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A Note on the Rate of Convergence of Poles of Generalized Hermite-Padé Approximants

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Abstract : We consider row sequences of three generalized Hermite-Padé approximations (orthogonal Hermite-Padé approximation, Hermite-Padé-Faber approximation, and multipoint Hermite-Padé approximation) of a vector of the approximated functions $\mathbf{F}$ and prove that if $\mathbf{F}$ has a system pole of order ν, then such system pole attracts at least ν zeros of denominators of these approximants at the rate of a geometric progression. Moreover, the rates of these attractions are estimated.

Keywords : orthogonal polynomials, Faber polynomials, interpolation, Hermite-Padé approximation, rate of convergence.
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1 Introduction

The purpose of this paper is to prove a direct statement of Gonchar's theorem (see Theorem A below) for three generalized Hermite-Padé approximations. Let us recall the definition of classical Padé approximants and state a known result on row sequences of classical Padé approximants related to our study in this paper. In

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what follows, $\mathbb{N} := \{1, 2, 3, \ldots\}$, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, and $\mathbb{P}_n$ is the set of all polynomials of degree at most n.

Definition 1.1. Let $F(z) = \sum_{k=0}^{\infty} f_k z^k$ be a formal power series. Fix $(n, m) \in \mathbb{N}_0 \times \mathbb{N}_0$. Then, there exist $P \in \mathbb{P}_n$ and $Q \in \mathbb{P}_m$ such that $Q \not\equiv 0$ and

$$(QF - P)(z) = \mathcal{O}(z^{n+m+1}), \qquad \text{as } z \to 0.$$

The rational function $R_{n,m} := P/Q$ is called the (n, m) *classical Padé approximant of* F.

It is well-known that for any $(n, m) \in \mathbb{N}_0 \times \mathbb{N}_0$, $R_{n,m}$ always exists and is unique. For a given pair $(n, m) \in \mathbb{N}_0 \times \mathbb{N}_0$, we write

$$R_{n,m} = \frac{P_{n,m}}{Q_{n,m}},$$

where $Q_{n,m}$ is the monic polynomial that has no common zero with $P_{n,m}$.

Let $F(z) = \sum_{k=0}^{\infty} f_k z^k$ be a formal power series. Denote by $R_0(F)$ the radius of the largest disk centered at the origin to which F can be extended analytically and by $R_m(F)$ the radius of the largest disk centered at the origin to which F can be extended so that F has at most m poles counting multiplicities.

Let us define two indicators of the asymptotic behavior of the zeros of $Q_{n,m}$. Fix $m \in \mathbb{N}$. Let

$$\mathcal{P}_{n,m} := \{\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,m_n}\}, \qquad m_n \le m, \qquad n \in \mathbb{N}_0,$$

denote the collection of zeros of $Q_{n,m}$ (repeated according to their multiplicity). Define

$$|z - w|_1 := \min\{1, |z - w|\}, \qquad z, w \in \mathbb{C}.$$

Fix $\lambda \in \mathbb{C}$. The first indicator is defined by

$$\Delta(\lambda) := \limsup_{n\to\infty} \prod_{j=1}^{m_n} |\lambda_{n,j} - \lambda|_1^{1/n} = \limsup_{n\to\infty} \prod_{|\lambda_{n,j}-\lambda|<1} |\lambda_{n,j} - \lambda|^{1/n}.$$

Clearly, $0 \le \Delta(\lambda) \le 1$ (when $m_n = 0$ or $|\lambda_{n,j} - \lambda| \ge 1$ for all $j = 1, 2, \ldots, m_n$, the product is taken to be 1). The second indicator, a nonnegative integer $\sigma(\lambda)$, is defined as follows. We suppose that for each n, the points in

$$\mathcal{P}_{n,m} = \{\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,m_n}\} \tag{1.1}$$

are enumerated in nondecreasing distance to the point λ. We set

$$\delta_j(\lambda) := \limsup_{n\to\infty} |\lambda_{n,j} - \lambda|_1^{1/n}. \tag{1.2}$$

These numbers are defined by (1.2) for $j = 1, 2, \ldots, m', m' = \liminf_{n\to\infty} m_n$; for $j = m' + 1, \ldots, n$, we define $\delta_j(\lambda) = 1$. We have $0 \le \delta_j(\lambda) \le 1$. If $\Delta(\lambda) = 1$ (in that case all $\delta_j(\lambda) = 1$), then $\sigma(\lambda) = 0$. If $\Delta(\lambda) < 1$, then for some ν, $1 \leq \nu \leq m$, we have that $\delta_1(\lambda) \leq \ldots \leq \delta_\nu(\lambda) < 1$ and $\delta_{\nu+1}(\lambda) = 1$ or $\nu = m$; in this case we take $\sigma(\lambda) = \nu$. Set

$$\mathbb{B}(a, R) := \{z \in \mathbb{C} : |z - a| < R\}.$$

The following theorem proved by Gonchar [1, Theorem 1] concerns the relation between the location of a pole of the approximated function and the rate of attraction of such pole to poles of classical Padé approximants.

Theorem A. *Let $F(z) = \sum_{k=0}^{\infty} f_k z^k$ be a formal power series, $m \in \mathbb{N}$ be fixed, and $\lambda \neq 0$ be a given point in $\mathbb{C}$. The following statements are equivalent:*

(a) $\lambda \in \mathbb{B}(0, R_m(F))$ and F has a pole at λ.

(b) $\Delta(\lambda) < 1$ (or equivalently $\sigma(\lambda) \geq 1$).

If either (a) or (b) holds, then

$$\Delta(\lambda) = \frac{|\lambda|}{R_m(F)} \qquad \text{and} \qquad \sigma(\lambda) = \nu,$$

where ν is the order of the pole at λ.

The direct part of this theorem refers to the statement: if F has a pole at $\lambda \in \mathbb{B}(0, R_m(F))$ of order ν, then

$$\Delta(\lambda) \leq \frac{|\lambda|}{R_m(F)} \qquad \text{and} \qquad \sigma(\lambda) \geq \nu.$$

On the other hand, the inverse result in this theorem is the statement: if $\Delta(\lambda) < 1$, then F has a pole at $\lambda \in \mathbb{B}(0, R_m(F))$,

$$\Delta(\lambda) \geq \frac{|\lambda|}{R_m(f)}, \qquad \text{and} \qquad \nu \geq \sigma(\lambda),$$

where ν is the order of the pole at λ.

The aim of this paper is to prove analogues of the direct part of Theorem A for orthogonal Hermite-Padé approximation, Hermite-Padé-Faber approximation, and multipoint Hermite-Padé approximation defined as follows.

Let E be an infinite compact subset of the complex plane $\mathbb{C}$ such that $\overline{\mathbb{C}} \setminus E$ is simply connected. In the whole paper, E will be described as above.

Let us define the first approximation constructed from orthogonal polynomials on a general compact set E. Let μ be a finite positive Borel measure with infinite support $\operatorname{supp}(\mu)$ contained in E. We write $\mu \in \mathcal{M}(E)$ and define the associated inner product,

$$\langle g, h \rangle_\mu := \int g(\zeta)\overline{h(\zeta)}d\mu(\zeta), \quad g, h \in L_2(\mu).$$

Let

$$p_n(z) := \kappa_n z^n + \cdots, \quad \kappa_n > 0, \quad n = 0, 1, 2, \ldots,$$

be the orthonormal polynomial of degree n with respect to μ with positive leading coefficient; that is, $\langle p_n, p_m\rangle_\mu = \delta_{n,m}$. Denote by $\mathcal{H}(E)$ the space of all functions holomorphic in some neighborhood of E. We define

$$\mathcal{H}(E)^d := \{(F_1, F_2, \dots, F_d) : F_\ell \in \mathcal{H}(E), \ell = 1, 2, \dots, d\}.$$

The definition of orthogonal Hermite-Padé approximants is the following:

Definition 1.2. Let $\mathbf{F} = (F_1, F_2, \dots, F_d) \in \mathcal{H}(E)^d$ and $\mu \in \mathcal{M}(E)$. Fix a multi-index $\mathbf{m} = (m_1, m_2, \dots, m_d) \in \mathbb{N}^d$ and $n \in \mathbb{N}$. Set $|\mathbf{m}| := m_1 + m_2 + \dots + m_d$. Then, there exists $Q^\mu_{n,\mathbf{m}} \in \mathbb{P}_{|\mathbf{m}|}$ such that $Q^\mu_{n,\mathbf{m}} \not\equiv 0$ and $\langle Q^\mu_{n,\mathbf{m}} z^k F_\ell,\, p_n\rangle_\mu = 0$ for all $k = 0, 1, \dots, m_\ell - 1$ and $\ell = 1, 2, \dots, d$. The corresponding vector of rational functions

$$\mathbf{R}^\mu_{n,\mathbf{m}} := (R^\mu_{n,\mathbf{m},1}, R^\mu_{n,\mathbf{m},2}, \dots, R^\mu_{n,\mathbf{m},d})$$

$$= \left(\frac{\sum_{j=0}^{n-1} \langle Q^\mu_{n,\mathbf{m}} F_1,\, p_j\rangle_\mu p_j}{Q^\mu_{n,\mathbf{m}}}, \frac{\sum_{j=0}^{n-1} \langle Q^\mu_{n,\mathbf{m}} F_2,\, p_j\rangle_\mu p_j}{Q^\mu_{n,\mathbf{m}}}, \dots, \frac{\sum_{j=0}^{n-1} \langle Q^\mu_{n,\mathbf{m}} F_d,\, p_j\rangle_\mu p_j}{Q^\mu_{n,\mathbf{m}}} \right)$$

is called an $(n, \mathbf{m})$ *orthogonal Hermite-Padé approximant of* $\mathbf{F}$ *with respect to* μ.

Now, we want to define the second approximation constructed from Faber polynomials associated with E. Let Φ be the unique Riemann mapping function from $\overline{\mathbb{C}} \setminus E$ to the exterior of the closed unit disk verifying $\Phi(\infty) = \infty$, $\Phi'(\infty) > 0$. For each $\rho > 1$, the *level curve of index* ρ and the *canonical domain of index* ρ are defined by

$$\Gamma_\rho := \{z \in \mathbb{C} : |\Phi(z)| = \rho\} \qquad \text{and} \qquad D_\rho := E \cup \{z \in \mathbb{C} : |\Phi(z)| < \rho\},$$

respectively. Given $F \in \mathcal{H}(E)$, we denote by $\rho_0(F)$ the largest index ρ such that F extends as a holomorphic function to D_ρ and by $\rho_m(F)$ the largest index ρ such that F extends as a meromorphic function with at most m poles counting multiplicities in D_ρ.

The *Faber polynomial* of E of degree n is

$$\Phi_n(z) := \frac{1}{2\pi i} \int_{\Gamma_\rho} \frac{\Phi^n(t)}{t - z} dt, \qquad z \in D_\rho, \qquad n = 0, 1, 2, \dots.$$

The n-th *Faber coefficient* of $F \in \mathcal{H}(E)$ with respect to Φ_n is defined by the formula

$$[F]_n := \frac{1}{2\pi i} \int_{\Gamma_\rho} \frac{F(t)\Phi'(t)}{\Phi^{n+1}(t)} dt,$$

where $\rho \in (1, \rho_0(F))$.

Definition 1.3. Let $\mathbf{F} = (F_1, F_2, \dots, F_d) \in \mathcal{H}(E)^d$. Fix $\mathbf{m} = (m_1, m_2, \dots, m_d) \in \mathbb{N}^d$ and $n \in \mathbb{N}$. Set $|\mathbf{m}| := m_1 + m_2 + \dots + m_d$. Then, there exists $Q^E_{n,\mathbf{m}} \in \mathbb{P}_{|\mathbf{m}|}$ such that $Q^E_{n,\mathbf{m}} \not\equiv 0$ and $[Q^E_{n,\mathbf{m}} z^k F_\ell]_n = 0$ for all $k = 0, 1, \dots, m_\ell - 1$ and $\ell = 1, 2, \dots, d$. The corresponding vector of rational functions

$$\mathbf{R}^E_{n,\mathbf{m}} := (R^E_{n,\mathbf{m},1}, R^E_{n,\mathbf{m},2}, \dots, R^E_{n,\mathbf{m},d})$$

$$= \left(\frac{\sum_{j=0}^{n-1}[Q_{n,\mathbf{m}}^E F_1]_j \Phi_j}{Q_{n,\mathbf{m}}^E}, \frac{\sum_{j=0}^{n-1}[Q_{n,\mathbf{m}}^E F_2]_j \Phi_j}{Q_{n,\mathbf{m}}^E}, \ldots, \frac{\sum_{j=0}^{n-1}[Q_{n,\mathbf{m}}^E F_d]_j \Phi_j}{Q_{n,\mathbf{m}}^E} \right)$$

is called an $(n, \mathbf{m})$ *Hermite-Padé-Faber approximant of* $\mathbf{F}$ *with respect to* E.

Let $\alpha \subset E$ be a table of points; more precisely, $\alpha = \{\alpha_{n,k}\}$, $k = 1, \ldots, n$, $n = 1, 2, \ldots$. We propose the following definition.

Definition 1.4. Let $\mathbf{F} = (F_1, F_2 \ldots, F_d) \in \mathcal{H}(E)^d$. Fix a multi-index $\mathbf{m} = (m_1, m_2, \ldots, m_d) \in \mathbb{N}^d$ and $n \in \mathbb{N}$. Set $|\mathbf{m}| := m_1 + m_2 + \cdots + m_d$. Then, there exist $Q_{n,\mathbf{m}}^\alpha \in \mathbb{P}_{|\mathbf{m}|}$ and $P_{n,\mathbf{m},\ell}^\alpha \in \mathbb{P}_{n-1}$, $\ell = 1, 2, \ldots, d$ such that $Q_{n,\mathbf{m}}^\alpha \not\equiv 0$ and for all $\ell = 1, 2, \ldots, d$,

$$(Q_{n,\mathbf{m}}^\alpha F_\ell - P_{n,\mathbf{m},\ell}^\alpha)/a_{n+1} \in \mathcal{H}(E),$$

where $a_n(z) = \prod_{k=1}^n (z - \alpha_{n,k})$. The corresponding vector of rational functions

$$\mathbf{R}_{n,\mathbf{m}}^\alpha = \left(R_{n,\mathbf{m},1}^\alpha, \ldots, R_{n,\mathbf{m},d}^\alpha\right) = \left(\frac{P_{n,\mathbf{m},1}^\alpha}{Q_{n,\mathbf{m}}^\alpha}, \frac{P_{n,\mathbf{m},2}^\alpha}{Q_{n,\mathbf{m}}^\alpha}, \ldots, \frac{P_{n,\mathbf{m},d}^\alpha}{Q_{n,\mathbf{m}}^\alpha} \right)$$

is called an $(n, \mathbf{m})$ *multipoint Hermite-Padé approximant of* $\mathbf{F}$ *with respect* α.

Finding $Q_{n,\mathbf{m}}^\mu, Q_{n,\mathbf{m}}^E$, or $Q_{n,\mathbf{m}}^\alpha$ is equivalent to solving $|\mathbf{m}| + 1$ unknowns from $|\mathbf{m}|$ linear system of equations. Then, $Q_{n,\mathbf{m}}^\mu, Q_{n,\mathbf{m}}^E$, and $Q_{n,\mathbf{m}}^\alpha$ always exist but they may not be unique. Since $Q_{n,\mathbf{m}}^\mu, Q_{n,\mathbf{m}}^E$, and $Q_{n,\mathbf{m}}^\alpha$ are not the zero function, we normalize them to be "monic" polynomials. Moreover, we would like to emphasize that for any $(n, \mathbf{m}) \in \mathbb{N} \times \mathbb{N}^d$, $\mathbf{R}_{n,\mathbf{m}}^\mu, \mathbf{R}_{n,\mathbf{m}}^E$, and $\mathbf{R}_{n,\mathbf{m}}^\alpha$ may not be unique. These vectors of rational approximants were recently introduced in [2, 3, 4] to solve a problem about locating $|\mathbf{m}|$ system poles of the vector of the approximated functions nearest E.

Before we summarize the results in [2, 3, 4], we need to define some notations and state one more definition. We say that $\mu \in \mathbf{Reg}_1(E)$ when

$$\lim_{n\to\infty} |p_n(z)|^{1/n} = |\Phi(z)| \tag{1.3}$$

uniformly on compact subsets of $\mathbb{C}\setminus E$. The second kind functions defined as follows

$$s_n(z) := \int \frac{\overline{p_n(\zeta)}}{z - \zeta} d\mu(\zeta), \qquad z \in \overline{\mathbb{C}} \setminus \operatorname{supp}(\mu).$$

play an important role in our proof. The measure $\mu \in \mathbf{Reg}_2(E)$ if and only if

$$\lim_{n\to\infty} |s_n(z)|^{1/n} = |\Phi(z)|^{-1} \tag{1.4}$$

uniformly on compact subsets of $\mathbb{C} \setminus E$. The classes $\mathbf{Reg}_1(E)$ and $\mathbf{Reg}_2(E)$ are more or less the same in some cases (see the details in [2, Section 1]). In particular, if E is convex, then $\mathbf{Reg}_1(E) = \mathbf{Reg}_2(E)$ and these two classes coincide with the regular class in the usual sense (see [7, Definition 3.1.2] for the definition of the regular class in the usual sense). Define

$$\mathbf{Reg}_{1,2}(E) := \mathbf{Reg}_1(E) \cap \mathbf{Reg}_2(E).$$

We say that $\mu \in \mathbf{Reg}^m_{1,2}(E)$ if it is in $\mathbf{Reg}_{1,2}(E)$ and there exists a positive constant c such that

$$\frac{\kappa_{n-m}}{\kappa_n} \geq c, \qquad n \geq n_0.$$

There are many examples of measures in $\mathbf{Reg}^m_{1,2}(E)$ (see [5, 6, 8] and references therein).

Let $\alpha \subset E$ be a table of interpolation points ($\alpha = \{\alpha_{n,k}\}$, $k = 1, \ldots, n$, $n = 1, 2, \ldots$). Recall that $a_n(z) := \prod_{k=1}^n (z - \alpha_{n,k})$. It is well known that there exist tables of points α satisfying the condition

$$\lim_{n\to\infty} |a_n(z)|^{1/n} = c|\Phi(z)|, \tag{1.5}$$

or the stronger condition

$$\lim_{n\to\infty} a_n(z)/c^n \Phi^n(z) = G(z) \neq 0, \tag{1.6}$$

uniformly on compact subsets of $\overline{\mathbb{C}} \setminus E$, where c denotes some positive constant, see [9, Chapters 8-9]. It is easy to check that (1.6) $\Rightarrow$ (1.5). The following is a definition of system pole.

Definition 1.5. Given $\mathbf{F} = (F_1, F_2, \ldots, F_d) \in \mathcal{H}(E)^d$ and $\mathbf{m} = (m_1, m_2, \ldots, m_d) \in \mathbb{N}^d$, we say that $\xi \in \mathbb{C}$ is a *system pole of order* τ *of* $\mathbf{F}$ *with respect to* $\mathbf{m}$ if τ is the largest positive integer such that for each $t = 1, 2, \ldots, \tau$, there exists at least one polynomial combination of the form

$$\sum_{\ell=1}^{d} v_\ell F_\ell, \qquad \deg(v_\ell) < m_\ell, \qquad \ell = 1, 2, \ldots, d, \tag{1.7}$$

which is holomorphic in a neighborhood of $\overline{D}_{|\Phi(\xi)|}$ except for a pole at $z = \xi$ of exact order t.

To each system pole ξ of $\mathbf{F}$ with respect to $\mathbf{m}$, we define a characteristic index as follows. Let τ be the order of ξ as a system pole of $\mathbf{F}$. For each $t = 1, \ldots, \tau$, denote by $\rho_{\xi,t}(\mathbf{F}, \mathbf{m})$ the largest of all the numbers $\rho_t(G)$ (the index of the largest canonical domain containing at most t poles of G), where G is a polynomial combination of type (1.7) that is holomorphic in a neighborhood of $\overline{D}_{|\Phi(\xi)|}$ except for a pole at $z = \xi$ of order t. There is only a finite number of such possible values so the maximum is indeed attained. Then, we define

$$\boldsymbol{\rho}_\xi(\mathbf{F}, \mathbf{m}) := \min_{k=1,\ldots,\tau} \rho_{\xi,k}(\mathbf{F}, \mathbf{m}).$$

Combining Theorem 1.2 in [2], Corollary 1.6. in [3], and Theorem 1.3 [4], we arrive at the following theorem.

Theorem B. *Let* $\mathbf{F} = (F_1, F_2, \ldots, F_d) \in \mathcal{H}(E)^d$, $\mathbf{m} \in \mathbb{N}^d$ *be a fixed multi-index,* $\mu \in \mathbf{Reg}_{1,2}^{|\mathbf{m}|}(E)$, *and* α *satisfy the condition* (1.6). *Denote by* $Q_{\mathbf{m}}^{\mathbf{F}}$ *the monic polynomial whose zeros are the system poles of* $\mathbf{F}$ *with respect to* $\mathbf{m}$ *taking account of their order and by* $\mathcal{P}(\mathbf{F}, \mathbf{m})$ *the set of all zeros of* $Q_{\mathbf{m}}^{\mathbf{F}}$. *Then, the following assertions are equivalent:*

(a) $\mathbf{F}$ *has exactly* $|\mathbf{m}|$ *system poles with respect to* $\mathbf{m}$ *counting multiplicities.*

(b) The polynomials $Q_{n,\mathbf{m}}^{\mu}$ *of* $\mathbf{F}$ *are uniquely determined for all sufficiently large* n, *and there exists a polynomial* $\hat{Q}_{|\mathbf{m}|}$ *of degree* $|\mathbf{m}|$ *such that*

$$\limsup_{n\to\infty} \|Q_{n,\mathbf{m}}^{\mu} - \hat{Q}_{|\mathbf{m}|}\|^{1/n} = \hat{\theta} < 1.$$

(c) The polynomials $Q_{n,\mathbf{m}}^{E}$ *of* $\mathbf{F}$ *are uniquely determined for all sufficiently large* n, *and there exists a polynomial* $\tilde{Q}_{|\mathbf{m}|}$ *of degree* $|\mathbf{m}|$ *such that*

$$\limsup_{n\to\infty} \|Q_{n,\mathbf{m}}^{E} - \tilde{Q}_{|\mathbf{m}|}\|^{1/n} = \tilde{\theta} < 1.$$

(d) The polynomials $Q_{n,\mathbf{m}}^{\alpha}$ *of* $\mathbf{F}$ *are uniquely determined for all sufficiently large* n, *and there exists a polynomial* $\check{Q}_{|\mathbf{m}|}$ *of degree* $|\mathbf{m}|$ *such that*

$$\limsup_{n\to\infty} \|Q_{n,\mathbf{m}}^{\alpha} - \check{Q}_{|\mathbf{m}|}\|^{1/n} = \check{\theta} < 1.$$

The norm $\|\cdot\|$ *in (b), (c), and (d) denotes (for example) the norm induced in the space of polynomials of degree at most* $|\mathbf{m}|$ *by the maximum of the absolute value of the coefficients. Moreover, if one of the assertions (a), (b), (c), or (d) takes place, then* $\hat{Q}_{|\mathbf{m}|} = \tilde{Q}_{|\mathbf{m}|} = \check{Q}_{|\mathbf{m}|} = Q_{\mathbf{m}}^{\mathbf{F}}$,

$$\hat{\theta} = \tilde{\theta} = \check{\theta} = \max\left\{ \frac{|\Phi(\xi)|}{\rho_\xi(\mathbf{F}, \mathbf{m})} : \xi \in \mathcal{P}(\mathbf{F}, \mathbf{m}) \right\}. \tag{1.8}$$

Since the space of all polynomials of degree at most $|\mathbf{m}|$ has a finite dimension, all of its norms are equivalent. This implies that the norm in (b), (c), and (d) can be replaced by any norm. The estimates of the rates of convergences of $(\mathbf{R}_{n,\mathbf{m}}^{\mu})_{n\in\mathbb{N}}$, $(\mathbf{R}_{n,\mathbf{m}}^{E})_{n\in\mathbb{N}}$, and $(\mathbf{R}_{n,\mathbf{m}}^{\alpha})_{n\in\mathbb{N}}$ can be found in [2, Theorem 1.2], [3, Theorem 1.3], and [4, Theorem 1.4.], respectively. Moreover, we would like to emphasize that in Theorem B the assumption that $\mu \in \mathbf{Reg}_{1,2}^{|\mathbf{m}|}(E)$ is only for the convergence of $(Q_{n,\mathbf{m}}^{\mu})_{n\in\mathbb{N}}$ and the assumption that α satisfies the condition (1.6) is only for the convergence of $(Q_{n,\mathbf{m}}^{\alpha})_{n\in\mathbb{N}}$. In other words, (a) and (c) are equivalent without assuming that $\mu \in \mathbf{Reg}_{1,2}^{|\mathbf{m}|}(E)$ or α satisfies the condition (1.6).

In the current paper, we are interested in studying further about the rates of convergences of zeros of $Q_{n,\mathbf{m}}^{\mu}$, $Q_{n,\mathbf{m}}^{E}$, and $Q_{n,\mathbf{m}}^{\alpha}$ (when $\mathbf{m}$ is fixed and $n \to \infty$) to the system poles of $\mathbf{F} \in \mathcal{H}(E)^d$.

An outline of this paper is as follows. The main results in this paper are stated in Section 2. The proofs of the main results are in Section 3.

2 Main Results

Given $\xi \in \mathbb{C}$ and $\mathbf{m} \in \mathbb{N}^d$, the notations Δ^μ, σ^μ, δ_j^μ, Δ^E, σ^E, δ_j^E, Δ^α, σ^α, and δ_j^α in Theorem 2.1 and Corollary 2.1 are defined as in Section 1 taking for $n \in \mathbb{N}$,

$$\mathcal{P}_{n,\mathbf{m}}^{\mu} := \{\hat{\xi}_{n,1}, \hat{\xi}_{n,2}, \ldots, \hat{\xi}_{n,(\mu,\mathbf{m})_n}\}, \qquad (\mu,\mathbf{m})_n \leq |\mathbf{m}|,$$

$$\mathcal{P}_{n,\mathbf{m}}^{E} := \{\tilde{\xi}_{n,1}, \tilde{\xi}_{n,2}, \ldots, \tilde{\xi}_{n,(E,\mathbf{m})_n}\}, \qquad (E,\mathbf{m})_n \leq |\mathbf{m}|,$$

$$\mathcal{P}_{n,\mathbf{m}}^{\alpha} := \{\check{\xi}_{n,1}, \check{\xi}_{n,2}, \ldots, \check{\xi}_{n,(\alpha,\mathbf{m})_n}\}, \qquad (\alpha,\mathbf{m})_n \leq |\mathbf{m}|,$$

to be the collections of zeros of $Q_{n,\mathbf{m}}^{\mu}$, $Q_{n,\mathbf{m}}^{E}$, and $Q_{n,\mathbf{m}}^{\alpha}$ enumerated in nondecreasing distance to ξ, respectively,

Theorem 2.1. *Let $\mathbf{F} \in \mathcal{H}(E)^d$ and fix $\mathbf{m} \in \mathbb{N}^d$. Assume that ξ is a system pole of order ν of $\mathbf{F}$ with respect to $\mathbf{m}$. Then, the following are true.*

(a) If $\mu \in \mathbf{Reg}_2(E)$, then

$$\Delta^\mu(\xi) \leq \frac{|\Phi(\xi)|}{\rho_\xi(\mathbf{F},\mathbf{m})} \qquad \text{and} \qquad \sigma^\mu(\xi) \geq \nu.$$

(b)

$$\Delta^E(\xi) \leq \frac{|\Phi(\xi)|}{\rho_\xi(\mathbf{F},\mathbf{m})} \qquad \text{and} \qquad \sigma^E(\xi) \geq \nu.$$

(c) If α satisfies the condition (1.5)*, then*

$$\Delta^\alpha(\xi) \leq \frac{|\Phi(\xi)|}{\rho_\xi(\mathbf{F},\mathbf{m})} \qquad \text{and} \qquad \sigma^\alpha(\xi) \geq \nu.$$

Corollary 2.1. *Let $\mathbf{F} \in \mathcal{H}(E)^d$ and fix $\mathbf{m} \in \mathbb{N}^d$. Assume that ξ is a system pole of order ν of $\mathbf{F}$ with respect to $\mathbf{m}$. Then, the following are true.*

(a) Assume further that $\mu \in \mathbf{Reg}_2(E)$ and $\liminf_{n\to\infty} |\xi - \hat{\xi}_{n,\nu+1}| > 0$. Then,

$$\delta_1^\mu(\xi) \leq \delta_2^\mu(\xi) \leq \ldots \leq \delta_\nu^\mu(\xi) \leq \left(\frac{|\Phi(\xi)|}{\rho_\xi(\mathbf{F},\mathbf{m})}\right)^{1/\nu}. \tag{2.1}$$

In particular, $\delta_1^\mu(\xi) = \delta_2^\mu(\xi) = \ldots = \delta_\nu^\mu(\xi) = (|\Phi(\xi)|/\rho_\xi(\mathbf{F},\mathbf{m}))^{1/\nu}$ if and only if $\Delta^\mu(\xi) = |\Phi(\xi)|/\rho_\xi(\mathbf{F},\mathbf{m})$.

(b) Assume further that $\liminf_{n\to\infty} |\xi - \tilde{\xi}_{n,\nu+1}| > 0$. Then,

$$\delta_1^E(\xi) \leq \delta_2^E(\xi) \leq \ldots \leq \delta_\nu^E(\xi) \leq \left(\frac{|\Phi(\xi)|}{\rho_\xi(\mathbf{F},\mathbf{m})}\right)^{1/\nu}. \tag{2.2}$$

In particular, $\delta_1^E(\xi) = \delta_2^E(\xi) = \ldots = \delta_\nu^E(\xi) = (|\Phi(\xi)|/\rho_\xi(\mathbf{F},\mathbf{m}))^{1/\nu}$ if and only if $\Delta^E(\xi) = |\Phi(\xi)|/\rho_\xi(\mathbf{F},\mathbf{m})$.

(c) *Assume further that* α *satisfies the condition* (1.5) *and* $\liminf_{n\to\infty} |\xi - \check{\xi}_{n,\nu+1}| > 0$. *Then,*

$$\delta_1^\alpha(\xi) \leq \delta_2^\alpha(\xi) \leq \ldots \leq \delta_\nu^\alpha(\xi) \leq \left(\frac{|\Phi(\xi)|}{\rho_\xi(\mathbf{F},\mathbf{m})}\right)^{1/\nu}. \tag{2.3}$$

In particular, $\delta_1^\alpha(\xi) = \delta_2^\alpha(\xi) = \ldots = \delta_\nu^\alpha(\xi) = (|\Phi(\xi)|/\rho_\xi(\mathbf{F},\mathbf{m}))^{1/\nu}$ *if and only if* $\Delta^\alpha(\xi) = |\Phi(\xi)|/\rho_\xi(\mathbf{F},\mathbf{m})$.

3 Proofs of Main Results

Before we give a proof of Theorem 2.1, let us state the main lemma (see [2, Equation (18)], [3, Equation (2.5)], and [4, Lemma 2.1]).

For each $n \in \mathbb{N}$, let $q_{n,\mathbf{m}}^\mu, q_{n,\mathbf{m}}^E$, and $q_{n,\mathbf{m}}^\alpha$ be the polynomials $Q_{n,\mathbf{m}}^\mu, Q_{n,\mathbf{m}}^E$, and $Q_{n,\mathbf{m}}^\alpha$ normalized so that

$$\sum_{k=0}^{|\mathbf{m}|} |\hat{\lambda}_{n,k}| = 1, \qquad q_{n,\mathbf{m}}^\mu(z) = \sum_{k=0}^{|\mathbf{m}|} \hat{\lambda}_{n,k} z^k,$$

$$\sum_{k=0}^{|\mathbf{m}|} |\tilde{\lambda}_{n,k}| = 1, \qquad q_{n,\mathbf{m}}^E(z) = \sum_{k=0}^{|\mathbf{m}|} \tilde{\lambda}_{n,k} z^k,$$

$$\sum_{k=0}^{|\mathbf{m}|} |\check{\lambda}_{n,k}| = 1, \qquad q_{n,\mathbf{m}}^\alpha(z) = \sum_{k=0}^{|\mathbf{m}|} \check{\lambda}_{n,k} z^k,$$

respectively.

Lemma 3.1. *Let* $\mathbf{F} \in \mathcal{H}(E)^d$ *and fix* $\mathbf{m} \in \mathbb{N}^d$. *Assume that* ξ *is a system pole of order* ν *of* $\mathbf{F}$ *with respect to* $\mathbf{m}$. *Then, the following are true.*

(a) *If* $\mu \in \mathbf{Reg}_2(E)$, *then for all* $j = 0, 1, \ldots, \nu - 1$,

$$\limsup_{n\to\infty} |(q_{n,\mathbf{m}}^\mu)^{(j)}(\xi)|^{1/n} \leq \frac{|\Phi(\xi)|}{\rho_\xi(\mathbf{F},\mathbf{m})}.$$

(b) *For all* $j = 0, 1, \ldots, \nu - 1$,

$$\limsup_{n\to\infty} |(q_{n,\mathbf{m}}^E)^{(j)}(\xi)|^{1/n} \leq \frac{|\Phi(\xi)|}{\rho_\xi(\mathbf{F},\mathbf{m})}.$$

(c) *If* α *satisfies the condition* (1.5), *then for all* $j = 0, 1, \ldots, \nu - 1$,

$$\limsup_{n\to\infty} |(q_{n,\mathbf{m}}^\alpha)^{(j)}(\xi)|^{1/n} \leq \frac{|\Phi(\xi)|}{\rho_\xi(\mathbf{F},\mathbf{m})}.$$

Proof of Theorem 2.1. We will show that

$$\Delta^{\mu}(\xi) \leq \frac{|\Phi(\xi)|}{\rho_{\xi}(\mathbf{F},\mathbf{m})}, \qquad \sigma^{\mu}(\xi) \geq \nu. \tag{3.1}$$

The proofs of

$$\Delta^{E}(\xi) \leq \frac{|\Phi(\xi)|}{\rho_{\xi}(\mathbf{F},\mathbf{m})}, \qquad \sigma^{E}(\xi) \geq \nu, \tag{3.2}$$

$$\Delta^{\alpha}(\xi) \leq \frac{|\Phi(\xi)|}{\rho_{\xi}(\mathbf{F},\mathbf{m})} \qquad \sigma^{\alpha}(\xi) \geq \nu. \tag{3.3}$$

are identical to the proof of (3.1). So, we skip the proofs of (3.2) and (3.3). From Lemma 3.1, we have for all $j = 0, 1, \ldots, \nu - 1$,

$$\limsup_{n\to\infty} |(q^{\mu}_{n,\mathbf{m}})^{(j)}(\xi)|^{1/n} \leq \frac{|\Phi(\xi)|}{\rho_{\xi}(\mathbf{F},\mathbf{m})}. \tag{3.4}$$

Now, we want to show that

$$\lim_{n\to\infty} \hat{\xi}_{n,j} = \xi, \qquad j = 1, 2, \ldots, \nu, \tag{3.5}$$

i.e., there exist at least ν zeros of $Q^{\mu}_{n,\mathbf{m}}$ converge to ξ. By the normalization of $q^{\mu}_{n,m}$, it suffices to show that for any subsequence of indices Ω such that

$$\lim_{n\in\Omega} q^{\mu}_{n,\mathbf{m}} = q_{\Omega},$$

q_{Ω} is a non-null polynomial with a zero of order at least ν at ξ. Due to the normalization of $q^{\mu}_{n,\mathbf{m}}$, $q_{\Omega} \not\equiv 0$. Computing Taylor's expansion of $q^{\mu}_{n,\mathbf{m}}$ around ξ, we obtain

$$q^{\mu}_{n,\mathbf{m}}(z) = \sum_{k=0}^{|\mathbf{m}|} \frac{(q^{\mu}_{n,\mathbf{m}})^{(k)}(\xi)}{k!}(z-\xi)^k.$$

Applying (3.4) and the Weierstrass approximation theorem for derivatives, we have

$$q_{\Omega}(z) = \lim_{n\in\Omega} q^{\mu}_{n,\mathbf{m}}(z) = \lim_{n\in\Omega} \sum_{k=0}^{|\mathbf{m}|} \frac{(q^{\mu}_{n,\mathbf{m}})^{(k)}(\xi)}{k!}(z-\xi)^k = \sum_{k=\nu}^{|\mathbf{m}|} \frac{(q_{\Omega})^{(k)}(\xi)}{k!}(z-\xi)^k,$$

which implies what we wanted.

Let $\varepsilon > 0$ be sufficiently small so that $\mathbb{B}(\xi, 2\varepsilon)$ contains no other system pole of $\mathbf{F}$ with respect to $\mathbf{m}$. Let $\hat{\xi}_{n,1}, \ldots, \hat{\xi}_{n,\sigma_n}$ be the zeros of $q^{\mu}_{n,\mathbf{m}}$ contained in $\mathbb{B}(\xi, 2\varepsilon)$. By (3.5), we have $\nu \leq \sigma_n \leq |\mathbf{m}|$ for all sufficiently large n. In the sequel, we only consider such values of n. Set

$$\hat{Q}_n(z) := \prod_{j=1}^{\sigma_n}(z - \hat{\xi}_{n,j}).$$

It is easy to check that the functions $\hat{Q}_n/q^{\mu}_{n,\mathbf{m}}$ are holomorphic in $\mathbb{B}(\xi, 2\varepsilon)$ and uniformly bounded on any compact subset of $\mathbb{B}(\xi, 2\varepsilon)$, in particular on $\overline{\mathbb{B}(\xi, \varepsilon)}$. Therefore, by Cauchy's integral formula, for any $j = 0, 1, \ldots, \nu - 1$, the sequence $((\hat{Q}_n/q^{\mu}_{n,\mathbf{m}})^{(j)})_{n\in\mathbb{N}}$ is bounded on $\overline{\mathbb{B}(\xi, \varepsilon)}$. Using Leibniz's formula and the inequalities in (3.4), we obtain

$$\limsup_{n\to\infty} |\hat{Q}_n^{(j)}(\xi)|^{1/n} = \limsup_{n\to\infty} \left| \left(q^{\mu}_{n,\mathbf{m}} \frac{\hat{Q}_n}{q^{\mu}_{n,\mathbf{m}}} \right)^{(j)} (\xi) \right|^{1/n}$$

$$= \limsup_{n\to\infty} \left| \sum_{k=0}^{j} \binom{j}{k} (q^{\mu}_{n,\mathbf{m}})^{(k)}(\xi) \left(\frac{\hat{Q}_n}{q^{\mu}_{n,\mathbf{m}}} \right)^{(j-k)} (\xi) \right|^{1/n} \le \frac{|\Phi(\xi)|}{\rho_\xi(\mathbf{F}, \mathbf{m})} < 1, \qquad (3.6)$$

for each $j = 0, \ldots, \nu - 1$.

Finally, we want to show that

$$\Delta^{\mu}(\xi) \le \frac{|\Phi(\xi)|}{\rho_\xi(\mathbf{F}, \mathbf{m})} \qquad \text{and} \qquad \sigma^{\mu}(\xi) \ge \nu. \qquad (3.7)$$

Using (3.6) for $j = 0$ and the ordering imposed on the indexing of zeros of $Q^{\mu}_{n,\mathbf{m}}$, it follows that

$$\Delta^{\mu}(\xi) = \limsup_{n\to\infty} |Q^{\mu}_{n,\mathbf{m}}(\xi)|_1^{1/n} = \limsup_{n\to\infty} |\hat{Q}_n(\xi)|^{1/n} \le \frac{|\Phi(\xi)|}{\rho_\xi(\mathbf{F}, \mathbf{m})} < 1$$

and

$$\limsup_{n\to\infty} |\xi - \hat{\xi}_{n,1}|^{1/n} < 1$$

so that $\sigma^{\mu}(\xi) \ge 1$. Assume that for each $j = 1, \ldots, k$, where $k \le \nu - 1$,

$$\limsup_{n\to\infty} |\xi - \hat{\xi}_{n,j}|^{1/n} < 1, \qquad (3.8)$$

and let us show that it is also true for $k + 1$. Consider $\hat{Q}_n^{(k)}(\xi)$. Notice that one of the terms thus obtained is $\prod_{j=k+1}^{\sigma_n} (\xi - \hat{\xi}_{n,j})$ and each one of the other terms contains at least one factor of the form $(\xi - \hat{\xi}_{n,j})$ for some $j = 1, \ldots, k$. Combining (3.6) for $j = k$ and (3.8), it follows that

$$\limsup_{n\to\infty} \Big| \prod_{j=k+1}^{\sigma_n} (\xi - \hat{\xi}_{n,j}) \Big|^{1/n} < 1,$$

and due to the ordering of the indices, we get

$$\limsup_{n\to\infty} |\xi - \hat{\xi}_{n,k+1}|^{1/n} < 1.$$

Therefore, $\sigma^{\mu}(\xi) \ge \nu$.

□

Proof of Corollary 2.1. Again we will prove only (a). The proofs of (b) and (c) are the same as the one of (a).

Let us use the same notation defined in the proof of Theorem 2.1. By our assumption, we can assume that

$$\hat{Q}_n(z) = \prod_{j=1}^{\nu}(z - \hat{\xi}_{n,j}).$$

Recall that for each $j = 0, 1, \ldots, \nu - 1$,

$$\limsup_{n\to\infty} |\hat{Q}_n^{(j)}(\xi)|^{1/n} \le \frac{|\Phi(\xi)|}{\rho_\xi(\mathbf{F}, \mathbf{m})}.$$

Combining these inequalities and the expression,

$$\tilde{Q}_n(z) = (z-\xi)^\nu + \sum_{k=0}^{\nu-1} \frac{\tilde{Q}_n^{(k)}(\xi)}{k!}(z-\xi)^k,$$

we have

$$\limsup_{n\to\infty} \|(z-\xi)^\nu - \tilde{Q}_n(z)\|_{\mathbb{B}(\xi,2\varepsilon)}^{1/n} \le \frac{|\Phi(\xi)|}{\rho_\xi(\mathbf{F}, \mathbf{m})}.$$

In particular, if we replace z by $\hat{\xi}_{n,\nu}$, then

$$\delta_\nu^\mu(\xi) = \limsup_{n\to\infty} |\hat{\xi}_{n,\nu} - \xi|^{1/n} \le \left(\frac{|\Phi(\xi)|}{\rho_\xi(\mathbf{F}, \mathbf{m})}\right)^{1/\nu}.$$

This clearly implies (2.1).

By Theorem 2.1, $\Delta^\mu(\xi) \le |\Phi(\xi)|/\rho_\xi(\mathbf{F}, \mathbf{m})$ is always true. Since $\liminf_{n\to\infty} |\xi - \hat{\xi}_{n,\nu+1}| > 0$, $\Delta^\mu(\xi) = \delta_1^\mu(\xi)\delta_2^\mu(\xi)\ldots\delta_\nu^\mu(\xi)$. Therefore, $\delta_1^\mu(\xi) = \delta_2^\mu(\xi) = \ldots = \delta_\nu^\mu(\xi) = (|\Phi(\xi)|/\rho_\xi(\mathbf{F}, \mathbf{m}))^{1/\nu}$ if and only if $\Delta^\mu(\xi) = |\Phi(\xi)|/\rho_\xi(\mathbf{F}, \mathbf{m})$. □

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