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An Alternative Multiple Hypotheses Testing Procedure Using Fuzzy Approach

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Abstract : Holm [1] introduced an improved multiple hypotheses testing procedure based on the classic Bonferroni's method. By changing a constant in each testing step, it produces more powerful than the classic Bonferroni's method. This kind of improvement is also employed in many methods, including the Hochberg method [2] and the Holm-Sidak method [3],[4] . However, without changing a constant in each testing step, there is another improvement approach called a fuzzy approach. In this work, those multiple hypotheses testing procedures were improved using the fuzzy approach. Under the situation of an uncertainty occurred on the critical values, the fuzzy approach was applied to the critical value in each step in form of right triangle left. Power of the original multiple hypotheses testing method and the fuzzy multiple hypotheses testing method was compared by simulation study. The data sets are composed of 3 treatments with equal means, equal variances and equal sample sizes by using R program. The results showed that the fuzzy approach method has more power than the original method. Therefore, the fuzzy approach can be an alternative improvement of the multiple hypotheses testing procedure.

Keywords : fuzzy approach; fuzzy hypothesis testing; Holm's multiple hypotheses testing; triangular fuzzy number; power of the test
2010 Mathematics Subject Classification : 47H09; 47H10 (2000 MSC)

1 Introduction

Multiple hypotheses testing is a procedure for testing many hypotheses simultaneously. Suppose k null hypotheses $H_{0_1}, H_{0_2}, ..., H_{0_k}$ are to be simultaneously tested and their p-values $P_1, P_2, ..., P_k$ are calculated. In the testing process, one of the hypotheses is chosen and tested using an ordinary testing procedure. Then, one of the remain hypotheses is chosen and processed as the first hypothesis. This process is continued through the last hypothesis. Each time hypothesis is tested, it is called a step and each step uses a constant in form of $c\alpha$ or $1-(1-\alpha)^c$ where $c \in [0,1]$ and α is a familywise testing significance level where $\alpha \in [0,1]$. Therefore, this method is able to control familywise error not to exceed a pre-defined familywise significance level α.

Holm [1] developed the new method based on the original Bonferroni's method. This method begins by sorting p-values $P_1, P_2, ..., P_k$ to $P_{(1)}, P_{(2)}, ..., P_{(k)}$ where $P_{(1)} < P_{(2)} < ... < P_{(k)}$ which implied to change the order of the null hypotheses $H_{0_1}, H_{0_2}, ..., H_{0_k}$ to $H_{0_{(1)}}, H_{0_{(2)}}, ..., H_{0_{(k)}}$ where $H_{0_{(i)}}; i = 1, 2, ..., k$

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is the null hypothesis corresponding to its p-value $P_{(i)}$. By using $\frac{\alpha}{k}, \frac{\alpha}{k-1}, ..., \alpha$ as a critical value set, the testing is defined as follows: (1) if $P_{(1)} > \frac{\alpha}{k}$, then stop the testing process and accept all null hypotheses, $H_{0_{(1)}}, H_{0_{(2)}}, ..., H_{0_{(k)}}$, otherwise, $H_{0_{(1)}}$ is rejected; (2) if $P_{(2)} > \frac{\alpha}{k-1}$, then stop the testing process and accept $H_{0_{(2)}}, H_{0_{(3)}}, ..., H_{0_{(k)}}$, otherwise, $H_{0_{(2)}}$ is rejected;...;(k) if $P_{(k)} > \alpha$, then accept $H_{0_{(k)}}$, otherwise, $H_{0_{(k)}}$ is rejected. This kind of procedure can be called a step-down procedure.

Hochberg [2] also proposed a method similar to the Holm's method but the testing process starts with the comparison of $P_{(k)}$ and its critical value α. By using a step-up procedure approach, this method rejects all null hypotheses with a smaller or equal to any p-value which is found less than its critical value. Hochberg claimed this makes his method more powerful than the Holm's method.

Holland and Copenhaver [3],[4] extended the Holm's method when the test statistics are independent or positive octant dependent. This method is usually called the Holm-Sidak method. Based on a modification which is analogous to the Sidak correction [5], the critical value in each testing step can be changed to $1-(1-\alpha)^{\frac{1}{k}}, 1-(1-\alpha)^{\frac{1}{k-1}}, ..., 1-(1-\alpha)^{1}$ respectively. This adjustment leads to a slightly more power than the original Holm's method [6].

Consequently, it can be seen that the simplest way to improve any multiple hypotheses testing procedure is to change a constant in every testing step. However, there is an alternative approach, which is called a fuzzy approach. It is applied in many statistical methods. In this study, an application of the fuzzy approach on the multiple hypotheses testing procedure, including the Holm's method, the Holm-Sidak's method and the Hochberg method, was presented and the efficiency of them was compared.

2 Fuzzy Concept

Zadeh [7] generalized the classical notation of set theory to a fuzzy concept. Suppose A be a set. According to the classic set theory, any element x can be described as a member of set A or not a member of set A by a characteristic function: $\psi_A(x) = 1$ and $\psi_A(x) = 0$ respectively. For the fuzzy concept, a member of fuzzy set A can be described by a membership function $\overline{A}(x) : \mathbb{R} \to [0, 1]$. Please note that the notation of a fuzzy set is usually written as $\overline{A}$. If x is certainly a member of $\overline{A}$ or not a member of $\overline{A}$, then it can be seen that $\overline{A}(x) = 1$ or $\overline{A}(x) = 0$ respectively. However, x can be a partially member of $\overline{A}$ since $0 < \overline{A} < 1$. This concept can be applied to be a fuzzy hypothesis testing method.

A fuzzy hypothesis test is a procedure which some components are estimated using the fuzzy approach. Suppose triangular fuzzy numbers are only considered. Again, a fuzzy number which can be called a fuzzy set $\overline{A}$ is a membership function mapping any real number x and producing a value in $[0, 1]$. Based on the fuzzy number, a δ-cut of a fuzzy set $\overline{A}$, denoted as $\overline{A}_\delta$, is defined as

$$\overline{A}_\delta = \{x|\overline{A}(x) \geq \delta; 0 < \delta \leq 1\}. \tag{2.1}$$

For any triangular fuzzy number $\overline{N}$, the $\overline{N}_\delta$ can also be written in a form

$$\overline{N}_\delta = [\, n_1(\,\delta\,)\,, n_2(\,\delta\,)\,]; \forall\delta \in [\,0, 1]\,. \tag{2.2}$$

It can be seen that a fuzzy number is defined in a functions form of δ [8].

Suppose $H_0 : \theta = \theta_0$ versus $H_1 : \theta > \theta_0$ are tested and let the statistic $\overline{T}$ and its critical value $\overline{CV}$ are in triangular fuzzy form. A conclusion can be made using graph as follows [9]:

(1) reject the null hypothesis if $\overline{T} > \overline{CV}$,

(2) do not reject the null hypothesis if $\overline{T} < \overline{CV}$,

(3) there is no decision on the null hypothesis if $\overline{T} \approx \overline{CV}$.

Parchami, et al. [10] proposed a p-value for the fuzzy statistic. Let a fuzzy p-value be a fuzzy set on $[0, 1]$. The δ-cut of the p-value, denoted as $\overline{P}_\delta$, can be computed as follows:
for the right-sided alternative hypothesis; $\overline{P}_\delta = [\,\overline{P}_{\theta_2(\,\delta)}\,(\,T \leq t\,)\,, \overline{P}_{\theta_1(\,\delta)}\,(\,T \leq t\,)\,]$,

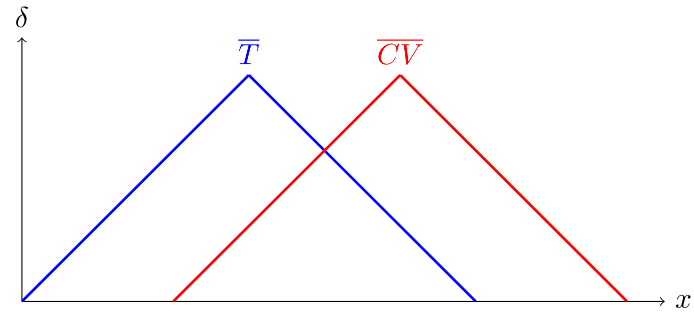


Figure 1: Fuzzy statistic $\overline{T}$ and fuzzy critical value $\overline{CV}$

for the left-sided alternative hypothesis; $\overline{P}_\delta = [\,\overline{P}_{\theta_1(\delta)}\,(T \geq t)\,, \overline{P}_{\theta_2(\delta)}\,(T \geq t)\,]$,

for the two-sided alternative hypothesis; $\overline{P}_\delta = \begin{cases} [\,2\overline{P}_{\theta_1(\delta)}\,(T \geq t)\,, 2\overline{P}_{\theta_2(\delta)}\,(T \geq t)\,]; t \geq m_r \\ [\,2\overline{P}_{\theta_2(\delta)}\,(T \leq t)\,, 2\overline{P}_{\theta_1(\delta)}\,(T \leq t)\,]; t \leq m_l \end{cases}$,

where

δ is a δ-cut level,

θ is an interested parameter,

m_l is defined as $\inf\{m, m \in \text{Supp}(\overline{M})\}$ and $\text{Supp}(\overline{M})$ is the support (or the base of the fuzzy set) of $\overline{M}$ which is defined as $\text{Supp}(\overline{M}) = \{x | \overline{M}(x) > 0\}$,

m_r is defined as $\sup\{m, m \in \text{Supp}(\overline{M})\}$,

$\overline{M}$ is a fuzzy set with membership function $\overline{M}(m) = \overline{H}_{0b}(\theta)$ and $\overline{H}_{0b}$ is a boundary of the fuzzy null hypothesis set where its δ-cut is $(\overline{H}_{0b})_\delta = [\,\theta_1(\delta)\,, \theta_2(\delta)\,]$.

Please note that in this study, the null hypothesis and the alternative hypothesis are in crisp form. Therefore, it can be implied that $\theta_1(\delta) = \theta_2(\delta) = \theta_0$.

Yuan [11] gave a criterion for the decision of the fuzzy approach which is evaluated by a degree of acceptance or a degree of rejection. It can be defined as follows: let be fuzzy sets and be normal and convex and let

$$\triangle_{AB} = \int_{a^+_{A_\delta} > a^-_{B_\delta}} (a^+_{A_\delta} - a^-_{B_\delta}) d\delta + \int_{a^-_{A_\delta} > a^+_{B_\delta}} (a^-_{A_\delta} - a^+_{B_\delta}) d\delta \tag{2.3}$$

where

$a^+_{A_\delta}$ is defined as $\sup\{x, x \in \overline{A}_\delta\}$,

$a^-_{A_\delta}$ is defined as $\inf\{x, x \in \overline{A}_\delta\}$,

$\overline{A}_\delta$ is a δ-cut of $\overline{A}$.

The degree of $\overline{A}$ is greater than $\overline{B}$ is defined as

$$D(\overline{A} > \overline{B}) = \frac{\triangle_{AB}}{\triangle_{AB} + \triangle_{BA}} \tag{2.4}$$

which can be applied to the fuzzy testing decision as follows: let $\overline{S}$ be a significance fuzzy set. The null hypothesis will be accepted if $\overline{P} > \overline{S}$ and $D(\overline{P} > \overline{S})$ is called a degree of acceptance. Otherwise, the null hypothesis is rejected with a degree of rejection $D(\overline{S} > \overline{P})$.

The decision of the fuzzy p-value is too complicated to calculate. Parchami [12] developed Fuzzy.p.value R package for calculating the fuzzy p-value and a degree of acceptance or a degree of rejection which can be obtained at the R CRAN network.

In this study, the fuzzy hypothesis testing procedure will be applied to each i^{th} step of the Holm's method, the Holm-Sidak method and the Hochberg's method. Suppose $H_{0_i} : \mu_i = \varphi$ and $H_{1_i} : \mu_i > \varphi$ where $\varphi \in \mathbb{R}$. Each μ_i is tested using the Holm's method, the Holm-Sidak method, the Hochberg method and the proposed fuzzy methods. Based on these methods, we then compared their performance.

3 Methodology

The data sets are simulated under the one-way model with k treatments [13], which is

$$Y_{ij} = \mu_i + \varepsilon_{ij}; i = 1, 2, ..., k; j = 1, 2, ..., n_i \tag{3.1}$$

where

Y_{ij} is a random sample of the i^{th} treatment,
μ_i is a mean response of the i^{th} treatment,
n_i is a sample size of the i^{th} treatment,
ε_{ij} is a random error term which is independent and identically normally distributed with mean 0 and unknown variance σ^2,

and let some model properties as follows:
(1) the model has 3 treatments $k = 3$ with equal sample sizes $n_i = 30, 40, ..., 80$ and ε_{ij} is identically normally distributed with mean = 0 and variance = σ^2,
(2) variance (σ^2) will be classified into 3 categories:
(3.1) $\sigma^2 = 1, 4$ for a small variance group,
(3.2) $\sigma^2 = 25, 36$ for a medium variance group, and
(3.3) $\sigma^2 = 81, 100$ for a high variance group,
(3) determine the true means response and the standard deviations of three treatments equally with $\mu_i = 0.05, 0.10, ..., 0.60$ and $\sigma^2 = 1, 4, 25, 36, 81, 100$ respectively.

Let γ_i be an alpha level in each i^{th} step of the Holm's method, the Holm-Sidak's method and the Hochberg method. According to Taweesapaya et al. [14], only a right triangular left shaped fuzzy critical value is considered. Its procedure is as follows:
(1) set $H_{0_i} : \mu_i = 0$ versus $H_{1_i} : \mu_i > 0$,
(2) set a critical value in each step of testing as a fuzzy form of $(\gamma_i, \gamma_i, 2 \times \gamma_i)$,
(3) the Holm's method, the Holm-Sidak's method and the Hochberg method are compared with their fuzzy approach methods by using the power of the test as the criteria with 5,000 replications. The simulation is done using R program [15].

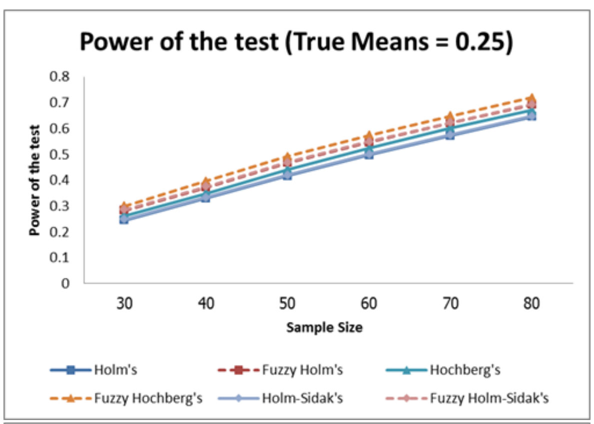
4 Results

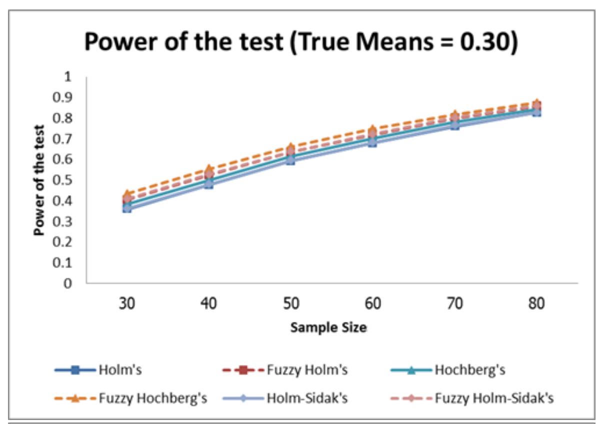
Our study is to compare power of the test based on 3 multiple hypotheses testing procedures and their fuzzy approach. The finding results will be present as following. The results for 3 variance groups, which are the low variance group, the medium group, and the high variance group, are consistent; their results have the same trend. Therefore, they will be presented partially; only the case of $\mu_i = 0.05, 0.10, ..., 0.60$, $\sigma^2 = 1$, and $n_i = 30, 40, ..., 80$ is presented. From the results, every fuzzy approach method has more power than its original method as seen in Table 1 and Figure 2. It can also be seen that the power of the test is increasing when the sample size increases.

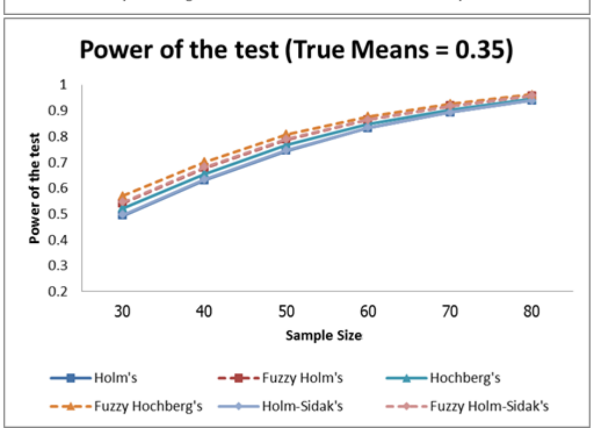
Table 1: Power of the test of the Holm's method, the Holm-Sidak's method, the Hochberg method and their fuzzy approach method with specific true means and sample sizes

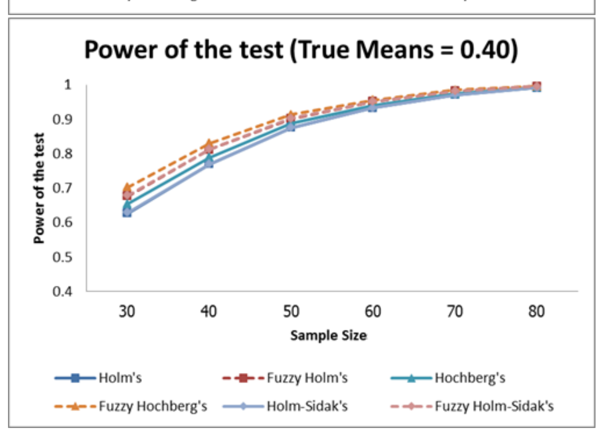
Mean	Sample Size	Method					
		Holm's	Fuzzy Holm's	Hochberg's	Fuzzy Hochberg's	Holm-Sidak's	Fuzzy Holm-Sidak's
0.25	30	0.2462	0.2825	0.2631	0.3007	0.2481	0.2855
	40	0.3309	0.3729	0.3499	0.3964	0.3334	0.3756
	50	0.4167	0.4674	0.4415	0.4924	0.4196	0.4697
	60	0.4983	0.5468	0.5242	0.5749	0.5013	0.5497
	70	0.5731	0.6222	0.6027	0.6495	0.5762	0.6237
	80	0.6458	0.6921	0.6712	0.7193	0.6479	0.6946
0.30	30	0.3603	0.4067	0.3840	0.4346	0.3623	0.4093
	40	0.4773	0.5258	0.5016	0.5548	0.4800	0.5287
	50	0.5937	0.6369	0.6175	0.6614	0.5950	0.6390
	60	0.6803	0.7214	0.7042	0.7489	0.6823	0.7235
	70	0.7618	0.8008	0.7834	0.8193	0.7639	0.8027
	80	0.8271	0.8588	0.8421	0.8738	0.8281	0.8599
0.35	30	0.4949	0.5427	0.5232	0.5715	0.4980	0.5461
	40	0.6320	0.6793	0.6560	0.7022	0.6345	0.6815
	50	0.7453	0.7887	0.7673	0.8083	0.7475	0.7902
	60	0.8333	0.8651	0.8477	0.8778	0.8351	0.8657
	70	0.8939	0.9185	0.9041	0.9276	0.8941	0.9187
	80	0.9401	0.9579	0.9479	0.9636	0.9403	0.9586
0.40	30	0.6275	0.6768	0.6541	0.7028	0.6298	0.6789
	40	0.7697	0.8123	0.7900	0.8304	0.7705	0.8135
	50	0.8757	0.9024	0.8893	0.9142	0.8766	0.9031
	60	0.9331	0.9519	0.9395	0.9559	0.9338	0.9523
	70	0.9714	0.9823	0.9750	0.9849	0.9715	0.9825
	80	0.9920	0.9962	0.9926	0.9965	0.9920	0.9962
0.45	30	0.7507	0.7889	0.7693	0.8059	0.7524	0.7907
	40	0.8803	0.9063	0.8913	0.9148	0.8810	0.9069
	50	0.9513	0.9662	0.9561	0.9697	0.9520	0.9666
	60	0.9847	0.9916	0.9856	0.9927	0.9848	0.9917
	70	0.9971	0.9985	0.9974	0.9987	0.9971	0.9985
	80	0.9999	1.0000	0.9999	1.0000	0.9999	1.0000
0.50	30	0.8455	0.8768	0.8581	0.8892	0.8459	0.8777
	40	0.9452	0.9596	0.9488	0.9644	0.9455	0.9597
	50	0.9881	0.9929	0.9902	0.9945	0.9882	0.9931
	60	0.9989	0.9996	0.9989	0.9997	0.9989	0.9996
	70	0.9998	0.9999	0.9999	0.9999	0.9998	0.9999
	80	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Mean	Sample Size	Method					
		Holm's	Fuzzy Holm's	Hochberg's	Fuzzy Hochberg's	Holm-Sidak's	Fuzzy Holm-Sidak's
0.55	30	0.9169	0.9381	0.9257	0.9453	0.9181	0.9385
	40	0.9809	0.9897	0.9843	0.9915	0.9811	0.9897
	50	0.9989	0.9995	0.9989	0.9995	0.9989	0.9995
	60	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	70	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	80	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.60	30	0.9623	0.9753	0.9658	0.9778	0.9623	0.9757
	40	0.9974	0.9988	0.9979	0.9989	0.9975	0.9988
	50	0.9998	1.0000	0.9999	1.0000	0.9998	1.0000
	60	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	70	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	80	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000









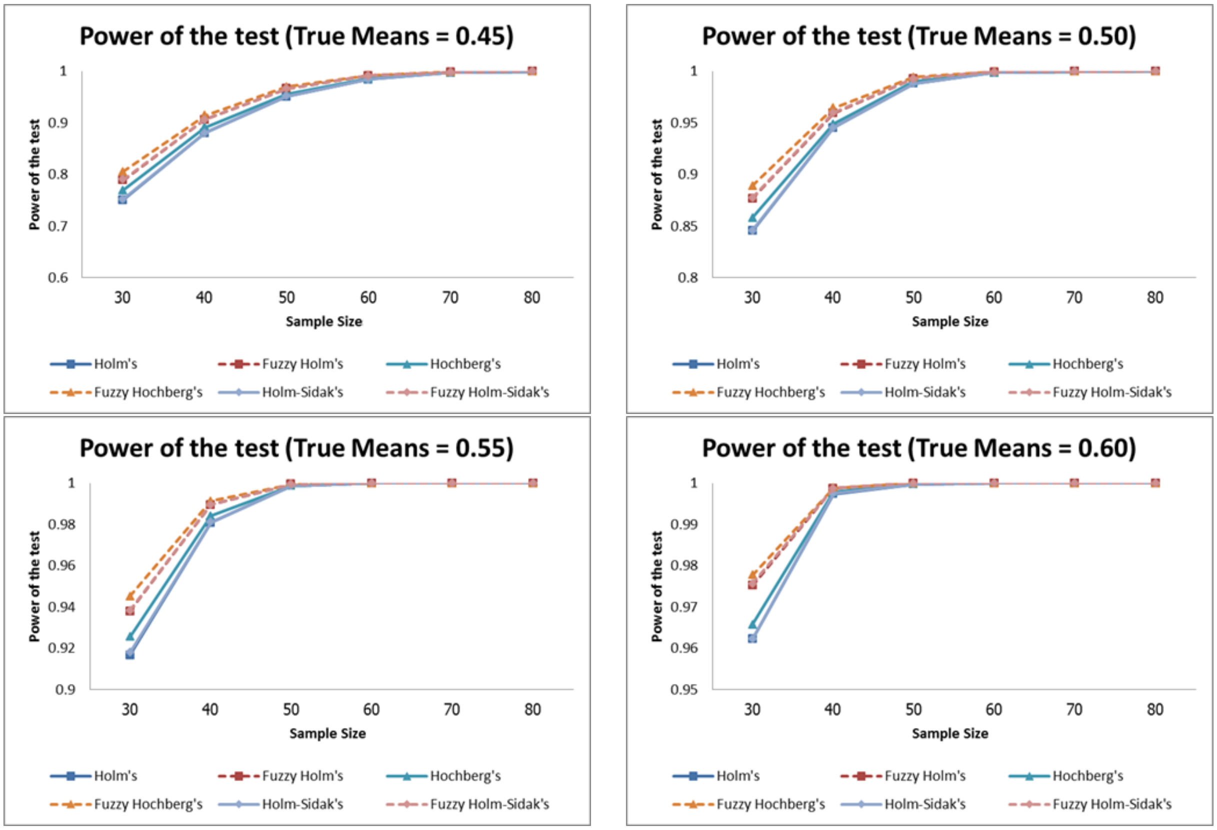


Figure 2: Power of the test in each case with a specific true mean and sample size

5 Conclusions and Discussions

From the results, all cases showed that the fuzzy approach method of the Holm's method, the Holm-Sidak's method and the Hochberg method have more power than their original method. Therefore, it can be concluded that a fuzzy approach with fuzzy critical value $(\gamma_i, \gamma_i, 2 \times \gamma_i)$ is an alternative way for improving multiple hypotheses testing procedure. Please note that the results for $\mu_i = 0.05, 0.10, 0.15, 0.20$ are unstable increasing due to the small difference between true mean and hypothesized mean which can be called a small effect size [16]. Therefore, they are all omitted in this work.

6 Suggestions

Further study can be done by changing the triangular fuzzy set to other forms, such as right triangular right fuzzy set, isosceles triangular fuzzy set or by changing the triangular fuzzy set base length.

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