, $\gamma, \beta \in \Gamma$ and $x\gamma y\beta z \in I$ implies $x\gamma y \in I$ or $x\beta z \in I$ or $y\beta z \in I$.

Definition 4.6. A proper ideal I in a commutative Γ-semiring R is called a **2-absorbing primary ideal** in R if whenever $x, y, z \in R$, $\gamma, \beta \in \Gamma$ and $x\gamma y\beta z \in I$, then $x\gamma y \in I$ or $x\beta z \in \sqrt{I}$ or $y\beta z \in \sqrt{I}$.

Definition 4.5 and Definition 4.6 lead to the following remark.

Remark 4.7. *Every 2-absorbing ideal in R is a 2-absorbing primary ideal in R.*

However, the converse of the above remark does not hold.

Example 4.8. From Example 2.2 (2), $\mathbb{Z}_0^+$ is a commutative $5\mathbb{Z}_0^+$-semiring. Then, $8\mathbb{Z}_0^+$ is a 2-absorbing primary ideal in $\mathbb{Z}_0^+$ but it is not a 2-absorbing ideal in $\mathbb{Z}_0^+$.

Proof. Let $x, y, z \in \mathbb{Z}_0^+$ and $\gamma, \beta \in 5\mathbb{Z}_0^+$ be such that $x\gamma y\beta z \in 8\mathbb{Z}_0^+$. If $8 \mid x\gamma y$, we are done. Suppose $8 \nmid x\gamma y$. Then, $2 \mid \beta z$. So, $8 \mid (x\beta z\alpha)^2 x\beta z$ for all $\alpha \in \Gamma$, that is $x\beta z \in \sqrt{8\mathbb{Z}_0^+}$. Thus, $8\mathbb{Z}_0^+$ is a 2-absorbing primary ideal in $\mathbb{Z}_0^+$.

Since $(2)(5)(2)(5)(2) \in 8\mathbb{Z}_0^+$ and $(2)(5)(2) \notin 8\mathbb{Z}_0^+$, it follows that $8\mathbb{Z}_0^+$ is not a 2-absorbing ideal in $\mathbb{Z}_0^+$. □

We can see from the next example that primary ideals need not be 2-absorbing ideals or prime ideals.

Example 4.9. From Example 2.2 (2), $\mathbb{Z}_0^+$ is a commutative $5\mathbb{Z}_0^+$-semiring. Then, the ideal $27\mathbb{Z}_0^+$ is a primary ideal in $\mathbb{Z}_0^+$ but it is not a 2-absorbing ideal so that it is not a prime ideal in $\mathbb{Z}_0^+$.

Proof. Let $x, y \in \mathbb{Z}_0^+$ be such that $x\Gamma y \subseteq 27\mathbb{Z}_0^+$. If $27 \mid x$, then $x \in 27\mathbb{Z}_0^+$. Suppose $27 \nmid x$. Since $x\Gamma y \subseteq 27\mathbb{Z}_0^+$, $3 \mid \alpha y$ for all $\alpha \in 5\mathbb{Z}_0^+$. Hence, $27 \mid (y\alpha)^3 y$ for all $\alpha \in 5\mathbb{Z}_0^+$. So, $y \in \sqrt{27\mathbb{Z}_0^+}$. Thus, $27\mathbb{Z}_0^+$ is a primary ideal in $\mathbb{Z}_0^+$.

Since $(3)(5)(3)(5)(3) \in 27\mathbb{Z}_0^+$ and $(3)(5)(3) \notin 27\mathbb{Z}_0^+$, it follows that $27\mathbb{Z}_0^+$ is not a 2-absorbing ideal in $\mathbb{Z}_0^+$. □

Next, we present a relationship between prime ideals and 2-absorbing ideals as well as a relationship between primary ideals and 2-absorbing primary ideals.

Proposition 4.10. *Every prime ideal in R is a 2-absorbing ideal in R and then it is a 2-absorbing primary ideal in R.*

Proof. Suppose that I is a prime ideal in R. Let $x, y, z \in R$ and $\gamma, \beta \in \Gamma$ be such that $x\gamma y\beta z \in I$. Then, $x\gamma y\Gamma y\beta z \subseteq I$. Since I is a prime ideal, we have $x\gamma y \in I$ or $y\beta z \in I$.

Therefore, I is a 2-absorbing ideal in R. □

Proposition 4.11. *Every primary ideal in R is a 2-absorbing primary ideal in R.*

Proof. Suppose that I is a primary ideal in R. Let $x, y, z \in R$ and $\gamma, \beta \in \Gamma$ be such that $x\gamma y\beta z \in I$. Then, $x\gamma y\Gamma y\beta z \subseteq I$. Since I is a primary ideal, we have $x\gamma y \in I$ or $y\beta z \in \sqrt{I}$. Thus, I is a 2-absorbing primary ideal in R. □

We can see from the next example that 2-absorbing ideals and 2-absorbing primary ideals need not be primary ideals.

Example 4.12. From Example 2.2 (2), $\mathbb{Z}_0^+$ is a commutative $5\mathbb{Z}_0^+$-semiring. Then, the ideal $10\mathbb{Z}_0^+$ is a 2-absorbing ideal in $\mathbb{Z}_0^+$ so that it is a 2-absorbing primary ideal. However, it is not a primary ideal in $\mathbb{Z}_0^+$.

Proof. Let $x, y, z \in \mathbb{Z}_0^+$ and $\gamma, \beta \in 5\mathbb{Z}_0^+$ be such that $x\gamma y\beta z \in 10\mathbb{Z}_0^+$. Then, $10 \mid x\gamma y\beta z$. So, $2 \mid x$ or $2 \mid \gamma$ or $2 \mid y$ or $2 \mid \beta$ or $2 \mid z$. If $2 \mid x$ or $2 \mid y$, then $10 \mid x\gamma y$. If $2 \mid z$, then $10 \mid y\beta z$. If $2 \mid \gamma$ or $2 \mid \beta$, then $10 \mid x\gamma y$ or $10 \mid x\beta z$. Hence, $x\gamma y \in 10\mathbb{Z}_0^+$ or $x\beta z \in 15\mathbb{Z}_0^+$ or $y\beta z \in 10\mathbb{Z}_0^+$. Thus, $10\mathbb{Z}_0^+$ is a 2-absorbing ideal in $\mathbb{Z}_0^+$ and then it is a 2-absorbing primary ideal.

Since $2(5\mathbb{Z}_0^+)1 \subseteq 10\mathbb{Z}_0^+$, $2 \notin 10\mathbb{Z}_0^+$ and $1 \notin \sqrt{10\mathbb{Z}_0^+}$, it follows that $10\mathbb{Z}_0^+$ is not a primary ideal in $\mathbb{Z}_0^+$. □

The following results are inspired by results in [7], [11] and [8].

Theorem 4.13. *Let R_1 and R_2 be commutative Γ-semirings and $g : R_1 \to R_2$ be a homomorphism. If I is a 2-absorbing primary ideal in R_2 such that $g^{-1}(I) \neq R_1$, then $g^{-1}(I)$ is a 2-absorbing primary in R_1.*

Proof. Suppose that I is a 2-absorbing primary ideal in R_2 such that $g^{-1}(I) \neq R_1$. Then, $g^{-1}(I)$ is a proper ideal in R_1. Let $x, y, z \in R$ and $\beta, \gamma \in \Gamma$ be such that $x\beta y\gamma z \in g^{-1}(I)$. Then, $g(x)\beta g(y)\gamma g(z) = g(x\beta y\gamma z) \in I$. Since I is a 2-absorbing primary ideal in R_2, we have $g(x\beta y) = g(x)\beta g(y) \in I$ or $g(x\gamma z) = g(x)\gamma g(z) \in \sqrt{I}$ or $g(y\gamma z) = g(y)\gamma g(z) \in \sqrt{I}$. Hence, $x\beta y \in g^{-1}(I)$ or $x\gamma z \in g^{-1}(\sqrt{I}) = \sqrt{g^{-1}(I)}$ or $y\gamma z \in g^{-1}(\sqrt{I}) = \sqrt{g^{-1}(I)}$.

Therefore, $g^{-1}(I)$ is a 2-absorbing primary ideal in R_1. □

Theorem 4.14. *Let R_1 and R_2 be commutative Γ-semirings, $g : R_1 \to R_2$ be an epimorphism and I be a k-ideal in R_1. If I is a 2-absorbing primary ideal in R_1 such that $g(I) \neq R_2$ and $\{ x \in R_1 \mid \exists\, a, b \in R_1$ such that $x = a + b$ and $g(a) = g(b) \} \subseteq I$, then $g(I)$ is a 2-absorbing primary ideal in R_2.*

Proof. Suppose that I is a 2-absorbing primary ideal in R_1 such that $g(I) \neq R_2$ and $\{ x \in R_1 \mid \exists\, a, b \in R_1$ such that $x = a + b$ and $g(a) = g(b) \} \subseteq I$. Then, $g(I)$ is a proper ideal in R_2. Let $x, y, z \in R_2$ and $\gamma, \beta \in \Gamma$ be such that $x\gamma y\beta z \in g(I)$. Then, there exists $t \in I$ such that $x\gamma y\beta z = g(t)$. Since g is surjective, there exist $p, q, r \in R$ such that $g(p) = x, g(q) = y$ and $g(r) = z$. Hence, $g(p\gamma q\beta r) = g(p)\gamma g(q)\beta g(r) = x\gamma y\beta z = g(t)$. So, $p\gamma q\beta r + t \in \{ x \in R_1 \mid \exists\, b, c \in R_1$ such that $x = b + c$ and $g(b) = g(c) \} \subseteq I$. Since $t \in I$ and I is a k-ideal, $p\gamma q\beta r \in I$. Since I is a 2-absorbing primary ideal, $p\gamma q \in I$ or $p\beta r \in \sqrt{I}$ or $q\beta r \in \sqrt{I}$. Hence, $x\gamma y = g(p)\gamma g(q) = g(p\gamma q) \in g(I)$ or $x\beta z = g(p)\beta g(r) = g(p\beta r) \in g(\sqrt{I}) = \sqrt{g(I)}$ or $y\beta z = g(p)\beta g(r) = g(q\beta r) \in g(\sqrt{I}) = \sqrt{g(I)}$.

Therefore, $g(I)$ is a 2-absorbing primary ideal in R_2. □

Theorem 4.15. *Let I be a proper ideal in R and P be a k-ideal in R such that $I \subsetneqq P$. Then P is a 2-absorbing primary ideal in R if and only if P/I is a 2-absorbing primary ideal in R/I.*

Proof. First, suppose that P is a 2-absorbing primary ideal in R. Then, P/I is a proper ideal in R/I. Let $x, y, z \in R$ and $\gamma, \beta \in \Gamma$ be such that $(x/I)\gamma(y/I)\beta(z/I) \in P/I$. Hence, $(x\gamma y\beta z)/I \in P/I$ and then by Lemma 2.15, $x\gamma y\beta z \in P$. Since P

is a 2-absorbing primary ideal, $x\gamma y \in P$ or $x\beta z \in \sqrt{P}$ or $y\beta z \in \sqrt{P}$. Hence, $(x/I)\gamma(y/I) = (x\gamma y)/I \in P/I$ or $(x/I)\beta(z/I) = (x\beta z)/I \in \sqrt{P}/I \subseteq \sqrt{P/I}$ or $(y/I)\beta(z/I) = (y\beta z)/I \in \sqrt{P}/I \subseteq \sqrt{P/I}$. Thus, P/I is a 2-absorbing primary ideal in R/I.

Next, suppose that P/I is a 2-absorbing primary ideal in R/I. By Lemma 2.15, P is a proper ideal in R. Let $x, y, z \in R$ and $\gamma, \beta \in \Gamma$ be such that $x\gamma y\beta z \in P$. Hence, $(x/I)\gamma(y/I)\beta(z/I) = (x\gamma y\beta z)/I \in P/I$. Since P/I is a 2-absorbing primary ideal, $(x\gamma y)/I = (x/I)\gamma(y/I) \in P/I$ or $(x\beta z)/I = (x/I)\beta(z/I) \in \sqrt{P/I}$ or $(y\beta z)/I = (y/I)\beta(z/I) \in \sqrt{P/I}$. If $(x\gamma y)/I \in P/I$, by Lemma 2.15, then $x\gamma y \in P$. Suppose $(x\beta z)/I \in \sqrt{P/I}$. Then there exists $n \in \mathbb{N}$ such that $((x\beta z\alpha)^{n-1}x\beta z)/I = (((x\beta z)/I)\alpha)^{n-1}((x\beta z)/I) \in P/I$ for all $\alpha \in \Gamma$. By Lemma 2.15, $(x\beta z\alpha)^{n-1}x\beta z \in P$ for all $\alpha \in \Gamma$. Hence, $x\beta z \in \sqrt{P}$. Similarly, if $(y\beta z)/I \in \sqrt{P/I}$, then $y\beta z \in \sqrt{P}$.

Therefore, P is a 2-absorbing primary ideal in R. □

If I is an ideal in R, then it can be shown that $(I : x) = \{ r \in R \mid r\gamma x \in I$ for all $\gamma \in \Gamma \}$ is an ideal in R containing I for all $x \in R$.

Theorem 4.16. *Let I be a 2-absorbing primary ideal in R and $\sqrt{I}$ be a prime ideal in R. Then $(I : x)$ is a 2-absorbing primary ideal in R for all $x \in R \setminus \sqrt{I}$.*

Proof. Let $x \in R\setminus\sqrt{I}$. Then $(I : x)$ is a proper ideal in R. Moreover, let $a, b, c \in R$ and $\gamma, \beta \in \Gamma$ be such that $a\gamma b\beta c \in (I : x)$. Hence, $a\gamma(b\beta c)\beta x \in I$. Since I is a 2-absorbing primary ideal, $a\gamma b\beta c \in I$ or $a\beta x \in \sqrt{I}$ or $b\beta c\beta x \in \sqrt{I}$.
Case 1. $a\gamma b\beta c \in I$. Since I is a 2-absorbing primary ideal, $a\gamma b \in I \subseteq (I : x)$ or $a\beta c \subseteq \sqrt{I} \subseteq \sqrt{(I : x)}$ or $b\beta c \subseteq \sqrt{I} \subseteq \sqrt{(I : x)}$.
Case 2. $a\beta x \in \sqrt{I}$. Hence, $a\beta c\Gamma x \subseteq \sqrt{I}$. Since $x \notin \sqrt{I}$ and $\sqrt{I}$ is a prime ideal, we have $a\beta c \in \sqrt{I} \subseteq \sqrt{(I : x)}$.
Case 3. $b\beta c\beta x \in \sqrt{I}$. Hence, $b\beta c\beta c\Gamma x \subseteq \sqrt{I}$. Since $x \notin \sqrt{I}$ and $\sqrt{I}$ is a prime ideal, $b\beta c\beta c \in \sqrt{I}$. Then there exists $n \in \mathbb{N}$ such that $(b\beta c\beta c\alpha)^{n-1}b\beta c\beta c \in I$ for all $\alpha \in \Gamma$. So, $(b\beta c\alpha)^{2n-1}b\beta c \in I$ for all $\alpha \in \Gamma$. It follows that $b\beta c \in \sqrt{I} \subseteq \sqrt{(I : x)}$.

Therefore, $(I : x)$ is a 2-absorbing primary ideal in R. □

Lemma 4.17. *Let I be a 2-absorbing primary ideal in R and $\sqrt{I}$ be a k-ideal in R. Suppose that there exist $a, b \in R$, an ideal J in R and $\gamma, \beta \in \Gamma$ such that $a\gamma b\beta J \subseteq I$. If $a\gamma b \notin I$, then $a\beta J \subseteq \sqrt{I}$ or $b\beta J \subseteq \sqrt{I}$.*

Proof. Assume $a\gamma b \notin I$. Suppose that $a\beta J \not\subseteq \sqrt{I}$ and $b\beta J \not\subseteq \sqrt{I}$. Then, there exist $j_1, j_2 \in J$ such that $a\beta j_1 \notin \sqrt{I}$ and $b\beta j_2 \notin \sqrt{I}$. Since $a\gamma b\beta j_1 \in I$, $a\gamma b \notin I$ and $a\beta j_1 \notin \sqrt{I}$, we have $b\beta j_1 \in \sqrt{I}$. Since $a\gamma b\beta j_2 \in I$, $a\gamma b \notin I$ and $b\beta j_2 \notin \sqrt{I}$, we have $a\beta j_2 \in \sqrt{I}$. Since $a\gamma b\beta(j_1 + j_2) \in I$ and $a\gamma b \notin I$, we have $a\beta(j_1 + j_2) \in \sqrt{I}$ or $b\beta(j_1 + j_2) \in \sqrt{I}$.
Case 1. $a\beta(j_1 + j_2) \in \sqrt{I}$. Since $\sqrt{I}$ is a k-ideal in R and $a\beta j_2 \in \sqrt{I}$, it follows that $a\beta j_1 \in \sqrt{I}$, which is a contradiction.
Case 2. $b\beta(j_1 + j_2) \in \sqrt{I}$. Since $\sqrt{I}$ is a k-ideal in R and $b\beta j_1 \in \sqrt{I}$, it follows

that $b\beta j_2 \in \sqrt{I}$, which is a contradiction.

Therefore, $a\beta J \subseteq \sqrt{I}$ or $b\beta J \subseteq \sqrt{I}$. □

Next, we present a characterization of 2-absorbing primary ideals.

Theorem 4.18. *Let I be a proper k-ideal in a commutative Γ-semiring R with zero and $\sqrt{I}$ be a k-ideal in R. Then I is a 2-absorbing primary ideal in R if and only if whenever ideals I_1, I_2, I_3 in R and $\gamma, \beta \in \Gamma$ with $I_1\gamma I_2\beta I_3 \subseteq I$, then $I_1\gamma I_2 \subseteq I$ or $I_1\beta I_3 \subseteq \sqrt{I}$ or $I_2\beta I_3 \subseteq \sqrt{I}$.*

Proof. First, suppose that I is a 2-absorbing primary ideal in R and let I_1, I_2 and I_3 be ideals in R and $\gamma, \beta \in \Gamma$ be such that $I_1\gamma I_2\beta I_3 \subseteq I$. Suppose to the contrary that $I_1\gamma I_2 \not\subseteq I$ and $I_1\beta I_3 \not\subseteq \sqrt{I}$ and $I_2\beta I_3 \not\subseteq \sqrt{I}$. Then, there exist $a, q_1 \in I_1$ and $b, q_2 \in I_2$ such that $a\gamma b \notin I$ and $q_1\beta I_3 \not\subseteq \sqrt{I}$ and $q_2\beta I_3 \not\subseteq \sqrt{I}$. Since $q_1\gamma q_2\beta I_3 \subseteq I$ and $q_1\beta I_3 \not\subseteq \sqrt{I}$ and $q_2\beta I_3 \not\subseteq \sqrt{I}$, by Lemma 4.17, we have $q_1\gamma q_2 \in I$. Since $a\gamma b\beta I_3 \subseteq I$ and $a\gamma b \notin I$, by Lemma 4.17, we have $a\beta I_3 \subseteq \sqrt{I}$ or $b\beta I_3 \subseteq \sqrt{I}$.
Case 1. $a\beta I_3 \subseteq \sqrt{I}$ and $b\beta I_3 \not\subseteq \sqrt{I}$. Since $q_1\gamma b\beta I_3 \subseteq I$ and $b\beta I_3 \not\subseteq \sqrt{I}$ and $q_1\beta I_3 \not\subseteq \sqrt{I}$, by Lemma 4.17, we have $q_1\gamma b \in I$. Since $\sqrt{I}$ is a k-ideal and $a\beta I_3 \subseteq \sqrt{I}$ and $q_1\beta I_3 \not\subseteq \sqrt{I}$, we have $(a + q_1)\beta I_3 \not\subseteq \sqrt{I}$. Since $(a + q_1)\gamma b\beta I_3 \subseteq I$, $(a + q_1)\beta I_3 \not\subseteq \sqrt{I}$ and $b\beta I_3 \not\subseteq \sqrt{I}$, by Lemma 4.17, we have $(a + q_1)\gamma b \in I$. Since I is a k-ideal and $q_1\gamma b \in I$, we have $a\gamma b \in I$, which is a contradiction.
Case 2. $a\beta I_3 \not\subseteq \sqrt{I}$ and $b\beta I_3 \subseteq \sqrt{I}$. This case is not possible similarly to Case 1.
Case 3. $a\beta I_3 \subseteq \sqrt{I}$ and $b\beta I_3 \subseteq \sqrt{I}$. Since $\sqrt{I}$ is a k-ideal, $b\beta I_3 \subseteq \sqrt{I}$ and $q_2\beta I_3 \not\subseteq \sqrt{I}$, we have $(b + q_2)\beta I_3 \not\subseteq \sqrt{I}$. Since $\sqrt{I}$ is a k-ideal, $a\beta I_3 \subseteq \sqrt{I}$ and $q_1\beta I_3 \not\subseteq \sqrt{I}$, we have $(a+q_1)\beta I_3 \not\subseteq \sqrt{I}$. Since $q_1\gamma(b+q_2)\beta I_3 \subseteq I$, $q_1\beta I_3 \not\subseteq \sqrt{I}$ and $(b + q_2)\beta I_3 \not\subseteq \sqrt{I}$, by Lemma 4.17, we have $q_1\gamma(b + q_2) \in I$. Since I is a k-ideal and $q_1\gamma q_2 \in I$, we have $q_1\gamma b \in I$. Since $(a + q_1)\gamma q_2\beta I_3 \subseteq I$, $q_2\beta I_3 \not\subseteq \sqrt{I}$ and $(a + q_1)\beta I_3 \not\subseteq \sqrt{I}$, by Lemma 4.17, we have $(a + q_1)\gamma q_2 \in I$. Since I is a k-ideal and $q_1\gamma q_2 \in I$, we have $a\gamma q_2 \in I$. Since $(a+q_1)\gamma(b+q_2)\beta I_3 \subseteq I$, $(a+q_1)\beta I_3 \not\subseteq \sqrt{I}$ and $(b + q_2)\beta I_3 \not\subseteq \sqrt{I}$, by Lemma 4.17, we have $(a + q_1)\gamma(b + q_2) \in I$. Since I is a k-ideal, $q_1\gamma q_2 \in I$, $a\gamma q_2 \in I$ and $q_1\gamma b \in I$, we have $a\gamma b \in I$, which is a contradiction.
Hence, $I_1\gamma I_2 \subseteq I$ or $I_1\beta I_3 \subseteq \sqrt{I}$ or $I_2\beta I_3 \subseteq \sqrt{I}$.

On the other hand, suppose that whenever ideals I_1, I_2, I_3 in R and $\gamma, \beta \in \Gamma$ with $I_1\gamma I_2\beta I_3 \subseteq I$, then $I_1\gamma I_2 \subseteq I$ or $I_1\beta I_3 \subseteq \sqrt{I}$ or $I_2\beta I_3 \subseteq \sqrt{I}$. Let $x, y, z \in I$ and $\gamma, \beta \in \Gamma$ be such that $x\gamma y\beta z \in I$. Then, $\langle x\rangle\gamma\langle y\rangle\beta\langle z\rangle \in I$. By assumption, $x\gamma y \in \langle x\rangle\gamma\langle y\rangle \in I$ or $x\beta z \in \langle x\rangle\beta\langle z\rangle \in \sqrt{I}$ or $y\beta z \in \langle y\rangle\beta\langle z\rangle \in \sqrt{I}$. So, I is a 2-absorbing primary ideal in R. □

Finally, 2-absorbing primary ideals in a commutative $(\Gamma_1 \times \Gamma_2)$-semiring $R_1 \times R_2$ are provided.

Theorem 4.19. *Let R_i be a commutative Γ_i-semiring for all $i \in \{1, 2\}$.*

(i) If I_1 is a 2-absorbing primary ideal in R_1, then $I_1 \times R_2$ is a 2-absorbing primary ideal in $R_1 \times R_2$.

(ii) If I_2 is a 2-absorbing primary ideal in R_2, then $R_1 \times I_2$ is a 2-absorbing primary ideal in $R_1 \times R_2$.

Proof. Note that $R_1 \times R_2$ is a commutative $(\Gamma_1 \times \Gamma_2)$-semiring.

(i) Suppose that I_1 is a 2-absorbing primary ideal in R_1. Then, $I_1 \times R_2$ is a proper ideal in $R_1 \times R_2$. Let $x_1, y_1, z_1 \in R_1$, $x_2, y_2, z_2 \in R_2$, $\gamma_1, \beta_1 \in \Gamma_1$ and $\gamma_2, \beta_2 \in \Gamma_2$ be such that $(x_1, x_2)(\gamma_1, \gamma_2)(y_1, y_2)(\beta_1, \beta_2)(z_1, z_2) \in I_1 \times R_2$. Hence, $x_1\gamma_1 y_1 \beta_1 z_1 \in I_1$. Since I_1 is a 2-absorbing primary ideal, $x_1\gamma_1 y_1 \in I_1$ or $x_1\beta_1 z_1 \in \sqrt{I_1}$ or $y_1\beta_1 z_1 \in \sqrt{I_1}$. If $x_1\gamma_1 y_1 \in I_1$, then $(x_1, x_2)(\gamma_1, \gamma_2)(y_1, y_2) \in I_1 \times R_2$. If $x_1\beta_1 z_1 \in \sqrt{I_1}$, then $(x_1, x_2)(\beta_1, \beta_2)(z_1, z_2) \in \sqrt{I_1} \times R_2 \subseteq \sqrt{I_1 \times R_2}$. Similarly, if $y_1\beta_1 z_1 \in \sqrt{I_1}$, then $(y_1, y_2)(\beta_1, \beta_2)(z_1, z_2) \in \sqrt{I_1} \times R_2 \subseteq \sqrt{I_1 \times R_2}$. Thus, $I_1 \times R_2$ is 2-absorbing primary ideal in $R_1 \times R_2$.

The proof of (ii) is similar to the proof of (i). □

Theorem 4.20. *Let R_i be a commutative Γ_i-semiring with zero 0_{R_i} and unity 1_{R_i} such that $0_{R_i} \neq 1_{R_i}$ for all $i \in \{1, 2\}$. If I is a 2-absorbing primary ideal in $R_1 \times R_2$, exactly one of these holds:*

(i) $I = I_1 \times R_2$ for some 2-absorbing primary ideal I_1 in R_1;

(ii) $I = R_1 \times I_2$ for some 2-absorbing primary ideal I_2 in R_2;

(iii) $I = I_1 \times I_2$ for some primary ideal I_i in R_i for all $i \in \{1, 2\}$.

Proof. Suppose that I is a 2-absorbing primary ideal in $R_1 \times R_2$. Then, $I = I_1 \times I_2$ for some ideals I_1 in R_1 and I_2 in R_2. Assume $I_2 = R_2$. Then I_1 must be a proper ideal in R_1. Let $x, y, z \in R_1$ and $\gamma, \beta \in \Gamma_1$ be such that $x\gamma y \beta z \in I_1$. Let $a \in R_2$ and $\delta \in \Gamma_2$. So $(x, a)(\gamma, \delta)(y, a)(\beta, \delta)(z, a) \in I_1 \times R_2$. Since $I_1 \times R_2$ is a 2-absorbing primary ideal, $(x, a)(\gamma, \delta)(y, a) \in I_1 \times R_2$ or $(x, a)(\beta, \delta)(z, a) \in \sqrt{I_1 \times R_2}$ or $(y, a)(\beta, \delta)(z, a) \in \sqrt{I_1 \times R_2}$. If $(x, a)(\gamma, \delta)(y, a) \in I_1 \times R_2$, then $x\gamma y \in I_1$. If $(x, a)(\beta, \delta)(z, a) \in \sqrt{I_1 \times R_2} = \sqrt{I_1} \times R_2$, then $x\beta z \in \sqrt{I_1}$. Similarly, if $(y, a)(\beta, \delta)(z, a) \in \sqrt{I_1 \times R_2}$, then, $y\beta z \in \sqrt{I_1}$. Thus, I_1 is a 2-absorbing primary ideal in R_1.

By similar argument, if $I_1 = R_1$, then I_2 is a 2-absorbing primary ideal in R_2.

Now, suppose that $I_1 \neq R_1$ and $I_2 \neq R_2$. Suppose that I_1 is not a primary ideal in R_1. If $1_{R_2} \in \sqrt{I_2}$, then $1_{R_2} \in I_2$, so $I_2 = R_2$ which is a contradiction. Hence, $1_{R_2} \notin \sqrt{I_2}$. Since I_1 is not a primary ideal in R_1, there exist $b, c \in R_1$ such that $b\Gamma c \subseteq I_1$ but neither $b \in I_1$ nor $c \in \sqrt{I_1}$. Since 1_{R_1} and 1_{R_2} are unities, there exist $\alpha, \alpha' \in \Gamma_1$ and $\alpha'' \in \Gamma_2$ such that $b\alpha 1_{R_1} = b$, $1_{R_1}\alpha' c = c$ and $1_{R_2}\alpha'' 1_{R_2} = 1_{R_2}$. Since $b\Gamma c \subseteq I_1$, we have $b\alpha c \in I_1$. Hence, $(b, 1_{R_2})(\alpha, \alpha')(1_{R_1}, 0_{R_2})(\alpha', \alpha'')(c, 1_{R_2}) = (b\alpha c, 0_{R_2}) \in I_1 \times I_2 = I$. Since I is a 2-absorbing primary ideal, we have

$$(b, 0_{R_2}) = (b, 1_{R_2})(\alpha, \alpha')(1_{R_1}, 0_{R_2}) \in I \text{ or}$$
$$(b\alpha c, 1_{R_2}) = (b, 1_{R_2})(\alpha, \alpha')(c, 1_{R_2}) \in \sqrt{I} = \sqrt{I_1} \times \sqrt{I_2} \text{ or}$$
$$(c, 0_{R_2}) = (1_{R_1}, 0_{R_2})(\alpha', \alpha'')(c, 1_{R_2}) \in \sqrt{I} = \sqrt{I_1} \times \sqrt{I_2}.$$

Hence $b \in I_1$ or $1_{R_2} \in \sqrt{I_2}$ or $c \in \sqrt{I_1}$, which is a contradiction. So I_1 is a primary ideal in R_1. Analogously, I_2 is a primary ideal in R_2. □

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