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The Beta Topp-Leone Gumbel Distribution for Modeling the Minimum Flow Data

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Abstract : In this work a flexible of the Gumbel distribution, called the beta Topp-Leone Gumbel distribution, is introduced. General properties of the distribution are also provided, such as, the linear representation for the probability density function and the cumulative distribution function, transformation, quantile function and moments. The method of maximum likelihood is used to estimate the proposed distribution's parameters. The applicability of the new distribution is illustrated by the minimum flow dataset.

Keywords : Gumbel distribution; Topp-Leone distribution; Beta generated family; Maximum likelihood estimation; Model selection criteria
2010 Mathematics Subject Classification : 62E10; 62F10; 62P99

1 Introduction

The extreme value theory can be applied in various researches, for example, hydrology, air pollution, engineering, finance and environmental science. The study of natural phenomena by modeling the extreme values, such as highest flood discharges, wind speed, earthquake magnitudes, river and sea levels, ocean surface wave height or temperature become an important factor for environmental statistics since the knowledge of extreme values is of the most importance in an avoidance of disaster. The Gumbel distribution [1] is a well-known statistical model from its wide applicability. It is also known as the extreme value type I distribution. In 2000, one of text books by Kotz and Nadarajah [2] described this distribution by listing more than fifty applications ranging from reliability testing through environmental science, such as floods, earthquakes, rainfall, wind speeds, sea levels and etc.. The Gumbel distribution is defined as follows: A random variable X has the cumulative distribution function (cdf) and the probability density function (pdf) are given by

$$G(x;\mu,\sigma)=\exp(-v) \quad \text{and} \quad g(x;\mu,\sigma)=\sigma^{-1}v\exp(-v), \quad x\in\mathbb{R},$$

respectively, where $v=\exp\{-(x-\mu)/\sigma\}$, $-\infty<\mu<\infty$ and $\sigma>0$.

Refer to the importance and applicability of the Gumbel distribution, many researcher's attempts were accomplished to improve this distribution to become more flexible for modeling data. In some generalizations of the Gumbel distribution, we list as follows:

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(i) The beta Gumbel (BetaGum) distribution: A random variable X has the pdf

$$f_{\text{BetaGum}}(x; a, b, \mu, \sigma) = \frac{1}{\sigma B(a,b)} v \exp(-av)[1 - \exp(-v)]^{b-1}, \quad x \in \mathbb{R}, \tag{1.1}$$

where $v = \exp\{-(x-\mu)/\sigma\}$, $-\infty < \mu < \infty$, $a, b, \sigma > 0$ and $Beta(\cdot, \cdot)$ is the complete beta function. If the pdf of X can be expressed as (1.1), we may write $X \sim \text{BetaGum}(a, b, \mu, \sigma)$. The BetaGum was introduced by Nadarajah and Kotz [3].

(ii) The exponentiated Gumbel (ExpGum) distribution: A random variable X has the pdf and cdf

$$f_{\text{ExpGum}}(x; a, \mu, \sigma) = a\sigma^{-1} v(\exp(-av)), \quad x \in \mathbb{R}, \tag{1.2}$$

and

$$F_{\text{ExpGum}}(x; a, \mu, \sigma) = \exp(-av), \tag{1.3}$$

respectively, where $v = \exp\{-(x-\mu)/\sigma\}$, $-\infty < \mu < \infty$ and $a, \sigma > 0$. If the pdf and cdf of X can be expressed as (1.2) and (1.3), respectively, we may write $X \sim \text{ExpGum}(a, \mu, \sigma)$. Some of the mathematical properties of the ExpGum distribution has been studied by Nadarajah [4].

(iii) The Kumaraswamy Gumbel (KumGum) distribution: A random variable X has the pdf

$$f_{\text{KumGum}}(x; a, b, \mu, \sigma) = ab\sigma^{-1} v \exp(-av)[1 - \exp(-av)]^{b-1}, \quad x \in \mathbb{R}, \tag{1.4}$$

where $v = \exp\{-(x-\mu)/\sigma\}$, $-\infty < \mu < \infty$ and $a, b, \sigma > 0$. If the pdf of X can be expressed as (1.4), we may write $X \sim \text{KumGum}(a, b, \mu, \sigma)$. The KumGum distribution was firstly proposed by Cordeiro and de Castro [5] and then discussed in detail by Cordeiro et al. [6].

(iv) The Topp-Leone Gumbel (TLGum) distribution: A random variable X has the pdf and cdf

$$f_{\text{TLGum}}(x; c, \mu, \sigma) = 2c\sigma^{-1} v \exp(-v)(1 - \exp(-v))[1 - (1 - \exp(-v))^2]^{c-1}, \quad x \in \mathbb{R}, \tag{1.5}$$

and

$$F_{\text{TLGum}}(x; c, \mu, \sigma) = [1 - (1 - \exp(-v))^2]^c, \tag{1.6}$$

respectively, where $v = \exp\{-(x-\mu)/\sigma\}$, $-\infty < \mu < \infty$ and $c, \sigma > 0$. If the pdf and cdf of X can be expressed as (1.5) and (1.6), respectively, we may write $X \sim \text{TLGum}(c, \mu, \sigma)$. The TLGum was introduced by Bodhisuwan [7].

(v) The McDonald Gumbel (McGum) distribution: A random variable X has the pdf

$$f_{\text{McGum}}(x; a, b, c, \mu, \sigma) = \frac{c}{\sigma B(a,b)} v \exp(-v) \exp(-(ac-1)u)[1 - \exp(-cv)]^{b-1}, \quad x \in \mathbb{R}, \tag{1.7}$$

where $v = \exp\{-(x-\mu)/\sigma\}$, $-\infty < \mu < \infty$, $a, b, c, \sigma > 0$ and $Beta(\cdot, \cdot)$ is the complete beta function. If the pdf of X can be expressed as (1.7), we may write $X \sim \text{McGum}(a, b, c, \mu, \sigma)$. The McGum was introduced by de Brito et al.[8].

In this work, a new extension of the TLGum distribution referred to the beta Topp-Leone (BTLGum) distribution introduced for improving flexibility of the Gumbel distribution.

2 Preliminaries

Several classical continuous distributions are widely used for modeling data in many other areas, including economics, engineering, biological studies and environmental sciences. Recent improvements mostly focus on constructing the new families of distributions that generalized classical distributions and, meanwhile, increasing its flexibility in modeling data. Hence, several families of distributions have been proposed by adding more (location, shape or scale) parameters to generate new distributions in the statistical literary work recently.

The beta generated (BetaG) family of distributions by using the inverse cdf (quantile function) to a beta random variable has been invented by Eugene et al.[9].

Definition 2.1. *Let $G(x;\boldsymbol{\xi})$ be a baseline cdf, let $g(x;\boldsymbol{\xi}) = dG(x;\boldsymbol{\xi})/dx$ be a baseline pdf of a random variable X and $\boldsymbol{\xi}$ the $p\times 1$ vector of associated parameter(s). The BetaG family pdf and cdf are expressed as*

$$f_{BetaG}(x;a,b,\boldsymbol{\xi}) = \frac{1}{B(a,b)}g(x;\boldsymbol{\xi})G(x;\boldsymbol{\xi})^{a-1}[1-G(x;\boldsymbol{\xi})]^{b-1}, \quad a,b>0, \tag{2.1}$$

and

$$F_{BetaG}(x;a,b,\boldsymbol{\xi}) = I_{G(x;\boldsymbol{\xi})}(a,b), \tag{2.2}$$

respectively, where the function $I_{G(x;\boldsymbol{\xi})}(a,b)$ denotes the incomplete beta ratio defined by,

$$I_y(a,b) = \frac{B_y(a,b)}{B(a,b)},$$

where $B_y(a,b) = \int_0^y t^{a-1}(1-t)^{b-1}dt$, $0<t<1$, is the incomplete beta function.

In point of fact, if B is distributed according to the beta distribution for $0 < x < 1$ and positive shape parameters a and b, then a random variable $X = G^{-1}(B)$ would implies the pdf (2.1) and the cdf (2.2).

3 Main Results

This section is separated into 4 subsections which consist of introduction of the BTLGum distribution, some of its general properties, i.e. linear representation, transformation, quantile function and moments, maximum likelihood estimation for estimating parameters and the flexibility of the proposed distribution applied by the minimum flow data, as following respectively.

3.1 The Beta Topp-Leone Gumbel Distribution

Proposition 3.1. *The pdf and cdf of the BTLGum distribution are expressed as*

$$\begin{aligned} f(x;a,b,c,\mu,\sigma) = &\frac{2c}{\sigma B(a,b)} v\exp(-v)(1-\exp(-v))[1-(1-\exp(-v))^2]^{ac-1} \\ &\times [1-[1-(1-\exp(-v))^2]^c]^{b-1}, \quad x\in\mathbb{R} \end{aligned} \tag{3.1}$$

and

$$F(x;a,b,c,\mu,\sigma) = I_{[1-(1-\exp(-v))^2]^c}(a,b), \quad x\in\mathbb{R} \tag{3.2}$$

respectively, where $v = \exp\{-(x-\mu)/\sigma\}$, $-\infty<\mu<\infty$, a,b,c and $\sigma>0$. If the pdf and cdf of X can be expressed as (3.1) and (3.2), respectively, we may write $X \sim BTLGum(a,b,c,\mu,\sigma)$.

Proof. Inserting (1.5) and (1.6) in (2.1) and (2.2), we obtain the pdf and cdf of the BTLGum distribution (for $-\infty < \mu < \infty$, a, b, c and $\sigma > 0$) as

$$f(x; a, b, c, \mu, \sigma) = \frac{2c}{\sigma \mathrm{B}(a, b)} v \exp(-v)(1 - \exp(-v))[1 - (1 - \exp(-v))^2]^{ac-1} \times [1 - [1 - (1 - \exp(-v))^2]^c]^{b-1},$$

and

$$F(x; a, b, c, \mu, \sigma) = I_{[1-(1-\exp(-v))^2]^c}(a, b),$$

respectively. □

The BTLGum distribution forms various possible shapes depended on specific parameter values which are illustrated in Figure 1.

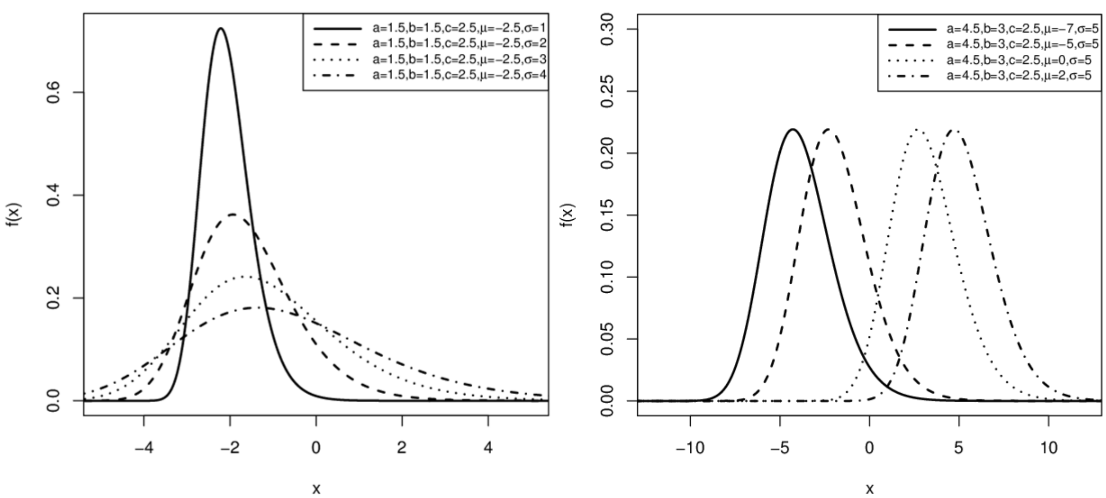

Figure 1: Some plots of BTLGum density function for some specific parameter values.

3.2 General Properties of the Beta Topp-Leone Gumbel Distribution

3.2.1 Linear representation

Some useful expansions for (3.1) and (3.2) can be derived by using the ExpGum distribution.

Proposition 3.2. *The cdf and pdf of the beta Topp-Leone Gumbel can be expressed as a linear representation of the ExpGum distribution, as follow*

$$F(x; a, b, c, \mu, \sigma) = \sum_{j,k=0}^{\infty} \sum_{m=0}^{2k} w_{j,k,m} F_{ExpGum}(x; m, \mu, \sigma), \tag{3.3}$$

and

$$f(x; a, b, c, \mu, \sigma) = \sum_{j,k=0}^{\infty} \sum_{m=0}^{2k} w_{j,k,m} f_{ExpGum}(x; m, \mu, \sigma), \tag{3.4}$$

respectively, where, for $b > 0$ real non-integer,

$$w_{j,k,m} = \frac{(-1)^{j+k+m}}{B(a,b)(a+j)}\binom{b-1}{j}\binom{c(a+j)}{k}\binom{2k}{m}.$$

If $b > 0$ is an integer, the index j in (3.3) and (3.4) stops at $b-1$, and if both a and c are integers, then the index k in the sum stops at $c(a+j)$.

Proof. First, we substitute $(1-t)^{b-1}$ under the integral by the binomial expansion and integrate to obtain

$$\int_0^x t^{a-1}(1-t)^{b-1}\mathrm{d}t = \sum_{j=0}^{\infty}(-1)^j\binom{b-1}{j}\int_0^x t^{a+j-1}\mathrm{d}t$$
$$= \sum_{j=0}^{\infty}\frac{1}{a+j}(-1)^j\binom{b-1}{j}x^{a+j},$$

where the binomial coefficient $\binom{b-1}{j} = \Gamma(b)/\Gamma(b-j)j!$ is determined by any real b. From (3.2), we obtain

$$F_{BTLGum}(x;a,b,c,\mu,\sigma) = \frac{1}{B(a,b)}\sum_{j=0}^{\infty}\frac{1}{a+j}(-1)^j\binom{b-1}{j}[1-(1-\exp(-v))^2]^{c(a+j)}.$$

Again using the binomial expansion, we can write

$$F_{BTLGum}(x;a,b,c,\mu,\sigma) = \sum_{j,k=0}^{\infty}\sum_{m=0}^{2k} w_{j,k,m}F_{\mathrm{ExpGum}}(x;m,\mu,\sigma) \tag{3.5}$$

where,

$$w_{j,k,m} = \frac{(-1)^{j+k+m}}{B(a,b)(a+j)}\binom{b-1}{j}\binom{c(a+j)}{k}\binom{2k}{m}.$$

By differentiating (3.5), we obtain

$$f_{BTLGum}(x;a,b,c,\mu,\sigma) = \sum_{j,k=0}^{\infty}\sum_{m=0}^{2k} w_{j,k,m}f_{\mathrm{ExpGum}}(x;m,\mu,\sigma).$$

□

3.2.2 Transformation

Lemma 3.3. *If a random variable B will follow beta distribution with corresponding parameters a and b, then a random variable*

$$X = \mu - \sigma\log\left\{-\log[1-(1-B^{1/c})^{1/2}]\right\} \tag{3.6}$$

follows a BTLGum distribution with parameters a, b, c, μ and σ.

3.2.3 Quantile Function

Lemma 3.4. *The quantile of the BTLGum random variable is given by*

$$x = Q(u) = \mu - \sigma\log\left\{-\log[1-(1-(I_u^{-1}(a,b))^{1/c})^{1/2}]\right\}, \quad 0 < u < 1, \tag{3.7}$$

where $I_u^{-1}(a,b)$ represents the inverse of the incomplete beta ratio function.

3.2.4 Moments

Proposition 3.5. *The rth moment of the BTLGum random variable is*

$$\mu_r^{'} = \sum_{j,k=0}^{\infty} \sum_{m=0}^{2k} w_{j,k,m} E(Z_m^r), \tag{3.8}$$

where the rth moment of the ExpGum random variable Z_θ with parameters θ, μ, and σ is

$$E[Z_\theta^r] = \theta \mu^n \sum_{j=0}^{r} \frac{(-1)^j \Gamma(r+1)}{\Gamma(r-j+1) j!} \left(\frac{\sigma}{\mu}\right)^j \left\{ \left(\frac{\partial}{\partial d}\right)^j [(\theta)^{-d} \Gamma(d)] \right\} \Bigg|_{d=1}, \tag{3.9}$$

where $\Gamma(\cdot)$ is the incomplete gamma function.

Proof. The rth moment of the ExpGum random variable (see, [4]) Z_θ with coressponding parameters θ, μ, and σ is

$$E[Z_\theta^r] = \theta \mu^n \sum_{j=0}^{r} \frac{(-1)^j \Gamma(r+1)}{\Gamma(r-j+1) j!} \left(\frac{\sigma}{\mu}\right)^j \left\{ \left(\frac{\partial}{\partial d}\right)^j [(\theta)^{-d} \Gamma(d)] \right\} \Bigg|_{d=1}.$$

From (3.4) and $E[X^r] = \int_{-\infty}^{\infty} x^r f(x) \mathrm{d}x$, we obtain

$$\mu_r^{'} = \sum_{j,k=0}^{\infty} \sum_{m=0}^{2k} w_{j,k,m} E(Z_m^r).$$

□

3.3 Maximum Likelihood Estimation

The maximum likelihood estimation (MLE) for estimating the model parameters is derived. The log-likelihood function, $\ell(\Theta; x)$, $\Theta = (a, b, c, \mu, \sigma)^T$, from a random sample $X_1, X_2, \ldots, X_n$ with pdf (3.1) reduces to

$$\begin{aligned} \ell(\Theta; x) =& n \log(2c) - n \log(\sigma) - n \log(\mathrm{B}(a, b)) - \sum_{i=1}^{n} \frac{(x_i - \mu)}{\sigma} - \sum_{i=1}^{n} v_i + \sum_{i=1}^{n} \log(1 - \exp(-v_i)) \\ &+ (ac - 1) \sum_{i=1}^{n} \log[1 - (1 - \exp(-v_i))^2] + (b - 1) \sum_{i=1}^{n} \log[1 - [1 - (1 - \exp(-v_i))^2]^c]. \end{aligned} \tag{3.10}$$

Table 1: Descriptive statistics.

Mean	Median	Std.Dev.	Variance	Skewness	Kurtosis	Min	Max
110.21	115.92	38.38	1473.134	-0.18	-1.15	43.86	186.43

The first derivative of log-likelihood function (the score function) $U(\Theta)$ is given by

$$U_a(\Theta) = n(\psi(a+b) - \psi(a)) + c\sum_{i=1}^{n} \log[1-(1-\exp(-v_i))^2],$$

$$U_b(\Theta) = n(\psi(a+b) - \psi(b)) + \sum_{i=1}^{n} \log[1-[1-(1-\exp(-v_i))^2]^c],$$

$$U_c(\Theta) = \frac{n}{c} + a\sum_{i=1}^{n} \log[1-[1-\exp(-v_i)]^2] - (b-1)\sum_{i=1}^{n} \frac{[1-(1-\exp(-v_i))^2]^c \log[1-(1-\exp(-v_i))^2]}{1-[1-(1-\exp(v_i))^2]^c},$$

$$U_\mu(\Theta) = \frac{1}{\sigma} - \frac{1}{\sigma}\sum_{i=1}^{n} v_i + \frac{1}{\sigma}\sum_{i=1}^{n} \frac{\exp\left\{-\frac{(x_i-\mu)}{\sigma} - v_i\right\}}{(1-\exp(-v_i))} + \frac{2(ac-1)}{\sigma}\sum_{i=1}^{n} \frac{v_i(1-v_i)}{(1-(1-v_i)^2)}$$
$$+ \frac{2c(b-1)}{\sigma}\sum_{i=1}^{n} \frac{\exp\left\{-\frac{(x_i-\mu)}{\sigma} - v_i\right\}(1-\exp(-v_i))[1-(1-\exp(-v_i))^2]^{c-1}}{[1-[1-(1-\exp(-v_i))^2]^c]},$$

$$U_\sigma(\Theta) = -\frac{n}{\sigma} - \frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i-\mu)v_i + \frac{1}{\sigma^2}\sum_{i=1}^{n} \frac{(x_i-\mu)\exp\left\{-\frac{(x_i-\mu)}{\sigma} - v_i\right\}}{(1-\exp(-v_i))}$$
$$+ \frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i-\mu) + \frac{2(ac-1)}{\sigma^2}\sum_{i=1}^{n} \frac{(x_i-\mu)v_i(1-v_i)}{[1-(1-v_i)^2]}$$
$$+ \frac{2c(b-1)}{\sigma^2}\sum_{i=1}^{n} \frac{(x_i-\mu)\exp\left\{-\frac{(x_i-\mu)}{\sigma} - u_i\right\}(1-\exp(-v_i))[1-(1-\exp(-v_i))^2]^{c-1}}{[1-[1-(1-\exp(-v_i))^2]^c]}.$$

where $v_i = \exp\{-(x_i-\mu)/\sigma\}$ and $\psi(s) = \mathrm{d}\log(s)/\mathrm{d}s$ is the digamma function. The MLE $\hat{\Theta}$ of Θ is obtained by solving systems of nonlinear equations $\partial\ell(\Theta; x)/\partial\Theta = 0$, which has no general explicit or closed-form solution in this study. However, $\hat{\Theta}$ can be well-performed in the `optimr` function [10] in `R` programming [11].

3.4 Application : The Minimum Flow Data

In the study regarding management and planning of the usage of water resources, the minimum flow, return level and return period are an important hydrological factor. This research aims to model the minimum flow (discharge) of at least seven consecutive days. The minimum flow data was surveyed and reported by Silveira et al.[12] in Cuiabá River, part of the Brazilian Pantanal. Descriptive statistics are provided to describe basic information about the data in Table 1.

Here, we compare the BTLGum, McGum, KumGum, BetaGum, TLGum, ExpGum and Gumbel distributions. Table 2 presents the MLEs of the parameters (the standard errors are given in parentheses). Table 3 presents the values of Akaike's information criterion (AIC), Bayesian information criterion (BIC), Akaike's information corrected criterion (AICC), Hannon and Quinn's information criterion (HQIC) and the Anderson-Darling (A^*), Cramér-von Mises (W^*) and Kolmogorov-Smirnov (K-S) goodness-of fit statistics (the p-values are given in parentheses) in order to confirm which model provide a greater fit to these data. The statistics A^*, W^* and K-S are explained by Chen and Balakrishnan [13]. The smaller the values of these statistics, the greater the fit to the data.

Table 2: MLEs of the model parameters.

Distribution	Estimates				
	a	b	c	μ	σ
BTLGum	0.093	15.543	1.708	220.750	56.645
	(0.060)	(21.202)	(0.986)	(0.950)	(1.061)
McGum	0.095	26.546	7.216	133.180	53.678
	(0.038)	(0.616)	(1.830)	(4.373)	(4.637)
KumGum	4.564	33.058	-	90.922	127.160
	(0.283)	(3.955)		(2.499)	(3.277)
BetaGum	28.254	54.610	-	129.980	267.210
	(1.470)	(1.916)		(3.083)	(6.235)
TLGum	-	-	0.307	152.970	43.543
			(0.499)	(0.446)	(0.154)
ExpGum	0.992	-	-	91.297	35.962
	(0.204)			(3.977)	(3.421)
Gumbel	-	-	-	91.003	35.962
				(6.582)	(4.246)

Based on the lowest value of the AIC statistic and the values of the statistics A^*, W^* and K-S, the BTLGum model fits this data greater than the other six models.

Figure 2 and Figure 3 illustrate the histogram and the empirical cdf of the minimum flow data. The fitted densities and cdfs are presented in Table 2.

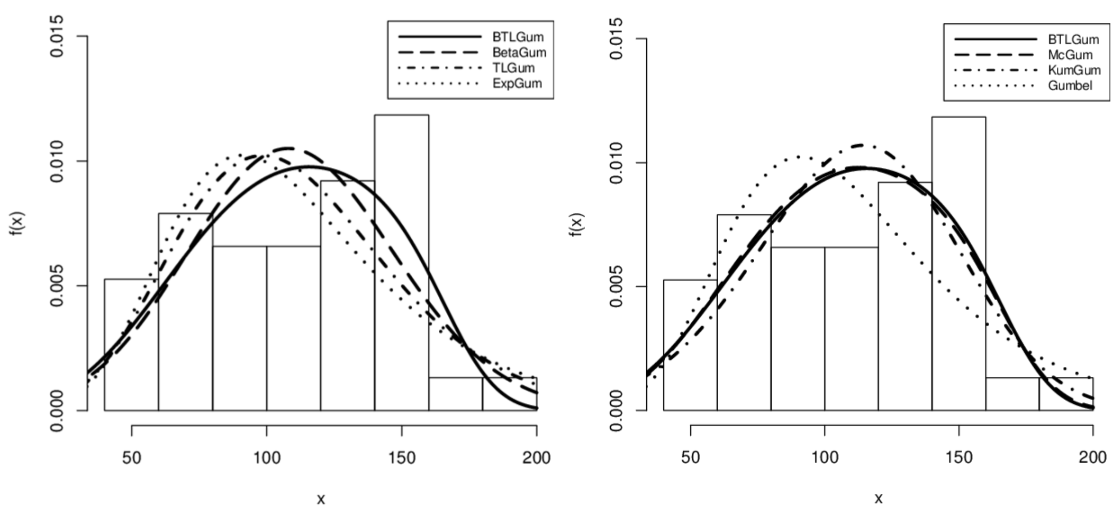

Figure 2: The histogram of the minimum flow data and plots estimated densities of the fitted models.

Table 3: The measures of $-\ell$, AIC, BIC, AICC, HQIC and goodness-of-fit tests.

Distribution	Statistics							
	$-\ell$	AIC	BIC	AICC	HQIC	A^*	W^*	K-S
BTLGum	190.480	390.960	399.140	392.830	393.870	0.439 (0.808)	0.066 (0.777)	0.127 (0.569)
McGum	190.510	391.020	399.210	392.900	393.930	0.464 (0.783)	0.072 (0.743)	0.134 (0.506)
KumGum	191.850	391.690	398.240	392.900	394.020	0.616 (0.631)	0.099 (0.592)	0.133 (0.511)
BetaGum	192.180	392.360	398.910	393.580	394.690	0.711 (0.549)	0.117 (0.511)	0.149 (0.372)
TLGum	192.970	391.930	396.850	392.640	393.680	0.893 (0.418)	0.148 (0.397)	0.163 (0.264)
ExpGum	194.430	394.870	399.780	395.570	396.620	1.121 (0.299)	0.186 (0.297)	0.165 (0.251)
Gumbel	194.430	392.870	396.140	393.210	394.030	1.121 (0.299)	0.186 (0.297)	0.165 (0.251)

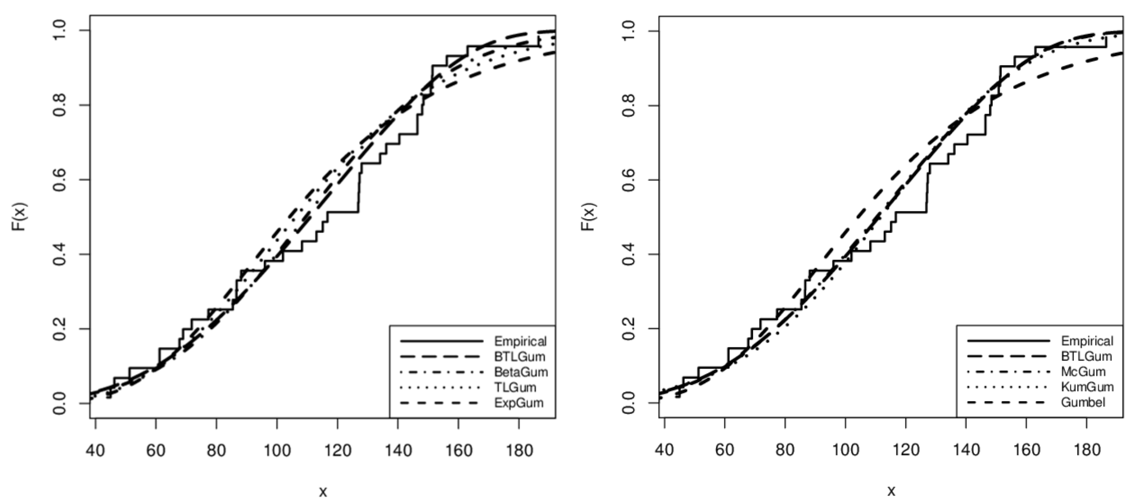

Figure 3: The empirical cdf of the minimum flow data and plots estimated cdfs of the fitted models.

4 Conclusion

The five-parameter generalized Gumbel distribution is proposed by taking the TLGum distribution in the BetaG family. Its cdf and pdf of the proposed distribution are derived. Some of its general properties, i.e. linear representation, transformation, quantile function and moments are presented. The MLE for parameters of the BTLGum distribution is described. Results of fitting the BTLGum, McGum, KumGum, BetaGum, TLGum, ExpGum and Gumbel distributions to the minimum flow data are evaluated. Considering AIC statistic and the values of the statistics A^*, W^* and K-S, it suggests that the BTLGum distribution outperform the other six distributions.

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