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Convergence in Hausdorff Content of Generalized Simultaneous Padé Approximants

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Abstract : Given a vector of the approximated functions analytic on a neighborhood of some compact subset of the complex plane with simply connected complement in the extended complex plane, we prove convergences in Hausdorff content of the corresponding two generalizations of type II Hermite-Padé approximants on some certain sequences. These two generalizations are based on orthogonal and Faber polynomial expansions. As consequences of these convergence results, we give alternate proofs of Montessus de Ballore type theorems for these generalizations.

Keywords : Hermite-Padé approximation; simultaneous Padé approximation; Faber polynomials; orthogonal polynomials; Montessus de Ballore's theorem; Hausdorff content.

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1 Introduction

Simultaneous Padé approximation (or Hermite-Padé approximation) has been studied for a long time. Making use of this approximation, Hermite [1] proved that e is transcendental in 1873. This approximation was systematically reintroduced for general vectors of approximated functions in [2]. Most of the studies of simultaneous Padé approximation were concentrated on diagonal sequences (for more information, see some important papers [3, 4, 5, 6] and some survey papers [7, 8] in this direction). There are very few papers [9, 10, 11, 12, 13] dedicated to the study of row sequences. The pioneering one in this direction is the work of Graves-Morris and Saff [10] where they proved an analogue of the Montessus de Ballore theorem. The other significant work in this direction is due to Cacoq, de la Calle, and López [9] where they proved some results on the inverse problem of row sequences. In last few years, these simulaneous Padé approximants were generalized in various forms such as orthogonal Hermite-Padé approximants, multipoint Hermite-Padé approximants, simultaneous Padé-Faber approximants, and simultaneous Padé-orthogonal approximants (see [14, 15, 16, 17, 18, 19]).

In this paper, we study convergences of two generalizations of simultaneous Padé approximation. The first approximation is based on orthogonal polynomials on a general compact set and is called simultaneous Padé-orthogonal approximation. The concept of simultaneous Padé-orthogonal approximation was first introduced by Cocoq and López in [20]. In their paper, those simultaneous Padé-orthogonal approximants are called simultaneous Fourier-Padé approximants and are constructed from orthogonal polynomial on the unit circle. They obtained convergence of row sequences of simultaneous Padé-orthogonal approximants. In [14], the definition of Cocoq and López was extended to more general compact set and convergence theorem for row sequences of the corresponding appriximants was proved. Simultaneous Padé-orthogonal approximation is defined as follows.

Let E be an infinite compact subset of the complex plane $\mathbb{C}$ such that $\overline{\mathbb{C}} \setminus E$ is simply connected. Denote by $\mathcal{K}$ the collection of these compact sets. Let μ be a finite positive Borel measure with an infinite support $\mathrm{supp}(\mu)$ contained in E. We write $\mu \in \mathcal{M}(E)$ and define the associated inner product

$$\langle g, h \rangle_\mu := \int g(\zeta)\overline{h(\zeta)}d\mu(\zeta), \qquad g, h \in L_2(\mu).$$

Let

$$p_n(z) := \kappa_n z^n + \cdots, \quad \kappa_n > 0, \quad n = 0, 1, 2, \ldots,$$

be the orthonormal polynomial of degree n with respect to μ with positive leading coefficient; that is $\langle p_n, p_m \rangle_\mu = \delta_{n,m}$. Define

$$\mathcal{H}(E)^d := \{(F_1, F_2, \ldots, F_d) : F_\alpha \in \mathcal{H}(E) \text{ for all } \alpha = 1, 2, \ldots, d\},$$

where $\mathcal{H}(E)$ is the space of all functions holomorphic in some neighborhood of E.

Definition 1.1. Let $E \in \mathcal{K}$, $\mathbf{F} = (F_1, F_2, \ldots, F_d) \in \mathcal{H}(E)^d$ and $\mu \in \mathcal{M}(E)$. Fix a multi-index $\mathbf{m} = (m_1, m_2, \ldots, m_d) \in \mathbb{N}_0^d \setminus \{\mathbf{0}\}$ where $\mathbf{0}$ is the zero vector in $\mathbb{N}_0^d$. Set $|\mathbf{m}| := m_1 + m_2 + \cdots + m_d$. Then, for each $n \geq \max\{m_1, m_2, \ldots, m_d\}$, there exist polynomials $q^\mu_{n,\mathbf{m}}$, $p^\mu_{n,\mathbf{m},\alpha}$, where $\alpha = 1, 2, \ldots, d$, such that

$$\deg(p^\mu_{n,\mathbf{m},\alpha}) \leq n - m_\alpha, \quad \deg(q^\mu_{n,\mathbf{m}}) \leq |\mathbf{m}|, \quad q^\mu_{n,\mathbf{m}} \not\equiv 0,$$
$$\langle q^\mu_{n,\mathbf{m}} F_\alpha - p^\mu_{n,\mathbf{m},\alpha}, p_j \rangle_\mu = 0, \quad j = 0, 1, \ldots, n.$$

The vector of rational functions

$$\mathbf{R}^\mu_{n,\mathbf{m}} = (R^\mu_{n,\mathbf{m},1}, R^\mu_{n,\mathbf{m},2}, \ldots, R^\mu_{n,\mathbf{m},d})$$
$$:= (p^\mu_{n,\mathbf{m},1}/q^\mu_{n,\mathbf{m}}, p^\mu_{n,\mathbf{m},2}/q^\mu_{n,\mathbf{m}}, \ldots, p^\mu_{n,\mathbf{m},d}/q^\mu_{n,\mathbf{m}})$$

is called an $(n, \mathbf{m})$ *simultaneous Padé-orthogonal approximant of* $\mathbf{F}$ *with respect to* μ.

Indeed, finding $q^\mu_{n,\mathbf{m}}$ is equivalent to solving a system of $|\mathbf{m}|$ homogeneous linear equations with $|\mathbf{m}|+1$ unknowns. Then, $q^\mu_{n,\mathbf{m}}$ always exists. Since $q^\mu_{n,\mathbf{m}} \not\equiv 0$, we normalize $q^\mu_{n,\mathbf{m}}$ to be a "monic" polynomial. Moreover, for each $\alpha = 1, 2, \ldots, d$, $p^\mu_{n,\mathbf{m},\alpha}$ is uniquely determined by $q^\mu_{n,\mathbf{m}}$. Therefore, for any pair $(n, \mathbf{m})$, a vector of rational functions $\mathbf{R}^\mu_{n,\mathbf{m}}$ always exists but may not be unique.

Now, we introduce a definition of poles for a vector of functions.

Definition 1.2. Let $\mathbf{\Omega} := (\Omega_1, \Omega_2, \ldots, \Omega_d)$ be a system of domains such that for each $\alpha = 1, 2, \ldots, d$, F_α is meromorphic in Ω_α. We say that the point λ *is a pole of* $\mathbf{F}$ *in* $\mathbf{\Omega}$ *of order* τ if there exists an index $\alpha \in \{1, 2, \ldots, d\}$ such that $\lambda \in \Omega_\alpha$ and it is a pole of F_α of order τ, and for $\beta \neq \alpha$ either λ is a pole of F_β of order less than or equal to τ or $\lambda \notin \Omega_\beta$. When $\mathbf{\Omega} := (\Omega, \Omega, \ldots, \Omega)$, we say that λ *is a pole of* $\mathbf{F}$ *in* Ω.

The second approximation is based on Faber polynomials defined as follows. Let $E \in \mathcal{K}$ and Φ be the exterior conformal mapping from $\overline{\mathbb{C}} \setminus E$ onto $\overline{\mathbb{C}} \setminus \{w \in \mathbb{C} : |w| \leq 1\}$ satisfying $\Phi(\infty) = \infty$ and $\Phi'(\infty) > 0$. For each $\rho > 1$, we define a *level curve with respect to* E *of index* ρ and a *canonical domain with respect to* E *of index* ρ by

$$\Gamma_\rho := \{z \in \mathbb{C} : |\Phi(z)| = \rho\} \quad \text{and} \quad D_\rho := E \cup \{z \in \mathbb{C} : |\Phi(z)| < \rho\},$$

respectively. Let $\mathbf{F} \in \mathcal{H}(E)^d$. Denote by $\rho_{|\mathbf{m}|}(\mathbf{F})$ the index $\rho > 1$ of the largest canonical domain D_ρ to which $\mathbf{F}$ has at most $|\mathbf{m}|$ poles. The *Faber polynomial of degree* n *for* E is defined by the formula

$$\Phi_n(z) := \frac{1}{2\pi i} \int_{\Gamma_\rho} \frac{\Phi^n(t)}{t - z} dt, \quad z \in D_\rho, \quad n = 0, 1, 2, \ldots \tag{1.1}$$

and the *Faber coefficient of* $F \in \mathcal{H}(E)$ *with respect to* Φ_n is given by

$$[F]_n := \frac{1}{2\pi i} \int_{\Gamma_\rho} \frac{F(t)\Phi'(t)}{\Phi^{n+1}(t)} dt, \tag{1.2}$$

where $\rho \in (1, \rho_0(F))$.

Definition 1.3. Let $E \in \mathcal{K}$ and $\mathbf{F} = (F_1, F_2, \ldots, F_d) \in \mathcal{H}(E)^d$. Fix a multi-index $\mathbf{m} = (m_1, m_2, \ldots, m_d) \in \mathbb{N}_0^d \setminus \{\mathbf{0}\}$. Set $|\mathbf{m}| := m_1 + m_2 + \cdots + m_d$. Then, for each $n \geq \max\{m_1, m_2, \ldots, m_d\}$, there exist polynomials $q^E_{n,\mathbf{m}}$, $p^E_{n,\mathbf{m},\alpha}$, where $\alpha = 1, 2, \ldots, d$, such that

$$\deg(p^E_{n,\mathbf{m},\alpha}) \leq n - m_\alpha, \quad \deg(q^E_{n,\mathbf{m}}) \leq |\mathbf{m}|, \quad q^E_{n,\mathbf{m}} \not\equiv 0,$$

$$[q^E_{n,\mathbf{m}} F_\alpha - p^E_{n,\mathbf{m},\alpha}]_j = 0, \quad j = 0, 1, \ldots, n.$$

The vector of rational functions

$$\begin{aligned}\mathbf{R}^E_{n,\mathbf{m}} &= (R^E_{n,\mathbf{m},1}, R^E_{n,\mathbf{m},2}, \ldots, R^E_{n,\mathbf{m},d}) \\ &:= (p^E_{n,\mathbf{m},1}/q^E_{n,\mathbf{m}}, p^E_{n,\mathbf{m},2}/q^E_{n,\mathbf{m}}, \ldots, p^E_{n,\mathbf{m},d}/q^E_{n,\mathbf{m}})\end{aligned}$$

is called an $(n, \mathbf{m})$ *simultaneous Padé-Faber approximant of* $\mathbf{F}$ *corresponding to* E.

Using the same line of reasoning, for any pair $(n, \mathbf{m})$, we normalize $q^E_{n,\mathbf{m}}$ to be a "monic" polynomial and a vector of rational functions $\mathbf{R}^E_{n,\mathbf{m}}$ always exists but may not be unique. In [15] and [16], the idea of simultaneous Padé-Faber approximants was introduced and analogues of Montessus de Ballore's theorem for simultaneous Padé-Faber approximants were proved.

Next, let us introduce the concept of convergence in Hausdorff content. Let B be a subset of the complex plane $\mathbb{C}$. By $\mathcal{U}(B)$, we denote the class of all coverings of B by at most a countable set of disks. Let $\beta > 0$ and set

$$h_\beta(B) := \inf \left\{ \sum_{j=1}^{\infty} |U_j|^\beta : \{U_j\} \in \mathcal{U}(B) \right\},$$

where $|U_j|$ is the radius of the disk U_j. This notation $h_\beta(B)$ is called the *β-dimensional Hausdorff content of the set B*. This set function is not a measure but it is subadditive and monotonic. Clearly, if B is a disk, then $h_\beta(B) = |B|^\beta$.

Definition 1.4. Let $\{g_n\}_{n\in\mathbb{N}}$ be a sequence of complex valued functions defined on a domain $D \subset \mathbb{C}$ and g another complex function defined on D. We say that $\{g_n\}_{n\in\mathbb{N}}$ converges in *β-dimensional Hausdorff content* to the function g inside D if for every compact subset K of D and for each $\epsilon > 0$, we have

$$\lim_{n\to\infty} h_\beta\{z \in K : |g_n(z) - g(z)| > \epsilon\} = 0.$$

Such a convergence will be denoted by $h_\beta - \lim_{n\to\infty} g_n = g$ in D.

In this paper, we prove convergences in Hausdorff content of those two generalizations when the sequences of indices $\{(n, \mathbf{m}_n)\}$ satisfy the limit below

$$\lim_{n\to\infty} \frac{|\mathbf{m}_n| \ln n}{n} = 0. \tag{1.3}$$

This type of sequences was introduced by Gonchar [21] for Padé (α, β)-approximants. We prove results analogous to Theorem 2 in [21] for two generalizations of simultaneous Padé approximants. As consequences of our main theorems, we give alternate proofs of the Montessus de Ballore type theorem for those generalizations.

The outline of this paper is as follows. Section 2 contains our main results. We collect needed auxiliary lemmas in Section 3. Section 4 is dedicated to the proofs of all results in Section 2.

2 Main Results

Before we state our results about the convergence of simultaneous Padé-orthogonal approximants, we need to define a class of measures and some more notation. A class of measures that we are interested is $\mathcal{R}(E) \subset \mathcal{M}(E)$. We write $\mu \in \mathcal{R}(E)$ when the corresponding sequence of orthonormal polynomials has *ratio asymptotics*; that is

$$\lim_{n\to\infty} \frac{p_n(z)}{p_{n+1}(z)} = \frac{1}{\Phi(z)},$$

uniformly on each compact subset of $\overline{\mathbb{C}} \setminus E$. Moreover, we restrict ourselves to a smaller collection of compact sets E defined as follows. Denote by $\mathcal{K}_1$ the collection of all sets $E \in \mathcal{K}$ such that the inverse function of Φ can be extended continuously to $\overline{\mathbb{C}} \setminus \{w \in \mathbb{C} : |w| < 1\}$.

The following theorem is our main result on simultaneous Padé-orthogonal approximants which is an analogue of Theorem 2 in [21].

Theorem 2.1. *Let $E \in \mathcal{K}_1$, $\rho > 1$, $\mu \in \mathcal{R}(E)$, and $\mathbf{F} = (F_1, F_2, \ldots, F_d) \in \mathcal{H}(E)^d$ be a vector of functions meromorphic in D_ρ. Suppose that for each $\alpha = 1, 2, \ldots, d$, F_α has exactly $\nu(F_\alpha, D_\rho)$ poles (counting multiplicities) in D_ρ and the sequence $\{\mathbf{m}_n\} := \{(m_{n,1}, m_{n,2}, \ldots, m_{n,d})\}$ satisfies the following conditions*

$$\liminf_{n\to\infty} m_{n,\alpha} \geq \nu(F_\alpha, D_\rho), \qquad \alpha = 1, 2, \ldots, d$$

and

$$\lim_{n\to\infty} \frac{|\mathbf{m}_n| \ln n}{n} = 0.$$

Then for any fixed numbers $\beta > 0$ and $\alpha = 1, 2, \ldots, d$, each sequence $\{R^{\mu}_{n,\mathbf{m}_n,\alpha}\}$ converges in β-dimensional Hausdorff content to F_α inside D_ρ as $n \to \infty$.

As a consequence of Theorem 2.1, we can prove a Montessus de Ballore type theorem for simultaneous Padé-orthogonal approximants which was earlier proved in [14, Theorem 2.4] stated below. Given $\mathbf{F} = (F_1, F_2, \ldots, F_d) \in \mathcal{H}(E)^d$ and a multi-index $\mathbf{m} := (m_1, m_2, \ldots, m_d) \in \mathbb{N}_0^d \setminus \{\mathbf{0}\}$, we define

$$\mathbf{D}_{\mathbf{m}}(\mathbf{F}) := (D_{\rho_{m_1}}(F_1), D_{\rho_{m_2}}(F_2), \ldots, D_{\rho_{m_d}}(F_d)).$$

Corollary 2.2. *Let $E \in \mathcal{K}_1$, $\mu \in \mathcal{R}(E)$, and $\mathbf{F} = (F_1, F_2, \ldots, F_d) \in \mathcal{H}(E)^d$. For each $\alpha = 1, 2, \ldots, d$, suppose that F_α has poles of total multiplicity m_α in $D_{\rho_{m_\alpha}(F_\alpha)}$ at the points $\lambda_{\alpha,1}, \lambda_{\alpha,2}, \ldots, \lambda_{\alpha,m_\alpha}$ and $\mathbf{F}$ has exactly $|\mathbf{m}|$ poles in $\mathbf{D_m(F)}$ where $\mathbf{m} := (m_1, m_2, \ldots, m_d)$. Then, $\mathbf{R}^\mu_{n,\mathbf{m}}$ is uniquely determined for all sufficiently large n and for any $\alpha = 1, 2, \ldots, d$, $\{R^\mu_{n,\mathbf{m},\alpha}\}$ converges uniformly to F_α on each compact subset of $D_{\rho_{m_\alpha}(F_\alpha)} \setminus \{\lambda_{\alpha,1}, \lambda_{\alpha,2}, \ldots, \lambda_{\alpha,m_\alpha}\}$ as $n \to \infty$. Moreover, for each $\alpha = 1, 2, \ldots, d$ and for any compact subset K of $D_{\rho_{m_\alpha}(F_\alpha)} \setminus \{\lambda_{\alpha,1}, \lambda_{\alpha,2}, \ldots, \lambda_{\alpha,m_\alpha}\}$,*

$$\limsup_{n\to\infty} \|F_\alpha - R^\mu_{n,\mathbf{m},\alpha}\|_K^{1/n} \leq \frac{\|\Phi\|_K}{\rho_{m_\alpha}(F_\alpha)},$$

where $\|\cdot\|_K$ denotes the sup-norm on K and if $K \subset E$, then $\|\Phi\|_K$ is replaced by 1.

Corollary 2.3. *Let $E \in \mathcal{K}_1$, $\mu \in \mathcal{R}(E)$, and $\mathbf{F} = (F_1, F_2, \ldots, F_d) \in \mathcal{H}(E)^d$. For each $\alpha = 1, 2, \ldots, d$, denote by*

$$D_{\rho_\infty(F_\alpha)} := \bigcup_{j=0}^{\infty} D_{\rho_j(F_\alpha)}$$

the maximal canonical domain in which F_α can be continued to a meromorphic function. Assume that

$$\liminf_{n\to\infty} m_{n,\alpha} = \infty, \qquad \alpha = 1, 2, \ldots, d$$

and

$$\lim_{n\to\infty} \frac{|\mathbf{m}_n| \ln n}{n} = 0.$$

Then for any fixed numbers $\beta > 0$ and $\alpha = 1, 2, \ldots, d$, each sequence $\{R^\mu_{n,\mathbf{m}_n,\alpha}\}$ converges in β-dimensional Hausdorff content to F_α inside $D_{\rho_\infty(F_\alpha)}$ as $n \to \infty$.

Note that the scalar case of the above two results were obtained in [22].

Similar results for simultaneous Padé-Faber approximants are stated below.

Theorem 2.4. *Let $E \in \mathcal{K}$, $\rho > 1$ and $\mathbf{F} = (F_1, F_2, \ldots, F_d) \in \mathcal{H}(E)^d$ be a vector of functions meromorphic in D_ρ. Suppose that for each $\alpha = 1, 2, \ldots, d$, F_α has exactly $\nu(F_\alpha, D_\rho)$ poles (counting multiplicities) in D_ρ and the sequence $\{\mathbf{m}_n\} := \{(m_{n,1}, m_{n,2}, \ldots, m_{n,d})\}$ satisfies the following conditions*

$$\liminf_{n\to\infty} m_{n,\alpha} \geq \nu(F_\alpha, D_\rho), \qquad \alpha = 1, 2, \ldots, d$$

and

$$\lim_{n\to\infty} \frac{|\mathbf{m}_n| \ln n}{n} = 0.$$

Then for any fixed numbers $\beta > 0$ and $\alpha = 1, 2, \ldots, d$, each sequence $\{R^E_{n,\mathbf{m}_n,\alpha}\}$ converges in β-dimensional Hausdorff content to F_α inside D_ρ as $n \to \infty$.

The following corollary coincides with Theorem 1 in [15].

Corollary 2.5. *Let $E \in \mathcal{K}$ and $\mathbf{F} = (F_1, F_2, \dots, F_d) \in \mathcal{H}(E)^d$. For each $\alpha = 1, 2, \dots, d$, suppose that F_α has poles of total multiplicity m_α in $D_{\rho_{m_\alpha}(F_\alpha)}$ at the points $\lambda_{\alpha,1}, \lambda_{\alpha,2}, \dots, \lambda_{\alpha,m_\alpha}$ and $\mathbf{F}$ has exactly $|\mathbf{m}|$ poles in $\mathbf{D_m}(\mathbf{F})$ where $\mathbf{m} := (m_1, m_2, \dots, m_d)$. Then, $\mathbf{R}^E_{n,\mathbf{m}}$ is uniquely determined for all sufficiently large n and for any $\alpha = 1, 2, \dots, d$, $\{R^E_{n,\mathbf{m},\alpha}\}$ converges uniformly to F_α on each compact subset of $D_{\rho_{m_\alpha}(F_\alpha)} \setminus \{\lambda_{\alpha,1}, \lambda_{\alpha,2}, \dots, \lambda_{\alpha,m_\alpha}\}$ as $n \to \infty$. Moreover, for each $\alpha = 1, 2, \dots, d$ and for any compact subset K of $D_{\rho_{m_\alpha}(F_\alpha)} \setminus \{\lambda_{\alpha,1}, \lambda_{\alpha,2}, \dots, \lambda_{\alpha,m_\alpha}\}$,*

$$\limsup_{n\to\infty} \|F_\alpha - R^E_{n,\mathbf{m},\alpha}\|_K^{1/n} \le \frac{\|\Phi\|_K}{\rho_{m_\alpha}(F_\alpha)},$$

where $\|\cdot\|_K$ denotes the sup-norm on K and if $K \subset E$; then $\|\Phi\|_K$ is replaced by 1.

Corollary 2.6. *Let $E \in \mathcal{K}$ and $\mathbf{F} = (F_1, F_2, \dots, F_d) \in \mathcal{H}(E)^d$. Assume that*

$$\liminf_{n\to\infty} m_{n,\alpha} = \infty, \qquad \alpha = 1, 2, \dots,$$

and

$$\lim_{n\to\infty} \frac{|\mathbf{m}_n| \ln n}{n} = 0.$$

Then for any fixed numbers $\beta > 0$ and $\alpha = 1, 2, \dots, d$, each sequence $\{R^E_{n,\mathbf{m}_n,\alpha}\}$ converges in β-dimensional Hausdorff content to F_α inside $D_{\rho_\infty(F_\alpha)}$ as $n \to \infty$.

3 Auxiliary Lemmas

In this section, we keep all needed notations and lemmas. Let $E \in \mathcal{K}$ and $\mu \in \mathcal{M}(E)$. We define the n-th Fourier coefficient of $G \in \mathcal{H}(E)$ with respect to p_n by

$$\langle G \rangle_n := \langle G, p_n \rangle_\mu = \int G(z)\overline{p_n(z)}d\mu(z).$$

We say that $\mu \in \mathbf{Reg}_1(E) \subset \mathcal{M}(E)$ when

$$\lim_{n\to\infty} |p_n(z)|^{1/n} = |\Phi(z)|, \tag{3.1}$$

uniformly on each compact subset of $\mathbb{C}\setminus E$. The following two lemmas (see, e.g., [17, Lemma 2.1] and [23]) concern the formulas for computing $\rho_0(G)$ and the domain of convergence of orthogonal and Faber polynomial expansions of holomorphic functions.

Lemma 3.1. *Let $E \in \mathcal{K}$, $G \in \mathcal{H}(E)$ and $\mu \in \mathbf{Reg}_1(E)$. Then,*

$$\rho_0(G) = \left(\limsup_{n\to\infty} |\langle G \rangle_n|^{1/n}\right)^{-1}.$$

Moreover, the series $\sum_{n=0}^{\infty} \langle G \rangle_n p_n(z)$ converges to $G(z)$ uniformly on each compact subset of $D_{\rho_0(G)}$.

Lemma 3.2. *Let $E \in \mathcal{K}$ and $G \in \mathcal{H}(E)$. Then,*

$$\rho_0(G) = \left(\limsup_{n\to\infty} |[G]_n|^{1/n} \right)^{-1}.$$

Moreover, the series $\sum_{n=0}^{\infty} [G]_n \Phi_n(z)$ converges to $G(z)$ uniformly on each compact subset of $D_{\rho_0(G)}$.

The second type functions s_n defined by

$$s_n(z) := \int \frac{\overline{p_n(\zeta)}}{z-\zeta} d\mu(\zeta), \quad z \in \overline{\mathbb{C}} \setminus \operatorname{supp}(\mu),$$

are very useful in our proofs. The next lemma (see [24, Lemma 3.1]) is the asymptotic relation between the orthogonal polynomials p_n and the second type functions s_n.

Lemma 3.3. *Let $E \in \mathcal{K}_1$. If $\mu \in \mathcal{R}(E)$, then*

$$\lim_{n\to\infty} p_n(z) s_n(z) = \frac{\Phi'(z)}{\Phi(z)},$$

uniformly on each compact subset of $\overline{\mathbb{C}} \setminus E$. Consequently, for any compact set $K \subset \mathbb{C} \setminus E$, there exists $n_0 \in \mathbb{N}$ such that $s_n(z) \neq 0$ for all $z \in K$ and $n \geq n_0$.

A simple relation (see [17, Lemma 2.2]) used frequently in this paper is contained in

Lemma 3.4. *Let $E \in \mathcal{K}$, $G \in \mathcal{H}(E)$, $k \in \mathbb{N}_0$, and $\rho \in (1, \rho_0(G))$. Then,*

$$\langle G \rangle_k = \frac{1}{2\pi i} \int_{\Gamma_\rho} G(w) s_k(w) dw. \tag{3.2}$$

The following lemma (see [25, page 43] or [26, page 583] for its proof) gives an estimate of Faber polynomials on on a level curve.

Lemma 3.5. *Let $\rho > 1$ be fixed. Then, there exists $c > 0$ such that*

$$\|\Phi_n\|_{\Gamma_\rho} \leq c\rho^n, \quad n \geq 0. \tag{3.3}$$

Indeed, by the maximum modulus principle, the inequalities (3.3) can be replaced by the inequalities

$$\|\Phi_n\|_{\overline{D}_\rho} \leq c\rho^n, \quad n \geq 0. \tag{3.4}$$

The following lemma is about the uniqueness of the common denominators of generalized simultaneous Padé approximants to polynomial expansions.

Lemma 3.6. *Let* $(n, \mathbf{m}) \in \mathbb{N} \times \mathbb{N}_0^d \setminus \{\mathbf{0}\}$ *be fixed. Then the following assertions hold:*

(b) If for all $q^{\mu}_{n,\mathbf{m}}$ *in Definition 1.1,* $\deg q^{\mu}_{n,\mathbf{m}} = |\mathbf{m}|$*, then* $q^{\mu}_{n,\mathbf{m}}$ *is unique.*

(b) If for all $q^{E}_{n,\mathbf{m}}$ *in Definition 1.3,* $\deg q^{E}_{n,\mathbf{m}} = |\mathbf{m}|$*, then* $q^{E}_{n,\mathbf{m}}$ *is unique.*

Proof of Lemma 3.6. Without loss of generality, we may consider only in the case of $q^{\mu}_{n,\mathbf{m}}$. From Definition 1.1, a polynomial $c_{|\mathbf{m}|}z^{|\mathbf{m}|}+c_{|\mathbf{m}|-1}z^{|\mathbf{m}|-1}+\ldots+c_0$ is $q^{\mu}_{n,\mathbf{m}}$ if and only if it is monic and the coefficients $c_{|\mathbf{m}|}, c_{|\mathbf{m}|-1}, \ldots, c_0$ satisfy the following system of $|\mathbf{m}|$ linear equations: for all $\alpha = 1, 2, \ldots, d$ and $j = n - m_\alpha + 1, j = n - m_\alpha + 2, \ldots, n$,

$$\sum_{k=0}^{|\mathbf{m}|} c_k \langle z^k F_\alpha \rangle_j = 0. \tag{3.5}$$

Suppose for a contradiction that there are 2 distinct monic polynomials q_1 and q_2 of degree $|\mathbf{m}|$ which satisfy all conditions of $q^{\mu}_{n,\mathbf{m}}$. Then there exists a polynomial $q := q_1 - q_2 \neq 0$ of degree less than $|\mathbf{m}|$. Obviously, all coefficients of q satisfies (3.5). By normalize q to be a monic polynomial, a contradiction on the degree of $q^{\mu}_{n,\mathbf{m}}$ occurs. ☐

The determinant of the matrix in the following lemma will be used in our proofs of the main theorems.

Lemma 3.7. *Let* Φ *be the exterior conformal mapping from* $\overline{\mathbb{C}} \setminus E$ *onto* $\overline{\mathbb{C}} \setminus \{w \in \mathbb{C} : |w| \leq 1\}$ *satisfying* $\Phi(\infty) = \infty$ *and* $\Phi'(\infty) > 0$*. Assume that* $\lambda_1, \lambda_2, \ldots, \lambda_q$ *are distinct points in* $\mathbb{C} \setminus E$ *and* $\tau_j \geq 0$ *for all* $j = 1, 2, \ldots, q$*. Define* $m := \sum_{j=1}^{q} \tau_j$ *and the* $m \times m$ *matrix as follows*

$$\Delta := \begin{bmatrix} \left(\Phi^{m-1}\right)(\lambda_j) & \cdots & \left(\Phi^{m-1}\right)^{(\tau_j-1)}(\lambda_j) \\ \left(\Phi^{m-2}\right)(\lambda_j) & \cdots & \left(\Phi^{m-2}\right)^{(\tau_j-1)}(\lambda_j) \\ \vdots & \cdots & \vdots \\ 1 & \cdots & 0 \end{bmatrix}_{j=1,2,\ldots,q}, \tag{3.6}$$

where the subindex on the determinant means that the indicated group of columns are successively written for $j = 1, 2, \ldots, q$*. Then,*

$$\det(\Delta) = \prod_{j=1}^{q} (\tau_j - 1)!!(\Phi'(\lambda_j))^{\tau_j(\tau_j-1)/2} \prod_{1\leq i<j\leq q} (\Phi(\lambda_j) - \Phi(\lambda_i))^{\tau_j\tau_i}, \tag{3.7}$$

where $n!!$ *stands for* $0!1!\cdots n!$*.*

The proof of the above lemma is similar to the one of Theorem 1 in [27].

The final lemma proved by Gonchar (see [21, Lemma 1]) allows us to derive uniform convergence on compact subsets of the region under consideration from convergence in h_1-content under appropriate assumptions.

Lemma 3.8. *Suppose that h_1-$\lim_{n\to\infty} g_n = g$ in D. Then the following assertions hold true:*

(i) If the functions $g_n, n \in \mathbb{N}$, are holomorphic in D, then the sequence $\{g_n\}_{n\in\mathbb{N}}$ converges uniformly inside D and g is holomorphic in D.

(ii) If each of the functions g_n is meromorphic in D and has no more than $k < +\infty$ poles in this domain, then the limit function g is also meromorphic and has no more than k poles in D.

(iii) If each function g_n is meromorphic and has no more than $k < +\infty$ poles in D and the function g is meromorphic and has exactly k poles in D, then all $g_n, n \geq N$, also have k poles in D; the poles of g_n tend to the poles $\lambda_1, \lambda_2, \ldots, \lambda_k$ of g (taking account of their orders) and the sequence $\{g_n\}_{n\in\mathbb{N}}$ tends to g uniformly inside the domain $D' = D \setminus \{\lambda_1, \lambda_2, \ldots, \lambda_k\}$.

4 Proofs of main results

Proof of Theorem 2.1. We normalize the polynomials $q^{\mu}_{n,\mathbf{m}_n}$ in terms of its zeros $\lambda_{n,j}$ such that

$$Q^{\mu}_{n,\mathbf{m}_n}(z) := \prod_{|\lambda_{n,j}|\leq 1}(z-\lambda_{n,j})\prod_{|\lambda_{n,j}|>1}\left(1-\frac{z}{\lambda_{n,j}}\right) \tag{4.1}$$

and for each $\alpha = 1, 2, \ldots, d$,

$$R^{\mu}_{n,\mathbf{m}_n,\alpha} = \frac{p^{\mu}_{n,\mathbf{m}_n,\alpha}}{q^{\mu}_{n,\mathbf{m}_n}} = \frac{P^{\mu}_{n,\mathbf{m}_n,\alpha}}{Q^{\mu}_{n,\mathbf{m}_n}}.$$

With this normalization, we can estimate upper and lower bounds on the normalized $Q^{\mu}_{n,\mathbf{m}_n}$.

Let $\alpha \in \{1, 2, \ldots, d\}$ be fixed. Denote by $m_\alpha := \nu(F_\alpha, D_\rho)$ the number of poles of F_α in D_ρ. We fix $\varepsilon > 0$ and cover each pole of F_α in $D_{\rho_{m_\alpha}(F_\alpha)}$ with an open disk of radius $(\varepsilon/(6m_\alpha))^{1/\beta}$ and denote by $J^{\beta}_{0,\varepsilon}(F_\alpha, m_\alpha)$ the union of these disks. We cover each zero of $Q^{\mu}_{n,\mathbf{m}_n}$ with an open disk of radius $(\varepsilon/(6|\mathbf{m}_n|n^2))^{1/\beta}$ and denote by $J^{\beta}_{n,\varepsilon}(\mathbf{F})$ the union of these disks. For each $k > 0$, we set

$$J^{\beta}_{\varepsilon}(F_\alpha, m_\alpha; k) := J^{\beta}_{0,\varepsilon}(F_\alpha, m_\alpha)\bigcup\left(\bigcup_{n=k}^{\infty} J^{\beta}_{n,\varepsilon}(\mathbf{F})\right).$$

By using the monotonicity and subadditivity of h_β, it easy to check that for any $k > 0$,

$$h_\beta(J^{\beta}_{\varepsilon}(F_\alpha, m_\alpha; k)) < \varepsilon$$

and $J^{\beta}_{\varepsilon_1}(F_\alpha, m_\alpha; k) \subset J^{\beta}_{\varepsilon_2}(F_\alpha, m_\alpha; k)$ for $\varepsilon_1 < \varepsilon_2$. For any set $B \subset D_{\rho_{m_\alpha}(F_\alpha)}$, we put $B(\varepsilon; k) := B \setminus J^{\beta}_{\varepsilon}(F_\alpha, m_\alpha; k)$. It is easy to check that if for any compact subset $K \subset D_{\rho_{m_\alpha}(F_\alpha)}$ and $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that $\{g_n\}_{n\in\mathbb{N}}$ converges uniformly to g on $K(\varepsilon; k)$, then $h_\beta - \lim_{n\to\infty} g_n = g$ in $D_{\rho_{m_\alpha}(F_\alpha)}$.

Due to the normalization in (4.1), for any compact subset $K \subset D_{\rho_{m_\alpha}(F_\alpha)}$, $\varepsilon > 0$ be fixed, and $k > 0$, there exist positive constants $C_1 > 0$ and $C_2 > 0$ independent of n such that for all sufficiently large n,

$$\left\| Q^\mu_{n,\mathbf{m}_n} \right\|_K \le C_1^{|\mathbf{m}_n|}, \tag{4.2}$$

and

$$\min_{z\in K\setminus J^\beta_\varepsilon(F_\alpha, m_\alpha; k)} |Q^\mu_{n,\mathbf{m}_n}(z)| \ge (C_2|\mathbf{m}_n|n^2)^{-2|\mathbf{m}_n|/\beta}. \tag{4.3}$$

Since $\mu \in \mathcal{R}(E)$, it follows that

$$\lim_{n\to\infty} \frac{p_n(z)}{p_{n+l}(z)} = \frac{1}{\Phi(z)^l}, \quad l = 0, 1, 2, \ldots, \tag{4.4}$$

uniformly on each compact subset of $\overline{\mathbb{C}} \setminus E$. Then from (4.4) and Lemma 3.3, we obtain

$$\lim_{n\to\infty} \frac{s_{n+l}(z)}{s_n(z)} = \lim_{n\to\infty} \frac{p_n(z)}{p_{n+l}(z)} \frac{p_{n+l}(z)s_{n+l}(z)}{p_n(z)s_n(z)} = \frac{1}{\Phi(z)^l} \frac{\Phi(z)'/\Phi(z)}{\Phi(z)'/\Phi(z)} = \frac{1}{\Phi(z)^l}, \tag{4.5}$$

uniformly on each compact subset of $\overline{\mathbb{C}} \setminus E$. Moreover, it follows from (4.4) and (4.5) that

$$\lim_{n\to\infty} |p_n(z)|^{1/n} = |\Phi(z)|, \tag{4.6}$$

and

$$\lim_{n\to\infty} |s_n(z)|^{1/n} = \frac{1}{|\Phi(z)|}, \tag{4.7}$$

uniformly on each compact subset of $\mathbb{C} \setminus E$, respectively.

Define

$$Q^{F_\alpha}(z) := \prod_{j=1}^{q} (z - \lambda_j)^{\tau_j},$$

where $\lambda_1, \lambda_2, \ldots, \lambda_q$ are distinct poles of F_α in $D_{\rho_{m_\alpha}(F_\alpha)}$ and $\tau_1, \tau_2, \ldots, \tau_q$ are their multiplicities. From the definition of simultaneous Padé-orthogonal approximants and Lemma 3.1, we have

$$Q^\mu_{n,\mathbf{m}_n}(z)F_\alpha(z) - P^\mu_{n,\mathbf{m}_n,\alpha}(z) = \sum_{k=n+1}^{\infty} a^{(\alpha)}_{k,n} p_k(z), \quad z \in D_{\rho_0(F_\alpha)}, \tag{4.8}$$

where

$$a^{(\alpha)}_{k,n} := \langle Q^\mu_{n,\mathbf{m}_n} F_\alpha \rangle_k, \quad k = 0, 1, 2, \ldots,$$

and $a_{k,n}^{(\alpha)}=0$, for all $k=n-m_{n,\alpha}+1, n-m_{n,\alpha}+2,\ldots,n$. Multiplying (4.8) by $Q^{F_\alpha}(z)$ and expanding the result in terms of the orthogonal system $\{p_\nu\}_{\nu=0}^{\infty}$ such that for $z\in D_{\rho_{m_\alpha}(F_\alpha)}$,

$$Q^{F_\alpha}(z)Q^{\mu}_{n,\mathbf{m}_n}(z)F_\alpha(z)-Q^{F_\alpha}(z)P^{\mu}_{n,\mathbf{m}_n,\alpha}(z)=\sum_{k=n+1}^{\infty}Q^{F_\alpha}(z)a_{k,n}^{(\alpha)}p_k(z)$$

$$=\sum_{\nu=0}^{\infty}b_{\nu,n}^{(\alpha)}p_\nu(z)=\sum_{\nu=0}^{n+|\mathbf{m}_n|-m_{n,\alpha}}b_{\nu,n}^{(\alpha)}p_\nu(z)+\sum_{\nu=n+|\mathbf{m}_n|-m_{n,\alpha}+1}^{\infty}b_{\nu,n}^{(\alpha)}p_\nu(z). \tag{4.9}$$

Let K be a compact subset of $D_{\rho_{m_\alpha}(F_\alpha)}$ and set

$$\sigma:=\max\{\|\Phi\|_K,1\} \tag{4.10}$$

($\sigma=1$ when $K\subset E$). Choose $\delta>0$ sufficiently small such that

$$\rho_1:=\rho_{m_\alpha}(F_\alpha)-\delta>\rho_{m_\alpha-1}(F_\alpha),\quad \rho_1-\delta>1,\quad \text{and}\quad \frac{\sigma+\delta}{\rho_1-\delta}<1. \tag{4.11}$$

First, we approximate $\sum_{\nu=n+|\mathbf{m}_n|-m_{n,\alpha}+1}^{\infty}|b_{\nu,n}^{(\alpha)}||p_\nu(z)|$ on $\overline{D}_\sigma$. Due to the inequality in (4.2) and Lemma 3.4, it follows that for $\nu\geq n+|\mathbf{m}_n|-m_{n,\alpha}+1$,

$$\begin{aligned}|b_{\nu,n}^{(\alpha)}|&=|\langle Q^{F_\alpha}Q^{\mu}_{n,\mathbf{m}_n}F_\alpha-Q^{F_\alpha}P^{\mu}_{n,\mathbf{m}_n,\alpha}\rangle_\nu|=|\langle Q^{F_\alpha}Q^{\mu}_{n,\mathbf{m}_n}F_\alpha\rangle_\nu|\\&=\left|\frac{1}{2\pi i}\int_{\Gamma_{\rho_1}}Q^{F_\alpha}(z)Q^{\mu}_{n,\mathbf{m}_n}(z)F_\alpha(z)s_v(z)dz\right|\leq c_1C_1^{|\mathbf{m}_n|}\|s_\nu\|_{\Gamma_{\rho_1}},\end{aligned} \tag{4.12}$$

where the constant c_1 does not depend on n (from now on, we will denote some constants that do not depend on n by $c_2,c_3,\ldots$). By using (4.7), there exists $n_0\in\mathbb{N}$ such that

$$\|s_\nu\|_{\Gamma_{\rho_1}}\leq\frac{c_2}{(\rho_1-\delta)^\nu},\quad \nu\geq n_0 \tag{4.13}$$

Moreover, from (4.6), it follows from maximum modulus principle that

$$\|p_\nu\|_{\overline{D}_\sigma}\leq c_3(\sigma+\delta)^\nu,\quad \nu\geq 0. \tag{4.14}$$

Therefore, by (4.12), (4.13), and (4.14), for $n>n_0$,

$$\begin{aligned}\sum_{\nu=n+|\mathbf{m}_n|-m_{n,\alpha}+1}^{\infty}|b_{\nu,n}^{(\alpha)}||p_\nu(z)|&\leq\sum_{\nu=n+|\mathbf{m}_n|-m_{n,\alpha}+1}^{\infty}c_4C_1^{|\mathbf{m}_n|}\left(\frac{\sigma+\delta}{\rho_1-\delta}\right)^\nu\\&\leq c_5C_1^{|\mathbf{m}_n|}\left(\frac{\sigma+\delta}{\rho_1-\delta}\right)^n,\quad z\in\overline{D}_\sigma.\end{aligned} \tag{4.15}$$

Next, we approximate $\sum_{\nu=0}^{n+|\mathbf{m}_n|-m_{n,\alpha}}|b_{\nu,n}^{(\alpha)}||p_\nu(z)|$ on $\overline{D}_\sigma$. To approximate $|b_{\nu,n}^{(\alpha)}|$, we need to approximate $|a_{k,n}^{(\alpha)}|$ first. Let $\rho_2\in(1,\rho_0(F_\alpha))$. Using Lemma 3.4

when $G = Q^{\mu}_{n,\mathbf{m}_n} F_{\alpha}$, we have

$$a^{(\alpha)}_{k,n} = \langle Q^{\mu}_{n,\mathbf{m}_n} F_{\alpha}\rangle_k = \frac{1}{2\pi i}\int_{\Gamma_{\rho_2}} Q^{\mu}_{n,\mathbf{m}_n}(z)F_{\alpha}(z)s_k(z)dz.$$

Define

$$\gamma^{(\alpha)}_{k,n} := \frac{1}{2\pi i}\int_{\Gamma_{\rho_1}} Q^{\mu}_{n,\mathbf{m}_n}(z)F_{\alpha}(z)s_k(z)dz.$$

Notice that for each $k \geq 0$, $Q^{\mu}_{n,\mathbf{m}_n} F_{\alpha} s_k$ is meromorphic on $\overline{D}_{\rho_1} \setminus D_{\rho_2}$ and has poles at $\lambda_1, \lambda_2, \ldots, \lambda_q$ with multiplicities at most $\tau_1, \tau_2, \ldots, \tau_q$, respectively. Applying Cauchy's residue theorem, we obtain

$$\gamma^{(\alpha)}_{k,n} - a^{(\alpha)}_{k,n} = \sum_{j=1}^{q} \mathrm{Res}(Q^{\mu}_{n,\mathbf{m}_n} F_{\alpha} s_k, \lambda_j). \tag{4.16}$$

Recall that the limit formula for the residue of $Q^{\mu}_{n,\mathbf{m}_n} F_{\alpha} s_k$ at λ_j is

$$\mathrm{Res}(Q^{\mu}_{n,\mathbf{m}_n} F_{\alpha} s_k, \lambda_j) = \frac{1}{(\tau_j - 1)!}\lim_{z\to\lambda_j}\left((z-\lambda_j)^{\tau_j} Q^{\mu}_{n,\mathbf{m}_n} F_{\alpha} s_k\right)^{(\tau_j-1)}(z).$$

Using Leibniz's rule and the fact that for n sufficiently large $s_n(z) \neq 0$ for $z \in \mathbb{C}\setminus E$ (see Lemma 3.3), we can transform the expression under the limit sign as follows

$$\begin{aligned} &((z-\lambda_j)^{\tau_j} Q^{\mu}_{n,\mathbf{m}_n} F_{\alpha} s_k)^{(\tau_j-1)}(z) \\ &= \sum_{t=0}^{\tau_j-1}\binom{\tau_j-1}{t}((z-\lambda_j)^{\tau_j} F_{\alpha} Q^{\mu}_{n,\mathbf{m}_n} s_n)^{(\tau_j-1-t)}(z)\left(\frac{s_k}{s_n}\right)^{(t)}(z). \end{aligned}$$

For $j = 1, 2, \ldots, q$ and $t = 0, 1, \ldots, \tau_j - 1$, set

$$\beta_n(j,t) := \frac{1}{(\tau_j-1)!}\binom{\tau_j-1}{t}\lim_{z\to\lambda_j}((z-\lambda_j)^{\tau_j} F_{\alpha} Q^{\mu}_{n,\mathbf{m}_n} s_n)^{(\tau_j-1-t)}(z)$$

(notice that $\beta_n(j,t)$ do not depend on k and α). Thus, we can rewrite (4.16) as

$$\gamma^{(\alpha)}_{k,n} - a^{(\alpha)}_{k,n} = \sum_{j=1}^{q}\sum_{t=0}^{\tau_j-1}\beta_n(j,t)\left(\frac{s_k}{s_n}\right)^{(t)}(\lambda_j). \tag{4.17}$$

Since $a^{(\alpha)}_{k,n} = 0$, $k = n - m_{n,\alpha} + 1, n - m_{n,\alpha} + 2, \ldots, n$, it follows from (4.17) and the assumption that $m_{n,\alpha} \geq m_{\alpha}$ (for n sufficiently large),

$$\gamma^{(\alpha)}_{k,n} = \sum_{j=1}^{q}\sum_{t=0}^{\tau_j-1}\beta_n(j,t)\left(\frac{s_k}{s_n}\right)^{(t)}(\lambda_j), \quad k = n - m_{\alpha} + 1, n - m_{\alpha} + 2, \ldots, n. \tag{4.18}$$

Now, we consider (4.18) as a system of m_α linear equations on the m_α unknowns $\beta_n(j,t)$ and the determinant Δ_n corresponding this system is

$$\begin{vmatrix} \left(\frac{s_{n-m_\alpha+1}}{s_n}\right)(\lambda_j) & \cdots & \left(\frac{s_{n-m_\alpha+1}}{s_n}\right)^{(\tau_j-1)}(\lambda_j) \\ \left(\frac{s_{n-m_\alpha+2}}{s_n}\right)(\lambda_j) & \cdots & \left(\frac{s_{n-m_\alpha+2}}{s_n}\right)^{(\tau_j-1)}(\lambda_j) \\ \vdots & \cdots & \vdots \\ 1 & \cdots & 0 \end{vmatrix}_{j=1,2,\ldots,q},$$

where the subindex on the determinant means that the indicated group of columns are successively written for $j = 1, 2, \ldots, q$. Using (4.5), we have

$$\lim_{n\to\infty} \Delta_n = \Delta := \begin{vmatrix} \left(\Phi^{m_\alpha-1}\right)(\lambda_j) & \cdots & \left(\Phi^{m_\alpha-1}\right)^{(\tau_j-1)}(\lambda_j) \\ \left(\Phi^{m_\alpha-2}\right)(\lambda_j) & \cdots & \left(\Phi^{m_\alpha-2}\right)^{(\tau_j-1)}(\lambda_j) \\ \vdots & \cdots & \vdots \\ 1 & \cdots & 0 \end{vmatrix}_{j=1,2,\ldots,q}.$$

Using Lemma 3.7, we can conclude that the determinant $\Delta c_6 \neq 0$.

To avoid long expressions, we define for all $j = 1, 2, \ldots, q$, and $t = 0, 1, \ldots, \tau_j - 1$,

$$h_{j,t} := \left(\sum_{l=0}^{j-1} \tau_l\right) + t + 1,$$

where $\tau_0 = 0$. Applying Cramer's rule to (4.18), we have

$$\beta_n(j,t) = \frac{\Delta_n(j,t)}{\Delta_n} = \frac{1}{\Delta_n}\sum_{y=1}^{m_\alpha} \gamma^{(\alpha)}_{n-m_\alpha+y,n} C_n[y, h_{j,t}],$$

where $\Delta_n(j,t)$ is the determinant obtained from Δ_n by replacing $h_{j,t}^{\text{th}}$ column with the column

$$[\gamma^{(\alpha)}_{n-m_\alpha+1,n}, \gamma^{(\alpha)}_{n-m_\alpha+2,n}, \cdots, \gamma^{(\alpha)}_{n,n}]^T$$

and $C_n[y,h]$ is the determinant of the $(y,h)^{\text{th}}$ cofacter matrix of $\Delta_n(j,t)$. Substituting $\beta_n(j,t)$ in (4.17), we obtain for $k \geq n+1$,

$$\gamma^{(\alpha)}_{k,n} - a^{(\alpha)}_{k,n} = \frac{1}{\Delta_n}\sum_{j=1}^{q}\sum_{t=0}^{\tau_j-1}\sum_{y=1}^{m_\alpha} \gamma^{(\alpha)}_{n-m_\alpha+y,n} C_n[y, h_{j,t}] \left(\frac{s_k}{s_n}\right)^{(t)}(\lambda_j). \tag{4.19}$$

Define

$$\mathbb{B}(\lambda, r) := \{z \in \mathbb{C} : |z - \lambda| < r\}.$$

Let $\varepsilon > 0$ be sufficiently small such that $\mathbb{B}(\lambda_j, \varepsilon) \subset \{z \in \mathbb{C} : |\Phi(z)| > \rho_2\}$ for all $j = 1, 2, \ldots, q$ and $\overline{\mathbb{B}(\lambda_j, \varepsilon)} \cap \overline{\mathbb{B}(\lambda_k, \varepsilon)} = \emptyset$ for all $k \neq j$. Using Cauchy's integral formula, we obtain

$$\left(\frac{s_k}{s_n}\right)^{(\ell)}(\lambda_j) = \frac{\ell!}{2\pi i} \int_{|z-\lambda_j|=\varepsilon} \frac{s_k(z)}{s_n(z)(z-\lambda_j)^{\ell+1}} dz. \tag{4.20}$$

Applying (4.5) on (4.20), there exists a constant c_7 such that for sufficiently large n,

$$\left|\left(\frac{s_k}{s_n}\right)^{(\ell)}(\lambda_j)\right| \leq \frac{c_7}{\rho_2^{k-n}}, \quad j = 1, 2, \ldots, q, \quad \ell = 0, 1, \ldots, \tau_j - 1, \quad k \geq n+1. \tag{4.21}$$

Moreover, by using Cauchy's integral formula as before, there exists a constant c_8 such that for all $k = n - m_\alpha + 1, n - m_\alpha + 2, \ldots, n$, $j = 1, 2, \ldots, q$, and $\ell = 0, 1, \ldots, \tau_j - 1$,

$$\left|\left(\frac{s_k}{s_n}\right)^{(\ell)}(\lambda_j)\right| \leq c_8 \tag{4.22}$$

for all sufficiently large n. From (4.22), we have

$$|C_n(g,h)| \leq c_9, \quad g, h = 1, 2, \ldots, m_\alpha. \tag{4.23}$$

Using (4.21), (4.22), (4.23), and $|\Delta| = |c_6| > 0$, it follows from (4.19) that

$$|a_{k,n}^{(\alpha)}| \leq |\gamma_{k,n}^{(\alpha)}| + \frac{c_{10}}{\rho_2^{k-n}} \sum_{y=1}^{m_\alpha} |\gamma_{n-m_\alpha+y,n}^{(\alpha)}|, \quad k \geq n+1. \tag{4.24}$$

By the definition of $\gamma_{k,n}^{(\alpha)}$, for all sufficiently large n, we have

$$|\gamma_{k,n}^{(\alpha)}| \leq \frac{c_{11} C_1^{|\mathbf{m}_n|}}{(\rho_1 - \delta)^k}.$$

This implies that

$$|a_{k,n}^{(\alpha)}| \leq \frac{c_{12} C_1^{|\mathbf{m}_n|}}{\rho_2^{k-n}(\rho_1 - \delta)^n}, \quad k \geq n+1. \tag{4.25}$$

Recall that $b_{\nu,n}^{(\alpha)} = \sum_{k=n+1}^{\infty} a_{k,n}^{(\alpha)} \langle Q^{F_\alpha} p_k \rangle_\nu$. By Cauchy-Schwarz inequality and the orthonormality of $\{p_k\}_{k=0}^{\infty}$, we have for all $\nu \geq 0$ and $k \geq 0$,

$$|\langle Q^{F_\alpha} p_k \rangle_\nu| = |\langle Q^{F_\alpha} p_k, p_\nu \rangle_\mu| \leq \left\|Q^{F_\alpha}\right\|_E \langle p_k, p_k \rangle_\mu^{1/2} \langle p_\nu, p_\nu \rangle_\mu^{1/2} \leq \left\|Q^{F_\alpha}\right\|_E \leq c_{13}. \tag{4.26}$$

Then,

$$|b_{\nu,n}^{(\alpha)}| \leq \sum_{k=n+1}^{\infty} |a_{k,n}^{(\alpha)}| |\langle Q^{F_\alpha} p_k \rangle_\nu| \leq \frac{c_{14} C_1^{|\mathbf{m}_n|}}{(\rho_1 - \delta)^n}.$$

Combining the above inequality and (4.14), for sufficiently large n,

$$\sum_{\nu=0}^{n+|\mathbf{m}_n|-m_{n,\alpha}} |b_{\nu,n}^{(\alpha)}||p_\nu(z)| \leq \sum_{\nu=0}^{n+|\mathbf{m}_n|-m_{n,\alpha}} c_{15} C_1^{|\mathbf{m}_n|} \frac{(\sigma+\delta)^\nu}{(\rho_1-\delta)^n}$$

$$\leq c_{16}(n+|\mathbf{m}_n|-m_{n,\alpha}+1)\tilde{C}_1^{|\mathbf{m}_n|}\left(\frac{\sigma+\delta}{\rho_1-\delta}\right)^n \tag{4.27}$$

where $\tilde{C}_1 := C_1(\sigma+\delta)$ and $z \in \overline{D}_\sigma$.

Combining (4.15) and (4.27), it follows from (4.9) that for sufficiently large n,

$$|Q^{F_\alpha}(z)Q_{n,\mathbf{m}_n}^\mu(z)F_\alpha(z) - P_{n,\mathbf{m}_n,\alpha}^\mu(z)| \leq c_{17}\tilde{C}_1^{|\mathbf{m}_n|}\theta^n, \qquad z \in \overline{D}_\sigma, \tag{4.28}$$

where θ is an arbitrary constant which satisfies

$$\frac{\sigma+\delta}{\rho_1-\delta} < \theta < 1.$$

Let $\beta > 0$ and $\varepsilon > 0$ be fixed. By the definition of $J_\varepsilon^\beta(F_\alpha, m_\alpha; k)$ and (4.3), the inequality (4.28) implies that for sufficiently large n,

$$|F_\alpha(z) - R_{n,\mathbf{m}_n,\alpha}^\mu(z)| \leq \frac{c_{18}\tilde{C}_1^{|\mathbf{m}_n|}\theta^n}{|Q_{m_\alpha}^{\mathbf{F}}(z)Q_{n,\mathbf{m}_n}^\mu(z)|}$$

$$\leq c_{18}\tilde{C}_1^{|\mathbf{m}_n|}\theta^n\left(\frac{6m_\alpha}{\varepsilon}\right)^{m_\alpha/\beta}(C_2|\mathbf{m}_n|n^2)^{2|\mathbf{m}_n|/\beta},$$

for all $z \in \overline{D}_\sigma \setminus J_\varepsilon^\beta(F_\alpha, m_\alpha; k)$ and k sufficiently large. Then, for sufficiently large n and k,

$$\left\|F_\alpha - R_{n,\mathbf{m}_n,\alpha}^\mu\right\|_{\overline{D}_\sigma\setminus J_\varepsilon^\beta(F_\alpha,m_\alpha;k)}^{1/n}$$

$$\leq \left(c_{18}\left(\frac{6m_\alpha}{\varepsilon}\right)^{m_\alpha/\beta}\right)^{1/n}\theta(\tilde{C}_1^{1/2}C_2^{1/\beta}|\mathbf{m}_n|^{1/\beta}(n)^{2/\beta})^{2|\mathbf{m}_n|/n}$$

$$\leq c_{18}^{1/n}\theta e^{(c_{19}+\frac{3}{\beta}\log(n))(2|\mathbf{m}_n|/n)},$$

which implies that

$$\limsup_{n\to\infty}\left\|F_\alpha - R_{n,\mathbf{m}_n,\alpha}^\mu\right\|_{\overline{D}_\sigma\setminus J_\varepsilon^\beta(F_\alpha,m_\alpha;k)}^{1/n} \leq \theta,$$

for sufficiently large k.

Letting $\delta \to 0$ and $\rho_1 \to \rho_{m_\alpha}(F_\alpha)$, we have

$$\frac{\sigma}{\rho_{m_\alpha}(F_\alpha)} < \theta < 1.$$

Since θ is arbitrary, we let $\theta \to \sigma/\rho_{m_\alpha}(F_\alpha)$, for sufficiently large k,

$$\begin{aligned}\limsup_{n\to\infty}\left\|F_\alpha - R^{\mu}_{n,\mathbf{m}_n,\alpha}\right\|^{1/n}_{K(\varepsilon;k)} &\le \limsup_{n\to\infty}\left\|F_\alpha - R^{\mu}_{n,\mathbf{m}_n,\alpha}\right\|^{1/n}_{\overline{D}_\sigma(\varepsilon;k)} \\ = \limsup_{n\to\infty}\left\|F_\alpha - R^{\mu}_{n,\mathbf{m}_n,\alpha}\right\|^{1/n}_{\overline{D}_\sigma\setminus J^{\beta}_{\varepsilon}(F_\alpha,m_\alpha;k)} &\le \frac{\sigma}{\rho_{m_\alpha}(F_\alpha)} < 1. \end{aligned} \tag{4.29}$$

This implies that for any $\beta > 0$ and $\alpha = 1, 2, \ldots, d$, each sequence $\{R^{\mu}_{n,\mathbf{m}_n,\alpha}\}$ converges in β-dimentional Hausdorff content to F_α inside $D_{\rho_{m_\alpha}(F_\alpha)}$, as $n \to \infty$. □

Proof of Corollary 2.2. By the assumption of Corollary 2.2, for each $\alpha = 1, 2, \ldots, d$, $m_{n,\alpha} = \nu(F_\alpha, D_\rho)$. Then, the conditions in Theorem 2.1 are obtained. By Theorem 2.1, we get $h_1 - \lim_{n\to\infty} R^{\mu}_{n,\mathbf{m}_n,\alpha} = F_\alpha$ in $D_{\rho_{m_\alpha}(F_\alpha)}$. Since $\deg Q^{\mu}_{n,\mathbf{m}} \le |\mathbf{m}|$, by applying Lemma 3.8, the inequality must becomes equality and each pole of F_α attracts as many zeros of $Q^{\mu}_{n,\mathbf{m}}$ as its order otherwise a contradiction on multiplicities of poles must occur. Therefore, for all sufficiently large n, $Q^{\mu}_{n,\mathbf{m}}$ is unique by Lemma 3.6. This implies that for such n, $\mathbf{R}^{\mu}_{n,\mathbf{m}}$ is unique.

Let $K \subset D_{\rho_{m_\alpha}(F_\alpha)}\setminus\{\lambda_{\alpha,1}, \lambda_{\alpha,2}, \ldots, \lambda_{\alpha,m_\alpha}\}$ be a compact set and $\sigma := \max\{\|\Phi\|_K, 1\}$. Since all points $\lambda_{\alpha,1}, \lambda_{\alpha,2}, \ldots, \lambda_{\alpha,m_\alpha}$ attract all zeros of $Q^{\mu}_{n,\mathbf{m}}$, for sufficiently small $\varepsilon > 0$ and for sufficiently large k,

$$K \in \overline{D}_\sigma \setminus J^{\beta}_{\varepsilon}(F_\alpha, m_\alpha; k).$$

By the inequality (4.29),

$$\limsup_{n\to\infty}\left\|F_\alpha - R^{\mu}_{n,\mathbf{m},\alpha}\right\|^{1/n}_{K} \le \limsup_{n\to\infty}\left\|F_\alpha - R^{\mu}_{n,\mathbf{m},\alpha}\right\|^{1/n}_{\overline{D}_\sigma\setminus J^{\beta}_{\varepsilon}(F_\alpha,m_\alpha;k)} \le \frac{\sigma}{\rho_{m_\alpha}(F_\alpha)}. \tag{4.30}$$

This implies that the sequence $\{R^{\mu}_{n,\mathbf{m},\alpha}\}$ converges uniformly to F_α on each compact subset of $D_{\rho_{m_\alpha}(F_\alpha)} \setminus \{\lambda_1, \lambda_2, \ldots, \lambda_q\}$ as $n \to \infty$. □

Proof of Corollary 2.3. Let $\alpha \in \{1, 2, \ldots, d\}$ be fixed and let K be a compact subset of $D_{\rho_\infty(F_\alpha)}$. Let $\varepsilon > 0$ and $\beta > 0$ be fixed. Since K is compact, $K \subset D_{\rho_t(F_\alpha)}$ for some $t \in \mathbb{N}_0$. By the assumption on $m_{n,\alpha}$, it is clear that $\lim_{n\to\infty} m_{n,\alpha} \ge \nu(F_\alpha, D_{\rho_t(F_\alpha)})$. Applying Theorem 2.1, we have $h_\beta - \lim_{n\to\infty} R^{\mu}_{n,\mathbf{m}_n,\alpha} = F_\alpha$ in $D_{\rho_t(F_\alpha)}$. Thus,

$$\lim_{n\to\infty} h_\beta\{z \in K : |R^{\mu}_{n,\mathbf{m}_n,\alpha}(z) - F_\alpha(z)| > \varepsilon\} = 0.$$

□

Proof of Theorem 2.4. Let $Q^{E}_{n,\mathbf{m}_n}$ be the polynomial $q^{E}_{n,\mathbf{m}_n}$ normalized as in (4.1). And we have for all $\alpha = 1, 2, \ldots, d$,

$$R^{E}_{n,\mathbf{m}_n,\alpha} = \frac{p^{E}_{n,\mathbf{m}_n,\alpha}}{q^{E}_{n,\mathbf{m}_n}} = \frac{P^{E}_{n,\mathbf{m}_n,\alpha}}{Q^{E}_{n,\mathbf{m}_n}}.$$

Let $\alpha \in \{1, 2, \dots, d\}$ be fixed. Denote by $m_\alpha := \nu(F_\alpha, D_\rho)$ the number of poles of F_α in D_ρ. Note that the notations $J_\varepsilon^\beta(F_\alpha, m_\alpha; k)$ and $B(\varepsilon; k)$ are defined as in the proof of Theorem 2.1 by replacing $Q^\mu_{n,\mathbf{m}_n}$ with $Q^E_{n,\mathbf{m}_n}$. Then, for any compact subset $K \subset D_{\rho_{m_\alpha}(F_\alpha)}$, for any $\varepsilon > 0$, and $k > 0$, there exist positive constants $C_1 > 0$ and $C_2 > 0$ independent of n such that for all sufficiently large n,

$$\left\| Q^E_{n,\mathbf{m}_n} \right\|_K \leq C_1^{|\mathbf{m}_n|}, \tag{4.31}$$

and

$$\min_{z \in K \setminus J_\varepsilon^\beta(F_\alpha, m_\alpha; k)} |Q^E_{n,\mathbf{m}_n}(z)| \geq (C_2 |\mathbf{m}_n| (n)^2)^{-2|\mathbf{m}_n|/\beta}. \tag{4.32}$$

Define

$$Q^{F_\alpha}(z) := \prod_{j=1}^{q} \left(z - \lambda_j\right)^{\tau_j},$$

where $\lambda_1, \lambda_2, \dots, \lambda_q$ are distinct poles of F_α in D_ρ and $\tau_1, \tau_2, \dots, \tau_q$ are their multiplicities, respectively. From the definition of simultaneous Padé-Faber approximants and Lemma 3.2, we have

$$Q^E_{n,\mathbf{m}_n}(z) F_\alpha(z) - P^E_{n,\mathbf{m}_n,\alpha}(z) = \sum_{k=n+1}^{\infty} a^{(\alpha)}_{k,n} \Phi_k(z), \quad z \in D_{\rho_0(F_\alpha)}, \tag{4.33}$$

where

$$a^{(\alpha)}_{k,n} := [Q^E_{n,\mathbf{m}_n} F_\alpha]_k, \quad k = 0, 1, 2, \dots,$$

and $a^{(\alpha)}_{k,n} = 0$, for all $k = n - m_{n,\alpha} + 1, n - m_{n,\alpha} + 2, \dots, n$. Multiplying (4.33) by $Q^{F_\alpha}(z)$ and expanding the result in terms of the Faber polynomial system $\{\Phi_\nu\}_{\nu=0}^\infty$ such that for $z \in D_{\rho_{m_\alpha}(F_\alpha)}$,

$$Q^{F_\alpha}(z) Q^E_{n,\mathbf{m}_n}(z) F_\alpha(z) - Q^{F_\alpha}(z) P^E_{n,\mathbf{m}_n,\alpha}(z) = \sum_{k=n+1}^{\infty} Q^{F_\alpha}(z) a^{(\alpha)}_{k,n} \Phi_k(z)$$

$$= \sum_{\nu=0}^{\infty} b^{(\alpha)}_{\nu,n} \Phi_\nu(z) = \sum_{\nu=0}^{n+|\mathbf{m}_n|-m_{n,\alpha}} b^{(\alpha)}_{\nu,n} \Phi_\nu(z) + \sum_{\nu=n+|\mathbf{m}_n|-m_{n,\alpha}+1}^{\infty} b^{(\alpha)}_{\nu,n} \Phi_\nu(z). \tag{4.34}$$

Let K be a compact subset of $D_{\rho_{m_\alpha}(F_\alpha)}$ and set

$$\sigma := \max\{\|\Phi\|_K, 1\} \tag{4.35}$$

($\sigma = 1$ when $K \subset E$). Choose $\delta > 0$ sufficiently small such that

$$\rho_1 := \rho_{m_\alpha}(F_\alpha) - \delta > \rho_{m_\alpha - 1}(F_\alpha), \quad \text{and} \quad \frac{\sigma + \delta}{\rho_1 - \delta} < 1. \tag{4.36}$$

First, we approximate $\sum_{\nu=n+|\mathbf{m}_n|-m_{n,\alpha}+1}^{\infty}|b_{\nu,n}^{(\alpha)}||\Phi_\nu(z)|$ on $\overline{D}_\sigma$. With the similar computation as (4.12). From (3.3) and (4.31), it follows that for $\nu \geq n + |\mathbf{m}_n| - m_{n,\alpha}+1$,

$$\begin{aligned} |b_{\nu,n}^{(\alpha)}| &= |[Q^{F_\alpha}Q_{n,\mathbf{m}_n}^E F_\alpha - Q^{F_\alpha}P_{n,\mathbf{m}_n,\alpha}^E]_\nu| = |[Q^{F_\alpha}Q_{n,\mathbf{m}_n}^E F_\alpha]_\nu| \\ &= \left|\frac{1}{2\pi i}\int_{\Gamma_{\rho_1}}\frac{Q^{F_\alpha}(z)Q_{n,\mathbf{m}_n}^E(z)F_\alpha(z)\Phi'(z)}{\Phi^{\nu+1}(z)}dz\right| \leq \frac{c_1C_1^{|\mathbf{m}_n|}}{\rho_1^\nu}, \end{aligned} \tag{4.37}$$

Therefore, by (3.4) and (4.37), we have for all $z\in\overline{D}_\sigma$,

$$\begin{aligned} &\sum_{\nu=n+|\mathbf{m}_n|-m_{n,\alpha}+1}^{\infty}|b_{\nu,n}^{(\alpha)}||\Phi_\nu(z)| \\ &\leq \sum_{\nu=n+|\mathbf{m}_n|-m_{n,\alpha}+1}^{\infty} c_2C_1^{|\mathbf{m}_n|}\left(\frac{\sigma}{\rho_1}\right)^\nu \leq c_3C_1^{|\mathbf{m}_n|}\left(\frac{\sigma}{\rho_1}\right)^n. \end{aligned} \tag{4.38}$$

Next, we approximate $\sum_{\nu=0}^{n+|\mathbf{m}_n|-m_{n,\alpha}}|b_{\nu,n}^{(\alpha)}||\Phi_\nu(z)|$ on $\overline{D}_\sigma$. Again, we begin by approximating $|a_{k,n}^{(\alpha)}|$. Choose $\rho_2\in(1,\rho_0(F_\alpha))$, we have

$$a_{k,n}^{(\alpha)} = [Q_{n,\mathbf{m}_n}^E F_\alpha]_k = \frac{1}{2\pi i}\int_{\Gamma_{\rho_2}}\frac{Q_{n,\mathbf{m}_n}^E(z)F_\alpha(z)\Phi'(z)}{\Phi^{k+1}(z)}dz.$$

Define

$$\gamma_{k,n}^{(\alpha)} := \frac{1}{2\pi i}\int_{\Gamma_{\rho_1}}\frac{Q_{n,\mathbf{m}_n}^E(z)F_\alpha(z)\Phi'(z)}{\Phi^{k+1}(z)}dz.$$

Arguing as (4.16), we have

$$\gamma_{k,n}^{(\alpha)} - a_{k,n}^{(\alpha)} = \sum_{j=1}^{q}\mathrm{Res}(Q_{n,\mathbf{m}_n}^E F_\alpha\Phi'/\Phi^{k+1},\lambda_j). \tag{4.39}$$

Now, we use Leibniz's formula to rewrite (4.39) in the same way as (4.17) such that

$$\gamma_{k,n}^{(\alpha)} - a_{k,n}^{(\alpha)} = \sum_{j=1}^{q}\sum_{t=0}^{\tau_j-1}\beta_n(j,t)\left(\Phi^{n-k}\right)^{(t)}(\lambda_j), \tag{4.40}$$

where

$$\beta_n(j,t) := \frac{1}{(\tau_j-1)!}\binom{\tau_j-1}{t}\lim_{z\to\lambda_j}\left((z-\lambda_j)^{\tau_j}F_\alpha Q_{n,\mathbf{m}_n}^E\frac{\Phi'}{\Phi^{n+1}}\right)^{(\tau_j-1-t)}(z),$$

for $j=1,2,\ldots,q$, and $t=0,1,\ldots,\tau_j-1$. Since $a_{k,n}^{(\alpha)}=0$ for $k=n-m_{n,\alpha}+1, n-m_{n,\alpha}+2,\ldots,n$ and the assumption that $m_{n,\alpha}\geq m_\alpha$ (for n sufficiently large), we have

$$\gamma_{k,n}^{(\alpha)} = \sum_{j=1}^{q}\sum_{t=0}^{\tau_j-1}\beta_n(j,t)\left(\Phi^{n-k}\right)^{(t)}(\lambda_j),\quad k=n-m_\alpha+1,n-m_\alpha+2,\ldots,n. \tag{4.41}$$

Next, we use the same technique in the proof of Theorem (2.1) to find $\beta_n(j,t)$ by replacing s_k/s_n with Φ^{n-k} in (4.18). Consider (4.41) as a system of m_α equations on the m_α unknowns $\beta_n(j,t)$ and the determinant Δ corresponding to this system is

$$\Delta := \left.\begin{vmatrix} \left(\Phi^{m_\alpha-1}\right)(\lambda_j) & \cdots & \left(\Phi^{m_\alpha-1}\right)^{(\tau_j-1)}(\lambda_j) \\ \left(\Phi^{m_\alpha-2}\right)(\lambda_j) & \cdots & \left(\Phi^{m_\alpha-2}\right)^{(\tau_j-1)}(\lambda_j) \\ \vdots & \cdots & \vdots \\ 1 & \cdots & 0 \end{vmatrix}\right|_{j=1,2,\ldots,q}.$$

It follows that for $k \geq n+1$,

$$\gamma_{k,n}^{(\alpha)} - a_{k,n}^{(\alpha)} = \frac{1}{\Delta}\sum_{j=1}^{q}\sum_{t=0}^{\tau_j-1}\sum_{y=1}^{m_\alpha}\gamma_{n-m_\alpha+y,n}^{(\alpha)} C[y,h_{j,t}]\left(\Phi^{n-k}\right)^{(t)}(\lambda_j), \tag{4.42}$$

where $C[y,h]$ is the determinant of the $(y,h)^{\text{th}}$ cofacter matrix of Δ. Arguing as (4.20-4.24) by replacing s_k/s_n with Φ^{n-k}, for sufficiently large n, we have

$$|a_{k,n}^{(\alpha)}| \leq |\gamma_{k,n}^{(\alpha)}| + \frac{c_4}{\rho_2^{k-n}}\sum_{y=1}^{m_\alpha}|\gamma_{n-m_\alpha+y,n}^{(\alpha)}|, \quad k \geq n+1. \tag{4.43}$$

By the definition of $\gamma_{k,n}^{(\alpha)}$, for all sufficiently large n, we have for $k \geq n+1$,

$$|\gamma_{k,n}^{(\alpha)}| \leq \frac{c_5 C_1^{|\mathbf{m}_n|}}{\rho_1^k}.$$

This implies that

$$|a_{k,n}^{(\alpha)}| \leq \frac{c_6 C_1^{|\mathbf{m}_n|}}{\rho_2^{k-n}\rho_1^n}, \quad k \geq n+1. \tag{4.44}$$

Now, we estimate $|[Q^{F_\alpha}\Phi_k]_\nu|$. Suppose that $\delta > 0$ is sufficiently small such that $\rho_2 - \delta > 1$. Then,

$$|[Q^{F_\alpha}\Phi_k]_\nu| = \left|\frac{1}{2\pi i}\int_{\Gamma_{\rho_2-\delta}}\frac{Q^{F_\alpha}(z)\Phi_k(z)\Phi'(z)}{\Phi^{\nu+1}(z)}dz\right| \leq c_7\frac{(\rho_2-\delta)^k}{(\rho_2-\delta)^\nu}. \tag{4.45}$$

Consequently, we get

$$|b_{\nu,n}^{(\alpha)}| \leq \sum_{k=n+1}^{\infty}|a_{k,n}^{(\alpha)}||[Q^{F_\alpha}\Phi_k]_\nu|$$

$$\leq \frac{c_8 C_1^{|\mathbf{m}_n|}}{(\rho_2-\delta)^\nu}\left(\frac{\rho_2}{\rho_1}\right)^n\sum_{k=n+1}^{\infty}\left(\frac{\rho_2-\delta}{\rho_2}\right)^k \leq \frac{c_9 C_1^{|\mathbf{m}_n|}}{(\rho_2-\delta)^\nu}\left(\frac{\rho_2-\delta}{\rho_1}\right)^n.$$

Therefore, sufficiently large n,

$$\sum_{\nu=0}^{n+|\mathbf{m}_n|-m_{n,\alpha}} |b_{\nu,n}^{(\alpha)}||\Phi_\nu(z)| \leq c_{10} C_1^{|\mathbf{m}_n|} \left(\frac{\rho_2-\delta}{\rho_1}\right)^n \sum_{\nu=0}^{n+|\mathbf{m}_n|-m_{n,\alpha}} \left(\frac{\sigma}{\rho_2-\delta}\right)^\nu$$

$$\leq c_{10} C_1^{|\mathbf{m}_n|} \left(\frac{\rho_2-\delta}{\rho_1}\right)^n \sum_{\nu=0}^{n+|\mathbf{m}_n|-m_{n,\alpha}} \sigma^\nu$$

$$\leq c_{11}(n+|\mathbf{m}_n|-m_{n,\alpha}+1)\tilde{C}_1^{|\mathbf{m}_n|}(\rho_2-\delta)^n \left(\frac{\sigma}{\rho_1}\right)^n, \tag{4.46}$$

where $z \in \overline{D}_\sigma$ and $\tilde{C}_1 := \sigma C_1$.

Combining (4.38) and (4.46), it follows from (4.34) that for each $k \geq n_2$,

$$|Q^{F_\alpha}(z)Q_{n,\mathbf{m}_n}^E(z)F_\alpha(z) - P_{n,\mathbf{m}_n,\alpha}^E(z)|$$

$$\leq c_{12}(n+|\mathbf{m}_n|-m_{n,\alpha}+1)\tilde{C}_1^{|\mathbf{m}_n|}(\rho_2-\delta)^n \left(\frac{\sigma}{\rho_1}\right)^n, \qquad z \in \overline{D}_\sigma, \quad n \geq k. \tag{4.47}$$

Using (4.32), we arrive for each $k \geq n_2$,

$$\limsup_{n\to\infty} \left\| F_\alpha - R_{n,\mathbf{m}_n,\alpha}^E \right\|_{\overline{D}_\sigma \setminus J_\varepsilon^\beta(F_\alpha, m_\alpha; k)}^{1/n} \leq (\rho_2-\delta)\left(\frac{\sigma}{\rho_1}\right). \tag{4.48}$$

Letting $\delta \to 0$, $\rho_1 \to \rho_{m_\alpha}(F_\alpha)$, and $\rho_2 \to 1$, we obtain that for for each $k \geq n_2$,

$$\limsup_{n\to\infty} \left\| F_\alpha - R_{n,\mathbf{m}_n,\alpha}^\mu \right\|_{K(\varepsilon;k)}^{1/n} \leq \limsup_{n\to\infty} \left\| F_\alpha - R_{n,\mathbf{m}_n,\alpha}^E \right\|_{\overline{D}_\sigma \setminus J_\varepsilon^\beta(F_\alpha, m_\alpha; k)}^{1/n} \leq \frac{\sigma}{\rho_{m_\alpha}(F_\alpha)}$$

(comparing this with (4.29)). This implies that for any $\beta > 0$, each sequence $\{R_{n,\mathbf{m}_n,\alpha}^E\}_{n\in\mathbb{N}}$ converges in β-dimentional Hausdorff content to F_α inside $D_{\rho_{m_\alpha}(F_\alpha)}$, as $n \to \infty$. □

Arguing as the proofs of Corollary 2.2 and 2.3 by replacing $R_{n,\mathbf{m}_n,\alpha}^\mu$ with $R_{n,\mathbf{m}_n,\alpha}^E$, we are obtain Corollary 2.5 and 2.6.

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