



Some Problems Connected with Diametrically Contractive Maps and Fixed Point Theory With Python Programming Language

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Abstract : In this paper we introduce nice properties of diametrically contractive maps with fixed point. We have given some counter examples to explain this idea and some interesting problems remain open. We have also discussed a concept of fixed point iteration used in Python programming language.

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1. Introduction

Contractive maps have nice properties concerning fixed point, and one can find volumes of literature devoted to fixed points of non-expansive maps. The class of shrinking i.e. strictly contractive maps is not studied largely. We have few specific results on these maps which are not applicable to all non expansive maps. In this paper we discuss shrinking maps diametrically contractive maps is also discussed in connection with shrinking maps.

Definition I : Let X be a Banach space and K a nonempty convex closed bounded subset of X . The class of contractive maps and the class of non expansive maps of $F : K \rightarrow K$ are defined as

$$\|F_x - F_y\| \leq \alpha \|x - y\|, \quad \forall x, y \in K \quad \text{and} \quad 0 < \alpha < 1 \quad (1.1)$$

and

$$\|F_x - F_y\| \leq \|x - y\|, \quad \forall x, y \in K \quad (1.2)$$

We define the class 'S' consists of the maps satisfying the following condition

$$\|F_x - F_y\| < \|x - y\| \quad \forall x \neq y, \quad x, y \in K. \quad (1.3)$$

These maps we will call them shrinking, see [1]. Some important results of maps $F : K \rightarrow K$ in the class 'S' are

1. The fixed point is unique, if it exists.
2. There is a map on the unit ball of Hilbert spaces with fixed point x_1 such that $F^n x$ does not converge to the fixed point for any $x(x \neq x_1)$. (see [1]).

3. If K is a compact set, then F has a fixed point x_1 .

Definition II : F is a diametrically contractive if $\delta(F(A)) \leq \delta(A)$ for every closed, convex, bounded non singleton subset A of K , where $\delta(A)$ denotes diameter of A .

Remark : 1. Diametrically contractive maps are shrinking.

2. If K is a compact set and F is shrinking, then it is diametrically contractive.

2. Open Problem (See [2])

2.1 Theorem : Let K be weakly compact subset of a Banach space X and $F : K \rightarrow K$ be diametrically contractive, then F has a fixed point.

The proof of the above theorem can be found in [3].

The following problems are open (see [2]).

1. Can we substitute weakly compact subset with closed convex bounded one in Theorem 1?
2. If F is diametrically contractive and x_1 is the fixed point of F , does $F^n x \rightarrow x_1$ for all (or atleast for some) $x \in K$?

In this paper we give two counter examples to ascertain Problem 1 and 2.

2.2 First Example : The following is an example of a fixed point free diametrically contractive self-map of a closed convex bounded set.

Consider the vector space X of all continuous real functions on the closed unit interval, with the norm

$$\|f\| = \|f\|_\infty + \|f\|_1 = \max_{0 \leq x \leq 1} |f(x)| + \int_0^1 |f(x)| dx. \quad (2.1)$$

Let $K = \{f \in X : f(0) = 0, f(1) = 1; 0 \leq f(x) \leq x; f \text{ is monotone nondecreasing}\}$
Define $F : K \rightarrow K$ as

$$Ff(x) = \begin{cases} 0 & 0 \leq x \leq 3/4 \\ (4x-3)f(4x-3) & 3/4 \leq x \leq 1 \end{cases} \quad (2.2)$$

Result 1 : The map F is fixed point free.

Proof : Assume that $f \in K$ is such that $Ff = f$. Notice, $f(x) = 0$ for $x \in [0, 3/4]$. If $x \in [3/4, 1]$ then $(4x-3)f(4x-3) = f(x)$ implies $f(x) = 0$ for every $x \in [0, 15/16] = [0, 1 - 1/4^2]$.

By repeatedly iterating the same reasoning, we can prove that $f(x) = 0$ for all $x \in [0, 1 - 1/4^n]$ where $n \in \mathbb{N}$. Thus we land up with a function f identically zero on $[0, 1)$ and $f(1) = 1$ which is discontinuous, this is a contradiction and hence proved.

Result 2 : The map F is diametrically contractive.

Proof : Consider a closed set T of K such that $\delta(T) > 0$. For two suitable subsequences f_n and g_n

$$\begin{aligned} \delta(F(T)) &= \lim_{n \rightarrow \infty} \|Ff_n - Fg_n\| = \lim_{n \rightarrow \infty} (\|Ff_n - Fg_n\|_\infty + \|Ff_n - Fg_n\|_1) \\ &\leq \lim_{n \rightarrow \infty} (\|f_n - g_n\|_\infty + \frac{3}{4} \|f_n - g_n\|_1) \\ &\leq \lim_{n \rightarrow \infty} \|f_n - g_n\| \leq \delta(A). \end{aligned} \quad (2.3)$$

Now we show that $\delta(F(T)) \neq \delta(T)$.

We prove by contradiction.

Let $\delta(F(T)) = \delta(T)$, then

$$\lim_{n \rightarrow \infty} \|f_n - g_n\|_1 = \lim_{n \rightarrow \infty} \|Ff_n - Fg_n\|_1 = 0$$

and

$$\lim_{n \rightarrow \infty} \|f_n - g_n\|_\infty = \lim_{n \rightarrow \infty} \|Ff_n - Fg_n\|_\infty = \delta(F(T)) = \delta(T). \tag{2.4}$$

Let us choose a sequence $\langle x_n \rangle$ such that $\|Ff_n - Fg_n\|_\infty = x_n|f_n(x_n) - g_n(x_n)|$. By considering eventually a subsequence, we may assume that $x_n \rightarrow x_0$ where $0 \leq x_0 \leq 1$. Then

$$\delta(T) = \lim_{n \rightarrow \infty} x_n|f_n(x_n) - g_n(x_n)| \leq \lim_{n \rightarrow \infty} x_n\|f_n - g_n\|_\infty = x_0\delta(T). \tag{2.5}$$

Thus $x_0 = 1$.

By considering subsequences, and by exchanging the sequences, we may assume that

$$f_n(x_n) \rightarrow \ell \text{ and } g_n(x_n) \rightarrow m; \quad m \leq \ell \leq 1.$$

Therefore (2.5) implies that

$$\ell - m = \delta(T)$$

Therefore, $f_n(x_n) \rightarrow \ell$, $g_n(x_n) \rightarrow \ell - \delta(T)$.

Take any $f \in T$; we have since $\lim_{n \rightarrow \infty} x_n = 1$

$$\delta(T) \geq |f(x_n) - g_n(x_n)| \rightarrow |1 - \ell + \delta(T)| \geq \delta(T).$$

This forces $\ell = 1$; $\lim_{n \rightarrow \infty} |f(x_n) - g_n(x_n)| = \delta(T)$ for every $f \in T$, and then

$$\lim_{n \rightarrow \infty} \|f - g_n\| = \delta(T). \tag{2.6}$$

Take $\epsilon \in (0, \delta(T))$, and there is a η such that for every $x \in [1 - \eta, 1]$ we have $1 + \epsilon \leq f(x) \leq 1$. For η large, $x_n > 1 - \eta$; and using monotonically assumption for these functions, we have a suitable points P_n

$$\begin{aligned} \int_0^1 |f(x) - g_n(x)|dx &\geq \int_{1-\alpha}^{x_n} |f(x) - g_n(x)|dx \\ &= (x_n - 1 + \eta)|f(c_n) - g_n(c_n)| \\ &\geq (x_n - 1 + \eta)(1 - \epsilon - g_n(x_n)) \\ &= \eta(\delta(T) - \epsilon). \end{aligned}$$

Hence, we obtain

$$\lim_{n \rightarrow \infty} \inf \|f - g\|_1 \geq \eta(\delta(T) - \epsilon) \tag{2.7}$$

which implies

$$\begin{aligned} \liminf_{n \rightarrow \infty} \|f - g_n\| &\geq \lim_{n \rightarrow \infty} \|f - g_n\|_\infty + \liminf_{n \rightarrow \infty} \|f - g_n\|_1 \\ &\geq \delta(T) + \eta(\delta(T) - \epsilon). \end{aligned}$$

This is a contradiction, therefore $\delta(F(A)) \neq \delta(T)$ and the map is diametrically contractive, and hence the proof.

3. The Map T is Shrinking.

Proof : Let f and $g \in K$ and $f \neq g$. Then

$$\begin{aligned} \|Ff - Fg\| &= \max_{0 \leq x \leq 1} |Ff(x) - Fg(x)| + \int_0^1 |Ff(x) - Fg(x)| dx \\ &= \max_{\frac{3}{4} \leq x \leq 1} (4x - 3)|f(4x - 3) - g(4x - 3)| \\ &\quad + \int_{3/4}^1 (4x - 3)|f(4x - 3) - g(4x - 3)| dx \\ &= \max_{0 \leq x \leq 1} |x(f(x) - g(x))| + \frac{3}{4} \int_0^1 x|f(x) - g(x)| dx \\ &\leq \|f - g\|_\infty + \frac{3}{4} \|f - g\|_1 \leq \|f - g\|. \end{aligned}$$

2.3 Second Example : Next we give an example of a diametrically contractive self map of a bounded closed convex set K , in which the existence of a fixed point does not imply convergence of iterates $T^n x$ to the fixed point. Consider the vector space P_0 , endowed with the following norm

$$\|x\| = \|x\|_\infty + \sum_{n=1}^{\infty} \frac{|x_n|}{4^n}. \quad (3.1)$$

Define $F : P^+ \rightarrow P^+$, where P^+ is the intersection of the positive cone with the unit closed ball, as follows:

$(Tx)_1 = (Tx)_2 = 0$ and $(Tx)_n = a_{n-1}x_{n-1}$ for $n \geq 3$, and sequence $\langle a_n \rangle$ for $n \geq 2$ is strictly increasing sequence of positive nonzero numbers such that

$$\prod_{n=1}^{\infty} a_n = \nu > 0 \quad (3.2)$$

Notice that F is linear and has a unique fixed which is the null vector.

The map F defined above is shrinking. For $x \neq y$, we have

$$\begin{aligned} \|Fx - Fy\| &= \|(0, 0, a_2(x_2 - y_2), a_3(x_3 - y_3), \dots)\| \\ &< \|(0, 0, (x_2 - y_2), (x_3 - y_3), \dots)\| \\ &< \|x - y\|. \end{aligned}$$

Now we look at the orbit F^n of non-zero elements in P^+ . Assume for $x, x_\alpha \neq 0$. Then

$$\|T^n x\| \geq |(T^n x)_{\alpha+n}| = a_\alpha a_{\alpha+1} \cdots a_{\alpha+n-1} x_\alpha \rightarrow \left(\prod_{n=\alpha}^{\infty} a_n \right) x_\alpha \neq 0.$$

2.4 Important Result : Map F is diametrically contractive.

Proof : Take a bounded closed convex set T contained in P^+ . Assume

$$\delta(T) = \delta(F(T)) > 0. \tag{3.3}$$

Consider two sequences $x^{(n)}$ and $y^{(n)}$ such that

$$\lim_{n \rightarrow \infty} \|Fx_n - Fy_n\| = \delta(F(T))$$

$$\begin{aligned} \delta(F(T)) &= \lim_{n \rightarrow \infty} \left(\|F(x^{(n)} - y^{(n)})\|_\infty + \sum_{k=1}^{\infty} \frac{|T(x^{(n)} - y^{(n)})_\alpha|}{4^\alpha} \right) \\ &= \lim_{n \rightarrow \infty} \left(\max_{\alpha \geq 2} |a_{\alpha-1}(x_{\alpha-1}^{(n)} - y_{\alpha-1}^{(n)})| + \sum_{k=1}^{\infty} \frac{a_{\alpha-1}|x_{\alpha-1}^{(n)} - y_{\alpha-1}^{(n)}|}{4^\alpha} \right) \end{aligned} \tag{3.4}$$

$$\begin{aligned} &= \lim_{n \rightarrow \infty} \left(\max_{\alpha \geq 1} a_\alpha |x_\alpha^{(n)} - y_\alpha^{(n)}| + \sum_{\alpha=1}^{\infty} \frac{a_\alpha |x_\alpha^{(n)} - y_\alpha^{(n)}|}{4^{\alpha+1}} \right) \\ &\leq \lim_{n \rightarrow \infty} \sup \left(\|x^{(n)} - y^{(n)}\|_\infty + \frac{1}{4} \sum_{\alpha=1}^{\infty} \frac{|x_\alpha^{(n)} - y_\alpha^{(n)}|}{4^\alpha} \leq \delta(T) \right). \end{aligned} \tag{3.5}$$

Thus $\lim_{n \rightarrow \infty} \|x^{(n)} - y^{(n)}\|_\infty = \delta(T)$ and $\lim_{n \rightarrow \infty} \sum_{\alpha=1}^{\infty} \frac{|x_\alpha^{(n)} - y_\alpha^{(n)}|}{4^\alpha} = 0$.

Notice that, there exists $\alpha(n)$ such that

$$\|x^{(n)} - y^{(n)}\|_\infty = |x_{\alpha(n)}^{(n)} - y_{\alpha(n)}^{(n)}|$$

and hence

$$\lim_{n \rightarrow \infty} |x_{\alpha(n)}^{(n)} - y_{\alpha(n)}^{(n)}| = \delta(T). \tag{3.6}$$

Let $Q = \{\alpha(n); n \in \mathbb{N}\}$. Therefore, $Q(n) = \alpha_0$ if α is finite, for infinitely many n . Hence

$$\sum_{\alpha=1}^{\infty} \frac{|x_\alpha^{(n)} - y_\alpha^{(n)}|}{4^\alpha} \geq \frac{|x_{\alpha_0}^{(n)} - y_{\alpha_0}^{(n)}|}{4^{\alpha_0}} \xrightarrow{n \rightarrow \infty} \frac{\delta(T)}{4^{\alpha_0}} \neq 0. \tag{3.7}$$

This is absurd as left hand side tends to zero. Therefore Q is infinite.

Consider a subsequence of $Q(n)$, which we denote by $\bar{\alpha}(n)$, such that

$$x_{\bar{\alpha}(n)}^{(n)} \rightarrow \delta(T) + \ell \quad \text{and} \quad y_{\bar{\alpha}(n)}^{(n)} \rightarrow \ell \quad (\geq 0).$$

For $x \in T$

$$\begin{aligned} \delta(T) + \ell &= \lim_{n \rightarrow \infty} |x_{\bar{\alpha}(n)} - x_{\bar{\alpha}(n)}^n| \leq \lim_{n \rightarrow \infty} \|x - x^{(n)}\|_{\infty} \\ &\leq \lim_{n \rightarrow \infty} \|x - x^{(n)}\| \leq \delta(T). \end{aligned}$$

This implies $\ell = 0$.

Thus, for every $x \in T$,

$$\lim_{n \rightarrow \infty} \|x - x^{(n)}\| = \delta(T). \quad (3.8)$$

Hence

$$\lim_{n \rightarrow \infty} \sum_{\alpha=1}^{\infty} \frac{|x_{\alpha} - x_{\alpha}^{(n)}|}{4^{\alpha}} = 0$$

and for every α

$$\lim_{n \rightarrow \infty} x_{\alpha}^{(n)} = x_{\alpha}.$$

The above should be true for every $x \in T$ and so T cannot contain two or more elements which implies $\delta(T) = 0$, this is a contradiction to assumption in (3.3). Therefore map F is diametrically contractive.

3. Fixed Point Iteration

Now we discuss a concept of fixed point iteration used in Python programming language.

What is Python? : Python is an interactive object oriented programming language, which is quick to learn, fast to program in, and the programs written in Python are easy to debug. Usually Python is used in small network related applications and system maintenance - scripting. The Python programming environment is free for both commercial and non-commercial use and it is downloadable from the web page <http://www.Python.org>.

Consider simple example of simple program in Python of fixed point iteration.

Let $f : [a, b] \rightarrow R$. We consider the function $g(x) = x - f(x)$ for $x \in [a, b]$. By using the function g instead of solving the equation $f(x) = 0$, we may solve the equation $g(x) = x$. The solutions of this equation are called the fixed points of the function 4g4. We start with any point x_0 of the internal $[a, b]$. We define the sequence of the iterations of g, x_0, x_1, x_2, \dots by the formula $x_{i+1} = g(x_i)$. It follows from the well known Banach fixed point theorem that this sequence converges to a fixed point of the function g if the following conditions are satisfied :

- (a) $g([a, b]) \subset [a, b]$
- (b) $|g(x) - g(y)| \leq L|x - y|$ for some $L < 1$ and for all $x, y \in [a, b]$.

In this example, convergence of the iteration can be studied by choosing

$$L = \max\{|f'(x)| : x \in [0, 1]\}.$$

Simple Program :

```
# Fixed point iteration
from math import *
from whrandom import *
# function
def fun (x) : return cos (x)
# function to be iterated
def iter f(x) : return x
# starting point
x = 20.0* random (x)
n = 10 # max. number of steps
Print "step point function value"
for i in range (n):
x = iter f(x)
Print "% 4d% 10.6g% 14.6g"% (i, x, fun(x))
```

Output :

Step	Point	function value
0	4.4175	-0.290636
1	4.70813	-0.00425545
2	4.71239	-1.284363-0.8
3	4.71239	-1.83691e-16
4	4.71239	-1.83691e-16
5	4.71239	-1.83691e-16
6	4.71239	-1.83691e-16
7	4.71239	-1.83691e-16
8	4.71239	-1.83691e-16

References

1. S. B. Nadier Jr., A note on an iterative test of Edelstein, Canadian Mathematical Bulletin, 15 (1972), 381-386.
2. H. K. Xu, Diametrically contractive mappings, Bulletin of the Australian Mathematical Society, 70 (2004), No. 3, 463-468.
3. K. R. Sastry and S. V. R. Naidu, Some fixed point theorems in normed linear spaces, Indian Journal of Pure and Applied Mathematics, 10 (1979), No. 8, 928-937.
4. B. Sims, Examples of fixed point free mappings, Handbook of Metric Fixed Point Theory, lwer Academic, Dordrecht, 2001, 35-48.
5. Python homepage (<http://www.Python.org/>)
6. Learning Python <http://www.Python.org/topics/learn/>
7. Python in the Mathematics Curriculum
<http://www.Python.org/Pycon/dc2004/papers/15/>
8. Python and Scientific Computing
<http://www.Python.org/topics/Scicom/>

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