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Max(Min)C_{11} Modules with Their Endomorphism Rings

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Abstract : In this paper, we consider the class of rings and modules with extending properties, and study an intensive class of max(min)C_{11} modules together with their endomorphism rings. An $R-$module M is maxC_{11} module if every maximal submodule with nonzero left annihilator has a complement which is a direct summand of M. M is called a minC_{11} if every minimal submodule has a complement which is a direct summand of M. We prove that if M is a finitely generated, quasi-projective self-generator, then M is C_{11} (resp. maxC_{11} , minC_{11} , max-minC_{11}) module if and only if its endomorphism ring S is a right C_{11} (resp. maxC_{11}, minC_{11}, max-minC_{11}) ring. If M is a prime module, then M is nonsingular, max-min C_{11} with a uniform submodule if and only if S is right and left nonsingular, right and left max-minC_{11} with uniform right and left ideals. Moreover, if M is a semiprime, weak duo module, then M is maxC_{11} if and only if it is minC_{11}.

Keywords : maxC_{11} module; minC_{11} module; max-minC_{11} module
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1 Introduction

Smith and Tercan [1], [2] defined C_{11} module as follows, an $R-$module M is called a C_{11} module, if every submodule of M has a complement which is a direct summand of M, i.e., for each submodule N of M there exists a direct summand K of M such that K is a complement of N in M. C_{11} modules were defined as a general of CS modules. They studied C_{11} modules and found many properties of C_{11} modules as follows, any direct sum of modules with C_{11} satisfies C_{11}. Moreover, a module M satisfies C_{11} if and only if $M = \mathbb{Z}_2(M) \oplus K$ for some nonsingular submodule K of M and both $\mathbb{Z}_2(M)$ and K satisfy C_{11}.

Throughout this paper R is an associative (not necessarily commutative) ring with identity and all modules are unitary. Let M be a right R-module. For a submodule (resp. essential submodule) X of M, we write $X \leqslant M$ (resp. $X \leqslant_e M$). According to [1], a submodule X of M is called a closed submodule if X has no proper essential extension in M, that is, for any submodule Y of M such that X is essential in Y then $X = Y$. Recall that for a given submodule X of M , a submodule Y of M is called a complement of X in M if Y is maximal with respect to $Y \cap X = 0$. Complements are exactly closed submodules.

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Let M be a right R-module with $S = End(M_R)$, its endomorphism ring. M is a *self-generator* if for every submodule X of M, we have $X = \sum_{f\in I} f(M)$ for some subset $I \subset S$. We denote $I_U = \{f \in S|f(M) \subseteq U\}$ for a submodule U of M, and $JM = J(M) = \sum_{f\in J} f(M)$ for a subset $J \subset S$. It is clear that I_U is a right ideal of S and JM is a submodule of M. A submodule $X \leqslant M$ is called a *fully invariant submodule* if $s(X) \subseteq X$ for every $s \in S$. M is called a *duo module* (resp. *weak duo module*) if every submodule (resp. every direct summand) is fully invariant. R is called a *right duo ring* (resp. *right weak duo ring*) if R_R is a duo module (resp. weak duo module), equivalently, every right ideal (resp. every right ideal generated by an idempotent) of R is two-sided.

For primeness in modules, we adopt the notions of N. V. Sanh et al. in [3],[4]. A fully invariant submodule X of M is called a *prime submodule* if for every fully invariant submodule U of M, any ideal K of $S, K(U) \subseteq X$ implies either $K(M) \subseteq X$ or $U \subseteq X$. A fully invariant submodule X of M is called a *semiprime submodule* if it is an intersection of prime submodules of M. A right $R-$module M is called a *prime (semiprime) module* if the zero submodule of M is prime (semiprime) in M.

We denote $r_X(Y)$ and $l_X(Y)$ for the right annihilator and the left annihilator of Y in X, respectively. If there is no ambiguity of the space X, then we simply write $r(Y), l(Y)$.

2 Preliminaries

First, we need to prepare some tools in order to develop our investigations in the next sections. Some results are employed from other authors.

S. M. Khuri [5], [6] investigated preservation of essentiality and closeness between submodules of a module M and corresponding ideals of its endomorphism ring. In particular, [6, Proposition 3.2] considered the case of nondegenerate modules. We do obtain similar results in the case of finitely generated, quasi-projective self-generators in the following lemmas.

Lemma 2.1. (Thuat, Hai, Nghiem and Chairat [Lemma 2.1][7]) *Let M be a finitely generated, quasi-projective right $R-$module which is a self-generator with the endomorphism ring S. The following statements hold for the module M.*

(1) X is a closed submodule of M if and only if $I_X = \{f \in S|f(M) \subseteq X\}$ is a closed right ideal of S.

(2) Conversely, K is a closed right ideal of S if and only if $KM = \sum_{s\in K} s(M)$ is a closed submodule of M.

Lemma 2.2. (Thuat, Hai, Nghiem and Chairat [Lemma 2.2][7]) *Let M be a finitely generated, quasi-projective right $R-$module which is a self-generator with the endomorphism ring S. The following statements hold:*

(1) U is a uniform submodule of M if and only if $I_U = \{f \in S|f(M) \subseteq U\}$ is a uniform right ideal of S.

(2) K is a uniform right ideal of S if and only if $KM = \sum_{f\in K} f(M)$ is a uniform submodule of M.

Lemma 2.3. (Thuat, Hai, Nghiem and Chairat [Lemma 2.3][7]) *Let M be a finitely generated, quasi-projective right $R-$module which is a self-generator with the endomorphism ring S. The following statements hold for the module M.*

(1) X is a maximal (resp. minimal) closed submodule of M if and only if $I_X = \{f \in S|f(M) \subseteq X\}$ is a maximal (resp. minimal) closed right ideal of S.

(2) Conversely, K is a maximal (resp. minimal) closed right ideal of S if and only if $KM = \sum_{s\in K} s(M)$ is a maximal (resp. minimal) closed submodule of M.

Lemma 2.4. (Thuat, Hai, Nghiem and Chairat [Lemma 2.4][7]) *X is a direct summand of the module M if and only if $I_X = \{f \in S|f(M) \subseteq X\}$ is a direct summand of S. In this case, $X = e(M)$ and $I_X = eS$ for some idempotent $e \in S$.*

Lemma 2.5. (Thuat, Hai, Nghiem and Chairat [Lemma 2.5][7]) *The module M is a duo (resp. weak duo) if and only if S is a right duo (resp. right weak duo) ring.*

3 MaxC_{11} modules and minC_{11} modules

H. Rayalong and S. Chairat [8] defined minC_{11} module and maxC_{11} module as follows, an $R-$module M is said to be minC_{11} module, if every minimal submodule has a complement which is a direct summand of M, i.e., for each minimal submodule N of M there exists a direct summand K of M such that K is a complement of N in M. A ring R is minC_{11} if it is minC_{11} $R-$module.

An $R-$module M is said to be maxC_{11} module, if every maximal submodule with nonzero right annihilator has a complement which is a direct summand of M, i.e., for each minimal submodule L of M with nonzero right annihilator there exists a direct summand K of M such that K is a complement of L in M. A ring R is maxC_{11} if it is maxC_{11} $R-$module.

Every C_{11}-module is minC_{11} and maxC_{11} because any submodule has a complement which is a direct summand. But conversely is not true in general. Every CS-module is minC_{11} and maxC_{11}, since every CS-module is C_{11}. Every simple module is minC_{11} and maxC_{11}. In particular, $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_6$, $\mathbb{Z}_{10}$ as a $\mathbb{Z}-$module is minC_{11} and maxC_{11}. Moreover, every uniform module is minC_{11} and maxC_{11}.

Theorem 3.1. *Let M be a finitely generated, quasi-projective right $R-$module which is a self-generator. Then M is a C_{11} (resp. maxC_{11}, minC_{11}, max-minC_{11}) module if and only if S is a right C_{11} (resp. right maxC_{11}, right minC_{11}, right max-minC_{11}) ring.*

Proof. Let M be a C_{11} module and K be a right ideal of S. Since $K(M)$ is a submodule of M, there exists a direct summand X of M such that X is a complement of $K(M)$ in M. We have X is closed, then by Lemma 2.1 and Lemma 2.4 I_X is a closed right ideal of S such that I_X is a direct summand of K. Conversely, let S be a right C_{11} ring, and N be any submodule of M. We have $K(N)$ is a submodule of M for any right ideal K of S. Then there is a complement I_X of K which is a direct summand for some submodule X of M. Then, by Lemma 2.1 and Lemma 2.4 X is a closed submodule of M which is a direct summand. Similarly, the case of minC_{11} property is deduced from Lemma 2.3, and 2.4.

We assume that M is maxC_{11}. For every maximal right ideal K of S with nonzero left annihilator in S, $K(M)$ is a maximal submodule of M by Lemma 2.3. Since K has nonzero left annihilator, there is some $0 \neq f \in S$ such that $fK = 0$, whence $K(M)$ has nonzero left annihilator in S (in deed, $fK(M) = 0$). Thus $K(M)$ is a direct summand of M, that is $K(M) = e(M)$ for some idempotent $e \in S$, by Lemma 2.4. Consequently, $K = eS$ is a direct summand of S, showing that S is right maxC_{11} . Conversely, for an arbitrary maximal submodule X of M with nonzero left annihilator in S, $I_X = \{s \in S | s(M) \subseteq X\}$ is a maximal right ideal of S with nonzero left annihilator in S. Therefore, if S is right maxC_{11} , then I_X is a direct summand of S, whence X is a direct summand of M. This implies that M is maxC_{11} . □

The next theorem extends this result to noncommutative rings, even more general, to modules over associative rings. Note that every commutative ring is right and left duo, and a commutative ring is semiprime if and only if it is nonsingular. The following lemma is needed to prove our next theorem.

Lemma 3.2. *For every closed submodule X of M and Y, a complement of X in M, X is a maximal (resp. minimal) closed if and only if Y is minimal (resp. maximal) closed.*

Proof. Let X be a closed submodule of M and Y, a complement of X. Then, Y is closed in M.

We suppose that X is maximal closed. Then $X \neq M$ implies $Y \neq 0$. In order to prove that Y is minimal closed, it is sufficient to show that Y is uniform. Assuming that A, B are nonzero submodules of Y. If $A \cap B = 0$, then there exists a closed submodule $0 \neq C \leqslant Y$ such that $A \subseteq C$ and $C \oplus B \leqslant_e Y$. We have $C \oplus B \oplus X \leqslant_e Y \oplus X \leqslant_e M$. Thus, there is a complement D of C, where $D \supseteq B \oplus X, D \neq X$. This is contradict to maximality of X. Therefore, $A \cap B \neq 0$ must be hold, proving that Y is uniform.

Next, we assume that X is minimal closed. For a closed submodule A of M such that $A \neq M$ and $Y \subseteq A$, if $B = A \cap X \neq 0$, then $B \leqslant_e X$ because of minimality of X. Thus $B \oplus Y \leqslant_e X \oplus Y \leqslant_e M$. Since $B \subseteq A, Y \subseteq A$, we have $B \oplus Y \subseteq A \leqslant_e M$, contradict to closeness of A. Therefore, $B = 0$ must be hold, that is $A = Y$. This shows that Y is maximal closed in M.

□

Theorem 3.3. *Let M be a finitely generated, quasi-projective right $R-$module which is a self-generator. If M is a semiprime, weak duo module, then the following conditions are equivalent:*

(1) M is maxC_{11} ;

(2) S is right maxC_{11} ;

(3) S is right minC_{11} ;

(4) M is minC_{11} .

Proof. Since M is semiprime, S is semiprime . Since M is weak duo, S is right weak duo by Lemma 2.5. Thus every right ideal generated by an idempotent of S in the following is two-sided.

The implications *(1)⇔(2)* and *(3)⇔(4)* follow from Theorem 3.1

(2)⇒(3) Let X be a minimal right ideal of S. Since S is semiprime, $Y = r(X)$ is the unique complement of X in S by [4, Theorem 3.2]. Moreover, Y is a maximal right ideal of S by the preceding lemma, and $Y \neq S$. We will show that $l(Y) \neq 0$. In contrary, if $l(Y) = 0$, then $Y = rl(Y) = S$, a contradiction. Thus we have $l(Y) \neq 0$. Since S is right max C_{11}, Y is a direct summand, writing $Y = eS$ for some idempotent $e \in S$. In addition, $r(Y)$ is again the unique complement of Y in S, and hence $X = r(Y)$. Since Y is two-sided, $Se = eS = Y$, so $(1 - e)S \subseteq r(Y) = X$. Therefore, we have $(1 - e)S = X$ by minimality of X. This means that X is a direct summand of S,, so Y is a complement of X which is a direct summand whence S is right minC_{11}.

(3)⇒(2) Let X be a maximal closed right ideal of S with nonzero left annihilator. Since S is semiprime, $Y = r(X)$ is the unique complement of X in S. Moreover, Y is a minimal right ideal of S by Lemma 2.3. Since S is right min C_{11}, Y is a direct summand. Note that S is right weak duo, hence $Y = eS = Se$ for some idempotent $e \in S$. In addition, $r(Y)$ is again the unique complement of Y in S, thus $X = r(Y)$. We observe that $Y \oplus (1-e)S = eS \oplus (1-e)S = S$, so $(1-e)S = r(Y) = X$. This implies that X is a direct summand of S, showing that S is right max C_{11}. The proof is now completed. □

Corollary 3.4. *Let R be a semiprime, right weak duo ring. Then, R is right maxC_{11} if and only if it is right minC_{11}.*

Proof. (⇒) Let R be a right maxC_{11} ring and I be a minimal right ideal in R. Since R is right weak duo, I is two-sided. By [4], $I = annannI$, hence $annI \neq 0$, but R semiprime, which implies $I \cap annI = 0$. Let J be a relative complement of I, so J is maximal ideal in R with respect to $I \cap J = 0$. Since R is maxC_{11} , I is a direct summand of R. Then we have , J is a complement of I which is a direct summand.

(⇐) Let R be a right minC_{11} ring and I be a maximal ideal in R, with $annI \neq 0$. By [9], $I = annannI$. Let J be a relative complement of I. Then J is closed in R, so I is a minimal in R by [9]. Since R is right minC_{11} , J is a direct summand of R. □

Corollary 3.5. *A commutative nonsingular ring is maxC_{11} if and only if it is minC_{11}.*

Proof. Since a nonsingular ring is semiprime and commutative ring implies a right weak duo ring , so the proof follow from Corollary 2.4. □

Corollary 3.6. *Let M be a right $R-$module and $S = End(M_R)$. Assuming that one of the following conditions is satisfied:*

(1) M is a free module which is a self-generator;

(2) R is semiprime, M is a torsionless or projective module which is a self-generator;

(3) M is a generator.

Then M is maxC_{11} (resp. minC_{11}, max-minC_{11}) if and only if S is right maxC_{11} (resp. right minC_{11}, right max-minC_{11}).

Proof. It is clear. □

By Theorem 3.3 , since maxC_{11} or minC_{11} are equivalent for a finitely generated, quasi-projective, semiprime, duo module which is a self-generator. Next, we provide a further study of maxC_{11} modules and minC_{11} modules. We consider some properties in minC_{11} modules that may not share with maxC_{11} modules and vice versa.

Proposition 3.7. *Let M be a max(resp. min)C_{11}, right $R-$module. If a right $R-$module N is isomorphic to M, then N is also max(resp. min)C_{11}.*

Proof. It is clear. □

Proposition 3.8. *Let M be a right $R-$module.*

(1) M is minC_{11} if and only if for every minimal submodule $A \leqslant M$, there exist submodules M_1, M_2 of M such that $A \leqslant M_1$, M_2 is a complement of A and $M_1 \oplus M_2 = M$.

(2) If M is minC_{11}, then so is every submodule, and hence every direct summand of M.

Proof. *(1)* It is clear.

(2) Let M be minC_{11} and A, a submodule of M. We need to prove that A is again minC_{11}. For every minimal submodule B of A, B is also minimal in M, so $B \oplus C = M$ for some $C \leqslant M$. Thus $A = A \cap (B \oplus C) = B \oplus (A \cap C)$, hence B is a direct summand of A. This implies that A is a min C_{11} module. The case of direct summands is obvious, completing the proof. □

Proposition 3.9. *Let M be a finitely generated right $R-$module. If every maximal submodule of M is a direct summand, then M is a maxC_{11} module.*

Proof. Let A be a maximal submodule of M. Since M is finitely generated, there is a maximal submodule B of M such that $A \subseteq B$. Then, by assumption B is a direct summand of M and $A = B$ since A is maximal. Then there exist a complement submodule C of M such that $A \bigoplus C = M$, proving that M is maxC_{11} . □

Example 3.10. *Let $\mathbb{Z}$ be the set of all integers. Then, for a given prime number p, we consider $\mathbb{Z}-$modules, $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}, \mathbb{Z}_{p^3} = \mathbb{Z}/p^3\mathbb{Z}, M = \mathbb{Z}_p \oplus \mathbb{Z}_{p^3}$. Clearly, $\mathbb{Z}_p$ and $\mathbb{Z}_{p^3}$ are CS modules so is C_{11} and min C_{11}. We observe that $A = (1 + p\mathbb{Z}, p + p^3\mathbb{Z})\mathbb{Z}$ is uniform and closed in M, but cannot be a direct summand because it has order p^2 (also see [10]). Thus M is not minC_{11} but A is uniform so is C_{11} and hence minC_{11}. This example also shows that a non-minC_{11} module may have minC_{11} submodules, and a direct sum of minC_{11} modules needs not to be minC_{11}. An other simple case is that Z is uniform hence is C_{11} but no maximal submodule of $\mathbb{Z}$ is direct summand. Thus the converse of Proposition* 3.9 *is not true.*

4 Max C_{11} and min C_{11} properties in nonsingular prime modules

In this section, M is a finitely generated, quasi-projective right $R-$module which is a self-generator with the endomorphism ring $S = End(M_R)$. Note that we regularly refer readers to some results in [5], [6] with requirement of retractability. This condition is automatically satisfied when M is a self-generator. By [3, Theorem 2.4], M is a prime module if and only if S is a prime ring. With the aid of results in section 3, we are going to generalize the results in [11] to nonsingular prime modules in this section.

Let M be a right $R-$module with $S = End(M_R)$, its endomorphism ring. *Uniform dimension* (or *Goldie dimension*) of M is denoted by $udim(M_R)$. M is a *self-generator* if for every submodule X of M, we have $X = \sum_{f \in I} f(M)$ for some subset $I \subset S$. According to [12], M is called *nonsingular* if the only submodule of M with essential right annihilator in R is zero, that is for any $X \leqslant M$, $r_R(X) \leqslant_e R$ implies $X = 0$. M is said to be *co-nonsingular* if the only submodule of M with essential left annihilator in S is zero, that is for any $X \leqslant M$, $l_S(X) \leqslant_e S$ implies $X = 0$. It is easy to see that if M is co-nonsingular, then every essential right ideal K of S has zero kernel (i.e. zero right annihilator) in M, that is $r_M(K) = \{m \in M | f(m) = 0, \forall f \in K\} = 0$.

Lemma 4.1. *If M is a minC_{11}, nonsingular and prime module with a uniform submodule, then S is right minC_{11}, right and left nonsingular.*

Proof. By [5, Theorem 3.1], S is right nonsingular. By Theorem 3.1, S is right minC_{11}. By assumption, M has a uniform submodule, namely U. Then $I = I_U = \{f \in S | fM \subseteq U\}$ is a uniform right ideal of S by Lemma 2.2. Since S is a right nonsingular, right minC_{11}, prime ring with a uniform right ideal, S is left nonsingular by Lemma 6. □

Proposition 4.2. *If M is a maxC_{11}, prime, nonsingular and co-nonsingular module with a uniform submodule, then S is right maxC_{11} and left minC_{11}.*

Proof. Firstly, we see that S is a prime ring. By Lemma 2, since M has a uniform submodule, S has a uniform right ideal. By [5, Theorem 3.1], nonsingularity of M implies that S is right nonsingular. By [12, Proposition 1], since M is co-nonsingular, S if left nonsingular hence is nonsingular. By Theorem 3.1, since M is a maxC_{11} module, S is a right max C_{11} ring. Therefore, S is a left minC_{11} ring by Lemma 7. □

Proposition 4.3. *If M is a C_{11}, nonsingular and prime module with a uniform submodule, then S is right C_{11} and left minC_{11}.*

Proof. Clearly, M is a minC_{11} module. Therefore, Lemma 4.1 claims that S is left nonsingular. Thus M is co-nonsingular by [12, Proposition 1]. It follows from Proposition 4.2 that S is left minC_{11}. Finally, S is right C_{11} by Theorem 3.1. □

Lemma 4.4. *If R is a non-domain ring, and M is a nonsingular, minC_{11} and prime module with a uniform submodule, then S is a right minC_{11} ring with uniform right and left ideals.*

Proof. It is an easy verification that S is prime due to [3] and is right nonsingular due to [5]. In addition, S is right minC_{11} by Theorem 3.1, and S has a uniform right ideal by Lemma 2.2. Consequently, S has a uniform left ideal by Lemma 8. Note that in this lemma 8, S is not a domain. □

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