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A New Selecting k Method of Hill's Estimator

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Abstract : Estimating the tail index parameter is one of the primal objectives in extreme value theory (EVT). The tail index was referred directly from extreme value index (EVI) or shape parameter (ξ) of a heavy-tailed distribution. Hill's estimator is the rst estimator which appeared and modified in the literature for the extreme value index since 1975 endless to nowadays such as smooHill estimator, Hill's estimator in altscale, smooHill estimator in altscale [1], weighted Hill's estimator [2]. For heavy-tailed distributions, the Hill's estimator is still the most popular way to estimate the tail index parameter (α). Its estimate is a measure of the heaviness of the underlying distribution of tail and also asymptotically as efficient based onthe optimal index k of order statistics. Therefore, selecting the optimal a value of index k will lead to an effective estimate of the Hill's estimator. In this research investigated the graphical methods of Hill's estimator, a very common way to determine a good choice for the index k is given by the so-called Hill plot. We purpose the method of quantile estimator for choice index k, to estimate the tail index of the series, which characterizes the tail behavior, especially the speed of the tail decay. The information in this study drawn from a Pareto distribution.

Keywords : Hill's estimator; Extreme value index; Quantile estimator
2010 Mathematics Subject Classification : 47H09; 47H10 (2000 MSC)

1 Introduction

The Extreme Value Theory (EVT) is proposed for administering with modelling the effect of extreme events. It is based on statistical methods which are designed to estimate probabilities of extreme events, which methods use a limited range of data of such events that occurred in the past and are suitable for predicting events even more extreme than those previously observed. Therefore it gives the necessary guidance to evaluate the possibility of risk in improbable events. Mainly data in many various branches such as finance, hydrology, climatology, environmental sciences, and insurance are characteristics of the

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information, is often extreme values event, which the events of those random variables may be the highest or lowest data and the probability of these events is at the tail end, the probability of extreme values event is relatively large are characteristics of phenomena of the heavy tails, namely heavy-tailed distribution. The heavy-tailed distribution has included left tail heavy, right tail heavy, and both of tails heavy. For this study, we are interested in the distribution of right tail heavy.

Let $X_1, X_2, ..., X_n$ be a sample of independent and identically distribution (i.i.d) function, $F(x) = P(X \leq x), x \in R$ therefore the distribution of this function is right heavy tail when,

$$P(X > x) = \overline{F}(x) = 1 - F(x) = L(x)x^{-\alpha}, \quad x > 0, \tag{1}$$

where $\alpha > 0$ is the index of regular variation and $L(x) > 0$ is a slowly varying function (McNeil et at, 2005 [3]).

$$\lim_{x \to \infty} \frac{L(tx)}{L(t)} = 1 \tag{2}$$

The tail exponent ($\alpha > 0$) controls the rate of decay of $F(x)$ and hence characterizes its tail behaviour. Another way to express equation (1) is that $1 - F(x)$ is regularly varying with index α. A distribution $F(x)$ concentrating on $[0, \infty)$ has a tail $1 - F(x)$ which is regularly varying with index α, $\alpha > 0$ if

$$\lim_{t \longrightarrow \infty} \frac{1 - F(tx)}{1 - F(t)} = x^{-\alpha}, x > 0 \tag{3}$$

The expression on the right characterizes regularly varying functions, that means they are power functions multiplied by slowly varying functions, see also [4], [5], and [6]. The above stated power function has a negative index of variation $-1/\xi$ which says that the tail decays with rate,

$$\alpha = \frac{1}{\xi}$$

therefore the parameter of interest is directly the tail index α which is the reciprocal of the shape parameter ξ. The estimators of the tail index were presented in several studies such as Pickands estimator [7], the moment estimator [8], smooHill estimator, Hill's estimator in alt scale, smooHill estimator in alt scale [1], which one of the most popular estimators is Hill's estimator [9]. Hill's estimator of the tail index has stressed the importance to draw inference about the behaviour of the tails. The method is based on the index k of order statistics and a non-parametric approach is conditioned upon the values of order statistics which exceed a certain threshold and a problem of choosing the appropriate number of index k of upper order statistics to construct the estimator arises. This problem becomes particularly critical because the Hill's estimator has proved to be very sensitive to the choice of index k, which selects the optimal a value of index k will lead to an effective estimate of the Hill's estimator. This research has proposed modified Hill's estimator by a new approach to selecting optimal k for the purpose an alternative for choosing optimal index k, which notion is using the method of quantile estimator on the interval of stability region. This research has proposed modified Hill's estimator by a new approach to selecting optimal index k for the purpose an alternative method when using Hill's estimator, which notion is using the method of quantile estimator on the interval of the stability region of Hill plot.

2 Preliminaries

In this section, some methods from the literature are presented which are used throughout the research.

2.1 Hill's Estimator

Hill's estimator was invented by a statistician Bruce M.Hill in 1975, see more in [9]. This estimator has introduced a simple general approach to inference about the tail behaviour of the Pareto type distribution

($\xi > 0$), which has a power law form with regularly varying tails as mentioned in equation (1) and (2). The Hill's estimator is one of the popular estimators of the extreme value index (EVI) or tail index which allowed exibility in the lower tail but that ensures the power law behaviour dominates the upper tail. Clearly, this model does not have such flexible upper tail behaviour as the Generalize Pareto distribution (GPD), but it is an important special case in many applications and since a wide range of techniques has been developed for both tail index and tail fraction estimation. From equation (3), in this case the parameter $\alpha = 1/\xi > 0$ is called the tail index of F. Theorem 1.2.2 in [10] gives an equivalent form of this condition:

$$\lim_{t\longrightarrow\infty} \frac{\int_t^\infty (1-F(x))\frac{dx}{x}}{1-F(t)} = \xi$$

Now partial integration yields

$$\int_t^\infty (1-F(s))\frac{ds}{s} = \int_t^\infty (\log u - \log t)dF(u)$$

Hence we have

$$\lim_{t\longrightarrow\infty} \frac{\int_t^\infty (\log u - \log t)dF(u)}{1-F(t)} = \xi \tag{4}$$

In order to develop an estimator based on this asymptotic result, replace in (4) the parameter t by the intermediate order statistic $X_{n-k,n}$ and F by the empirical distribution function F_n. This will give the following result in Hill's estimator (1975) [9],

$$\begin{aligned}
\widehat{\xi}_{n,k}^{Hill} &:= \frac{\int_{X_{n-k,n}}^\infty \log x - \log X_{n-k,n} dF - n(X)}{1-F_n(X_{n-k,n})} \\
&= \frac{n}{k}.\frac{1}{n}\sum_{i=1}^{k} \log X_{(n-i+1,n)} - \log X_{(n-k,n)} \\
&= \frac{1}{k}\sum_{i=1}^{k} \log X_{(n-i+1,n)} - \log X_{(n-k,n)}
\end{aligned}$$

This leads to the following definition of Hill's estimator,

Definition 2.1. *Let $X_1, X_2, ..., X_n$ are i.i.d. random variables from a distribution $F(x)$ and $X_{(1)} > X_{(2)} > ... > X_{(n)}$ the order statistics. The Hill's estimator is determined by*

$$\widehat{\xi}_{n,k}^{Hill} = \frac{1}{k}\sum_{i=1}^{k} \log X_{(n-i+1,n)} - \log X_{(n-k,n)} \tag{5}$$

The estimator thus strongly depends on the select for index k, $k = 1, \ldots, n$ which represents the cut-off between the observations considered as belonging to the centre of the distribution and those pertaining to the upper tail, so that order statistics $X_{i,n}$ with $1 \le i \le k$ can be considered as extreme realizations. It is important to mention that $\widehat{\xi}_{n,k}^{Hill}$ is a consistent estimator for the tail index only if $k \to \infty$ and $k/n \to 0$ as $n \to \infty$ was proved by [11], and asymptotically normal with mean ξ and variance ξ^2/k, $N(\xi, \xi^2/k)$. If one uses a too small k, the estimator has large variance, however for too large k, the estimator is likely to be biased.

$$\sqrt{k}(\widehat{\xi}_{n,k}^{Hill} - \xi) \xrightarrow{d} N(0, \xi^2) \tag{6}$$

In this research we have written the Hill's estimator as an estimator of the shape parameter (ξ). Anyway, $\widehat{\alpha}_{n,k}^{Hill} = \frac{1}{\xi_{n,k}^{Hill}}$ can be used as an estimator of the tail index (α), so that we will talk indifferently of the Hill's estimator as estimating either the tail index or the shape parameter.

2.2 Hill Plot

In practice, we can select index k in Hill's estimator by the graph of varying Hill shape estimates depending on the cut-off level, as already described by [9] so-call Hill plot. Hill plot is the graph make of tail index estimator $\widehat{\alpha}_{n,k}^{Hill}$ or $1/\widehat{\xi}_{n,k}^{Hill}$ [5], therefore we can make the Hill plot as follows

$$\{(k, \widehat{\xi}_{n,k}^{Hill}), 1 \leq k \leq n\} \tag{7}$$

The stability of the estimate is desired thus a stable region in the graph will be considered. Markéta explained that is suggested to look for the stable part in the upper $1\% - 5\%$ of the order statistics or we can pick out a value of $\widehat{\alpha}_{n,k}^{Hill} = 1/\widehat{\xi}_{n,k}^{Hill}$ from the stable regime of this plot. Hill plot is helpful when the data comes directly from Pareto or close to Pareto distributions [12] .

The Hill plot provides a clear evidence of the value of the estimator [13]. The problem is that the graph is volatile and it is not easy to decide what the estimate should be. The sample size may just be too small. The dificulties when using the Hill estimator was summarized by [5] as follows:

1. One must get a point estimate from a graph. What value of index k should one use?
2. The graph may exhibit considerable volatility or the true answer may be hidden in the graph.
3. The Hill estimator has optimality properties only when the underlying distribution is close to Pareto. If the distribution is far from Pareto, there may be outrageous error.
4. The Hill estimator is not location invariant. A shift in location does not theoretically affect the tail index but may throw the Hill estimate way off. The lack of location in variance means the Hill estimator can be surprisingly sensitive to changes in location. In this study we use the Hill plot for selecting optimal index k of Hill's estimator.

2.3 Quantile Estimator

For selecting optimal index k in Hill's estimator, in practice we use Hill plot, which is described in section of Hill's estimator as the flexible and widely to use. In this research, we are interesting for using quantile estimator finding optimal index k on Hill plot because k is order statistics which is quantile estimator. It can explain the position of order statistic, used when those data are not the normal distribution, therefore, that is considered suitable for use with extreme value data. In this research, we used quantile estimator type 8 or $\widehat{Q}_8(p)$[14], which is the quantile of a distribution is defined as

$$Q(p) = F^{-1}(p) = \inf\{x : F(x) \geq p\}, \qquad 0 < p < 1,$$

where $F(x)$ is the distribution function. Sample quantiles provide nonparametric estimators of their population counterparts based on a set of independent observations $\{X_1, ..., X_n\}$ from the distribution $F(x)$. Let $\{X_{(1)}, ..., X_{(n)}\}$ denote the order statistics of $\{X_1, ..., X_n\}$, and let $\widehat{Q}_i(p)$ denote the ith sample quantile definition.

One difficulty in comparing quantile definitions is that there is a number of equivalent ways of defining them. However, the sample quantiles that are used in statistical packages are all based on one or two order statistics, and can be written as

$$\widehat{Q}_i(p) = (1 - \gamma)X_{(j)} + \gamma X_{(j+1)} \tag{8}$$

where

$$\frac{j - m}{n} \leq p \leq \frac{j - m + 1}{n}$$

for some $m \in \mathbb{R}$ and $0 \leq \gamma \leq 1$. The value of γ is a function of $j = \lfloor pn + m \rfloor$ and $g = pn + m - j$. Where, $\lfloor u \rfloor$ denotes the largest integer not greater than u. We also use $\lceil u \rceil$ to denote the smallest integer not less than u.

Hyndman and Yanan Fan has investigated their motivation and some of their properties. There concluded by recommending that the median-unbiased estimator is used because it has most of the desirable properties of a quantile estimator and can be defined independently of the underlying distribution [14].

The six desirable properties (P1 to P6) for a sample quantile are as follow: P1 is based on the common assumption that $Q(p)$ is a continuous function of p. P2 is the sample analog of the result $F(Q(u)) \geq u$ (with equality when $F(\cdot)$ is continuous) P3 and P4 are symmetry properties that require that the tails of the underlying distribution are treated equally. P3 is equivalent to [15] criterion B for quartiles. P5 reflects the result that for a continuous distribution, we expect there to be positive probability for value beyond the range of the data. P6 is sensible given the widespread use of the sample median.

The definitions of sample quantiles can be divided into the discontinuous functions ($\widehat{Q}_1(p)$, $\widehat{Q}_2(p)$, $\widehat{Q}_3(p)$) and the piecewise linear continuous functions ($\widehat{Q}_4(p)$, $\widehat{Q}_5(p)$, $\widehat{Q}_6(p)$, $\widehat{Q}_7(p)$, $\widehat{Q}_8(p)$, $\widehat{Q}_9(p)$). Hyndman and Yanan Fan compared the most commonly implemented sample quantile definitions (type 1 to type 9) by writing them in a common notation from equation (8) [14]. For this study, we are the focus on the piecewise linear continuous functions. The result proposes that $\widehat{Q}_8(p)$ it the best choice because $\widehat{Q}_8(p)$ satisfies in five properties and it gives (approximately) median-unbiased estimates of $\widehat{Q}(p)$ regardless of the distribution $F(x)$.

For the quantile estimator type 8 or $\widehat{Q}_8(p)$, can be explained that by definition [14] is the median position, $MF(X_{(j)})$, where M denotes the median, is more difficult to obtain. Using an approximation to the incomplete beta function ratio [16] we find

$$MF(X_{(k)}) \approx \frac{\left(k - \frac{1}{3}\right)}{\left(n + \frac{1}{3}\right)}$$

Therefore, we define the sample quantile by setting

$$p_k = \frac{\left(k - \frac{1}{3}\right)}{\left(n + \frac{1}{3}\right)}$$

In fact, the resulting sample quantile is median unbiased of order $o(n^{-1/2})$. Reiss also states that the resulting sample quantile is optimal in the class of all estimators that are median unbiased of order $o(n^{-1/2})$ and equivalent under translations (shifting the observations amounts to shifting the distribution of $\widehat{Q}(p)$ [17].

Hoaglin shows that when p is an integer multiple of $(.5)^l$ where l is an integer, $\widehat{Q}_8(p)$ gives the same results as "letter values" [18].

Benard and Bos-Levenbach also argue for $p_k = \mathrm{M}F(X_{(k)})$, but use the approximation $p_k = (k - 0.3)/(n + 0.4)$ [19].

Therefore, by the properties of quantiles estimator we use the $\widehat{Q}_8(p)$ for selecting k on Hill plot for this study.

3 Main Results

In this section, we describe the proposed methods about a new selecting index k in Hill's estimator. We incorporate the algorithm with Hill's estimator which index k from the Hill plot. We also choose index k with the method of quantile estimator type 8 as follows:

1. Investigation of the Hill's estimator and Hill plot.
2. Deriving a new approach for select index k in Hill's estimator for heavy-tailed distribution.
3. Generating random variate of the Pareto distribution for simulation study.

4. Application with real data sets for hill's estimator by using a new approach for select index k.

5. The algorithm of a new approach for select index k used in the estimation of the tail index consider the following stepwise procedure:

Step 1. Generate the random sample $X_i, i = 1, ..., n$ where each random variable has a Pareto distribution with $\lambda = 1$ and $\xi = 1, 2, 3,$ and 4. In this study, the sample sizes are determined as $n = 1000$, show in the Figure 1.

Step 2. Estimate the tail index ($\alpha = 1/\xi$) of a random sample with Pareto distribution (in Step 1) by using Hill's estimator ($\widehat{\xi}^{Hill}_{n,k}$) from Equation (5).

Step 3. Plot the graph of value k against Hill's estimator $\widehat{\xi}^{Hill}_{n,k}$, Hill plot of X is illustrated in Figure 2.

$$\{(k, \widehat{\xi}^{Hill}_{n,k}), 1 \leq k \leq n\}$$

Step 4. Consider a stable region of Hill plot derived from Step 3 to select k. An interval of index k on a stable region for the lead to optimal index k, show in the Figure 3.

Step 5. Use the quantile estimator type 8 or $\widehat{Q}_8(p)$ [14] base on quantile function derived as follows, Equation (8) for finding optimal of value index k where $p = 0.25$ and 0.75. Then we will obtain the value index k from this approach.

Step 6. Use optimal index k from Step 5 to find cut-point of the plot with Hill's estimator for describing the tail index $\widehat{\xi}^{Hill}_{n,k}$, Show in the Figure 4 and 5, respectively.

Therefore, from the algorithm presented, we will have a new approach to selecting the optimal value index k in Hill's estimator.

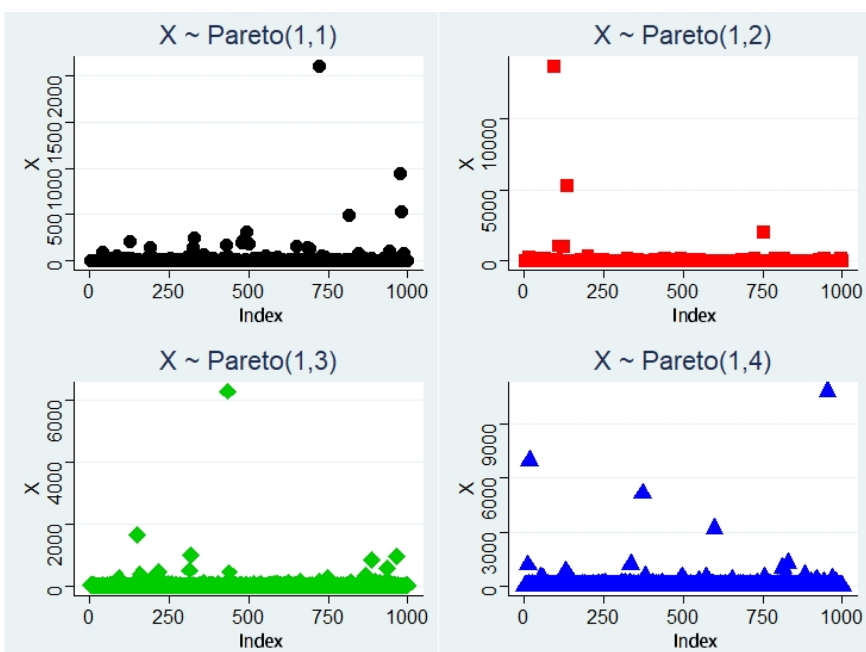

Figure 1: $X \sim Pareto\ (\lambda, \xi)\ where\ \lambda = 1,\ \xi = 1, 2, 3, 4\ and\ n = 1000$

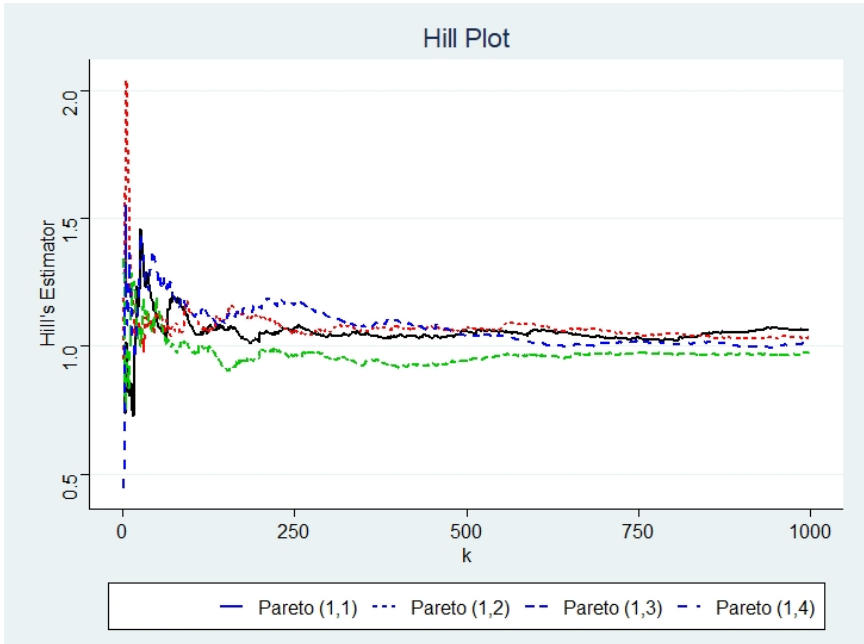

Figure 2: *Hill plot of* $X \sim$ *Pareto* (λ, ξ) *where* $\lambda = 1,\ \xi = 1, 2, 3, 4$

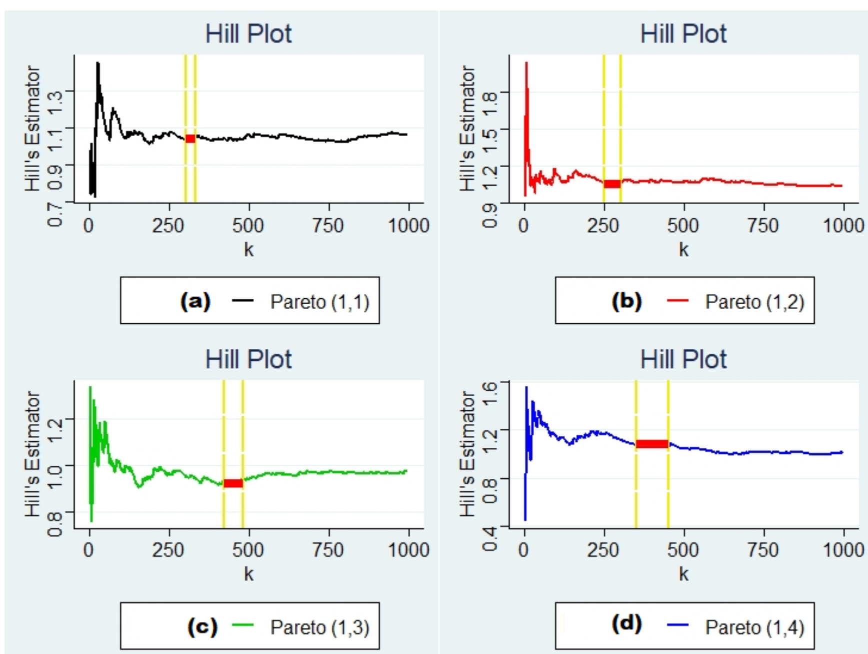

Figure 3: *The stable region on Hill plot of* $X \sim$ *Pareto* (λ, ξ)*. This Figure can be classified based on* $X \sim$ *Pareto* (λ, ξ) *where* $\lambda = 1,\ \xi = 1, 2, 3, 4$*. (a)* $X \sim$ *Pareto* $(1, 1)$ *shows a stable region for about* $[300, 330]$*. (b)* $X \sim$ *Pareto* $(1, 2)$ *shows a stable region for about* $[250, 300]$*. (c)* $X \sim$ *Pareto* $(1, 3)$ *shows a stable region for about* $[420, 480]$ *and (d)* $X \sim$ *Pareto* $(1, 4)$ *shows a stable region for about* $[374, 450]$*.*

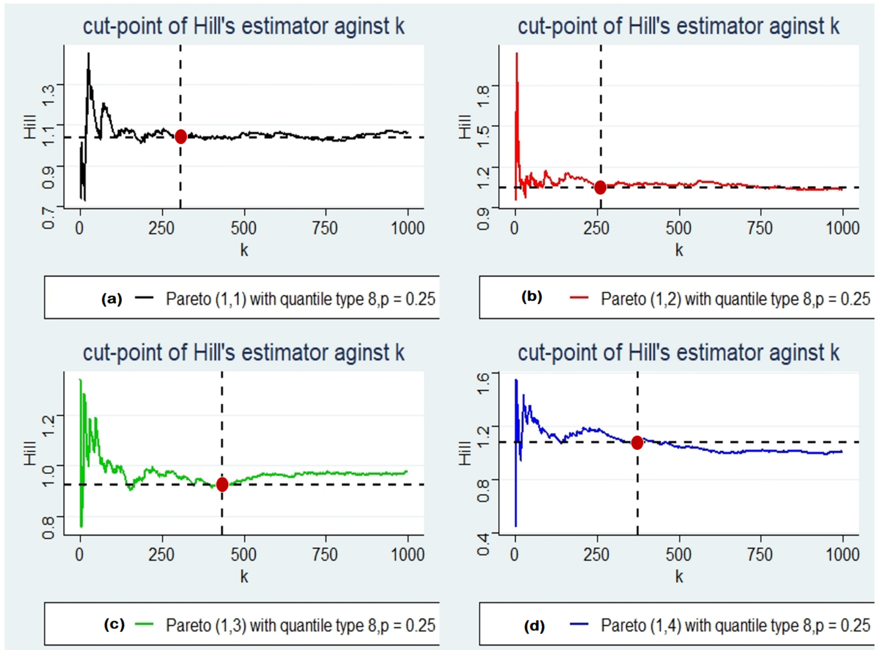

Figure 4: *The cut-point of the plot with Hill's estimator for describing the tail index where* $p = 0.25$.

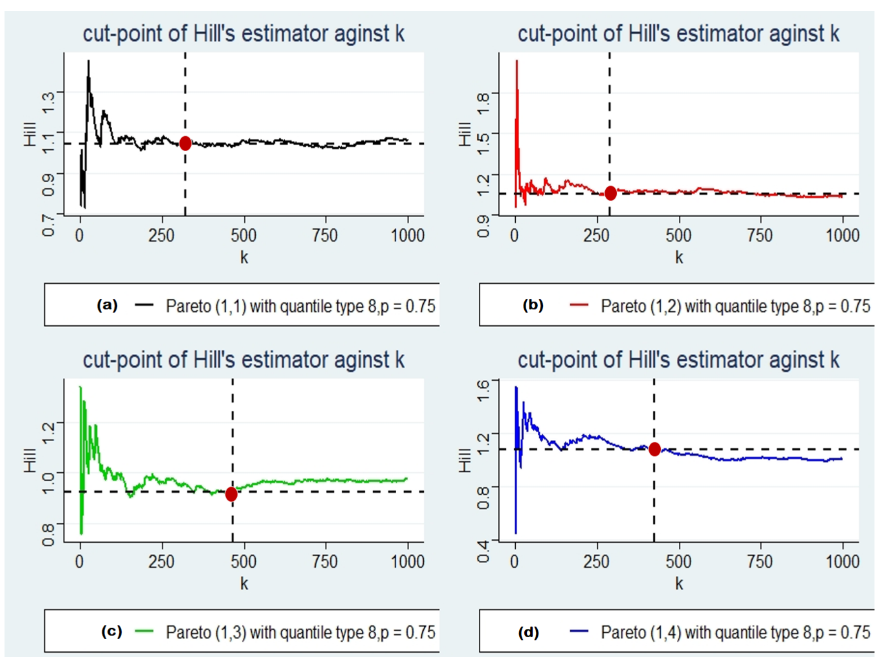

Figure 5: *The cut-point of the plot with Hill's estimator for describing the tail index where* $p = 0.75$.

See Figure 4 and 5. The cut-point of the plot with Hill's estimator after using $\widehat{Q}_8(p)$ for find optimal index k on the stable region (Figure 3.) of the plot can infer a value of $\widehat{\xi}^{Hill}_{n,k}$ and $\widehat{\alpha}^{Hill}_{n,k}$. The example of cut-point with $\widehat{Q}_8(0.25)$, Figure 4, in case of $X \sim$ Pareto $(1,1)$ shows a stable region for about $[300, 330]$ we found index k is 307.167 and the corresponding $\widehat{\xi}^{Hill}_{n,k}$ and $\widehat{\alpha}^{Hill}_{n,k}$ are approximately 1.039 and 0.962, respectively. In case of $X \sim$ Pareto $(1,2)$ shows a stable region for about $[250, 300]$ we found index k is 262.167 and the corresponding $\widehat{\xi}^{Hill}_{n,k}$ and $\widehat{\alpha}^{Hill}_{n,k}$ are approximately 1.045 and 0.957, respectively. In case of $X \sim$ Pareto $(1,3)$ shows a stable region for about $[420, 480]$ we found index k is 434.667 and the corresponding $\widehat{\xi}^{Hill}_{n,k}$ and $\widehat{\alpha}^{Hill}_{n,k}$ are approximately 0.925 and 1.081, respectively. and in case of $X \sim$ Pareto $(1,4)$ shows a stable region for about $[374, 450]$ we found index k is 374.667 and the corresponding $\widehat{\xi}^{Hill}_{n,k}$ and $\widehat{\alpha}^{Hill}_{n,k}$ are approximately 1.081 and 0.925, respectively.

The example of cut-point use $\widehat{Q}_8(0.75)$, Figure 5,in case of $X \sim$ Pareto $(1,1)$ shows a stable region for about $[300, 330]$ we found index k is 322.833 and the corresponding $\widehat{\xi}^{Hill}_{n,k}$ and $\widehat{\alpha}^{Hill}_{n,k}$ are approximately 1.041 and 0.961, respectively. In case of $X \sim$ Pareto $(1,2)$ shows a stable region for about $[250, 300]$ we found index k is 287.833 and the corresponding $\widehat{\xi}^{Hill}_{n,k}$ and $\widehat{\alpha}^{Hill}_{n,k}$ are approximately 1.053 and 0.950, respectively. But in case of $X \sim$ Pareto $(1,3)$ and in case of $X \sim$ Pareto $(1,4)$ that results as same as the case of $X \sim$ Pareto $(1,3)$ and $X \sim$ Pareto $(1,4)$ when we use $\widehat{Q}_8(0.25)$.

From the Figure 4 and 5, we can summarise and present in Table 1.

Table 1: Hill's estimator from selecting optimal index k by using cut-point of a new method which is obtained of $\widehat{Q}_8(p)$ method and Hill plot

(λ, ξ)	stable region	$\widehat{Q}_8(0.25)$	$\widehat{\xi}^{Hill}_{n,k}$	$\widehat{\alpha}^{Hill}_{n,k}$	$\widehat{Q}_8(0.75)$	$\widehat{\xi}^{Hill}_{n,k}$	$\widehat{\alpha}^{Hill}_{n,k}$
(1,1)	[300,330]	$k = 307.167$	1.039	0.962	$k = 322.833$	1.041	0.961
(1,2)	[250,300]	$k = 262.167$	1.045	0.957	$k = 287.833$	1.053	0.950
(1,3)	[420,480]	$k = 434.667$	0.925	1.081	$k = 465.333$	0.925	1.081
(1,4)	[350,450]	$k = 374.667$	1.081	0.925	$k = 425.333$	1.081	0.925

Therefore, from the algorithm presented, we will have a new approach to selecting the optimal value k of Hill's estimator.

4 Conclusion

The aim of this research is to propose a new method by using graphics for selecting index k in Hill's estimator by using a quantile estimator type 8 from the stable region of Hill plot, which will be a more flexible alternative for use in Hill's estimator. Expected results: a new approach for select index k is compatible with Hill's estimator and compatible with another estimator, which depends on the order statistical index k for heavy-tailed distributions is lead to classifying behaviour of the tail probability.

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