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Unitary Analogues of Some Arithmetic Functions

Pussadee Yangklan[†,1], **Vichian Laohakosol**[‡], **Pattira Ruengsinsub**[‡] **and Sukrawan Mavecha**[†]

[†]Department of Mathematics, Faculty of Science,
King Mongkut's Institute of Technology Ladkrabang, Bangkok 10520, Thailand
e-mail : pussadee_9@hotmail.com and sukrawan.ta@kmitl.ac.th
[‡]Department of Mathematics, Faculty of Science,
Kasetsart University, Bangkok 10900, Thailand
e-mail : fscivil@ku.ac.th and fscipar@ku.ac.th

Abstract : A number of arithmetic functions, referred to as the generalized unitary-Euler's totient, generalized unitary-Cohen's totient, generalized unitary-divisor, generalized unitary-Liouville, odd-phi, and even-phi functions which generalize the classical totient, divisor and Liouville functions, are introduced in the setting of unitary convolution. Basic properties of these functions extending the existing ones are established. Results related to the problem of counting exponentially odd and exponentially even numbers are derived as applications.

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1 Introduction

An arithmetic function, [1], is a complex-valued function defined on the set of positive integers. The set of arithmetic functions, $\mathcal{A}$, with addition $+$, and unitary convolution $\sqcup$ defined, respectively, by

$$(f+g)(n) = f(n) + g(n), \quad (f \sqcup g)(n) = \sum_{d||n} f(n/d)\, g(d),$$

where $d||n$ denotes the *unitary divisor* (i.e., those divisors d of n for which $\gcd(d, n/d) = 1$), is a commutative ring with zero divisors [2]. The identity element under the unitary convolution is the function

$$I(n) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1. \end{cases}$$

A non-zero arithmetic function f is said to be multiplicative if

$$f(mn) = f(m)f(n) \quad \text{whenever } \gcd(m, n) = 1.$$

Note that a unitary convolution of two multiplicative functions is a multiplicative function.

[1]Corresponding author.

A modern systematic study of unitary convolution seems to be started by Cohen. In [3], mimicking the classical case, he introduced the unitary Euler's totient as follows: for $a, b \in \mathbb{Z}$ with $b > 0$, denote by $(a, b)_*$ the greatest divisor of a which is a unitary divisor of b; when $(a, b)_* = 1$, the integer a is said to be *semiprime* to b. *The unitary Euler's totient*, $\overline{\varphi}$, is defined to be the number of positive integers that are semiprime to n, i.e.,

$$\overline{\varphi}(n) = \sum_{\substack{x \leq n \\ (x,n)_*=1}} 1.$$

A semi-reduced residue system mod n, denoted by $SRRS(n)$, is a set of integers in a residue system mod n that are semiprime to n. In the same paper, Cohen introduced a unitary Ramanujan sum $c^*(m, n)$ by

$$c^*(m,n) = \sum_{x \in SRRS(n)} \exp(2\pi i m x/n)$$

where the summation extends over the integers x in a semi-reduced residue system mod n. Based upon this notion, we observe that

$$\overline{\varphi}(n) = c^*(0, n).$$

Cohen also defined

$$\overline{\mu}(n) = c^*(1, n)$$

and proved a unitary analogue of the Möbius inversion formula which states that for $f, g \in \mathcal{A}$, we have

$$f(n) = \sum_{d||n} g(d) \iff g(n) = \sum_{d||n} \overline{\mu}(d) f(n/d),$$

as well as several identities relating to these functions such as

$$\sum_{d||n} \overline{\varphi}(d) = n, \qquad \sum_{d||n} \overline{\mu}(d) = I(n), \qquad \overline{\varphi}(n) = (\zeta_1 \sqcup \overline{\mu})(n), \qquad \overline{\mu}(n) = (-1)^{\omega(n)} \tag{1.1}$$

where $\zeta_1(n) := n$, and $\omega(n)$ denotes the number of distinct prime factors of n with $\omega(1) = 0$. In addition, Cohen introduced the concept of an exponentially odd number which is an integer $n \in \mathbb{N}$ whose prime factorization takes the form $n = p_1^{a_1} \ldots p_s^{a_s}$ with all powers a_i being odd positive integers. Denote by E_o the set of all exponentially odd numbers. Observe that $n \in E_o$ whenever its greatest unitary square divisor (the largest unitary divisor which is a square) is 1, and so $1 \in E_o$ (because its greatest unitary square divisor is 1).

Apart from Cohen's work, Rao in [4] gave the following extension of Cohen's totient. For $n, m, k \in \mathbb{N}$, let $(n, m^k)_k^*$ denote the largest unitary divisor of m^k that divides n and is a kth power. Denote *the unitary analogue of Cohen's totient*, $\varphi_k^*(m)$, as the number of positive integers $n \leq m^k$ such that $(n, m^k)_k^* = 1$. Rao proved that

$$\varphi_k^*(m) = \sum_{d||m} d^k \overline{\mu}(m/d) = (\zeta_k \sqcup \overline{\mu})(m) = m^k \prod_{p|n} \left(1 - \frac{1}{p^{k\nu_p(m)}}\right), \tag{1.2}$$

where $\zeta_k(n) := n^k$ and $\nu_p(m)$ denotes the highest power of the prime p that divides m. This result implies at once that φ_k^* is a multiplicative function and obviously that $\varphi_1^* = \overline{\varphi}$. Let $\overline{d}(n)$ denote the number of unitary divisors of n, and let $\overline{\sigma}_k(n)$ denote the sum of the kth power of the unitary divisors of n (see also [5]), i.e.,

$$\overline{d}(n) = \sum_{d||n} 1 = 2^{\omega(n)}, \quad \overline{\sigma_k}(n) = \sum_{d||n} d^k = (\zeta_k \sqcup U)(n).$$

where $U(n) := 1$ for all $n \in \mathbb{N}$. Rao also established the identities

$$\overline{\sigma}_k(n) = \sum_{d||n} \varphi_k^*(n/d) \overline{d}(d), \quad \sum_{d||n} \overline{\sigma}_{s+k}(d) \varphi_k^*(n/d) = n^k \overline{\sigma}_s(n).$$

Recently, in [6], a *generalized unitary Möbius function* was introduced via

$$\overline{\mu}_\alpha(n) = (-\alpha)^{\omega(n)} \quad (n \in \mathbb{N},\ \alpha \in \mathbb{C} \setminus \{0\}),\ \ \overline{\mu}_0 := I,\ \overline{\mu}_1 := \overline{\mu},\ \overline{\mu}_{-1} = U.$$

The main objectives in [6] were
(i) to prove the following generalized unitary Möbius inversion formula: for $f, g \in \mathcal{A}$ and $\alpha \in \mathbb{C} \setminus \{0\}$, we have

$$f(n) = \sum_{d||n} g(d)\overline{\mu}_{-\alpha}\left(\frac{n}{d}\right) \Longleftrightarrow g(n) = \sum_{d||n} f\left(\frac{n}{d}\right)\overline{\mu}_\alpha(d),\ \text{or symbolically,}\ f = g \sqcup \overline{\mu}_{-\alpha} \Longleftrightarrow g = f \sqcup \overline{\mu}_\alpha,$$

(ii) to derive some characterizations of multiplicative function using generalized unitary Möbius function.

Here, we continue our unitary investigation by further generalizing the unitary totient, unitary divisor and unitary Liouville functions. This is done by introducing the so-called gu-Euler's totient, gu-Cohen's totient, gu-divisor, gu-Liouville, odd-phi, and even-phi functions (the abbreviation gu stands for generalized unitary). Apart from deriving properties of these functions which extend the existing ones, in the last section, we apply these results to functions that are related to the problem of counting exponentially odd and exponentially even numbers.

2 Preliminaries

There are two parts in this section; generalized unitary analogues are defined in the first part and their properties are established in the second part.

2.1 Definitions

Let $\alpha \in \mathbb{R}$.
1. The **gu-Euler's totient**, $\overline{\varphi}_\alpha$, is defined by

$$\overline{\varphi}_\alpha = \zeta_1 \sqcup \overline{\mu}_\alpha.$$

2. The **gu-Cohen's totient**, φ^*_α, is defined by

$$\varphi^*_\alpha = \zeta_\alpha \sqcup \overline{\mu}.$$

3. The **gu-divisor function**, $\overline{\sigma}^\alpha(n)$, is defined to be the sum of the α^{th} power of positive unitary divisor d of n, i.e.,

$$\overline{\sigma}^\alpha(n) = \sum_{d||n} d^\alpha \qquad (n \in \mathbb{N}).$$

4. The **gu-Liouville function**, $\overline{\lambda}_\alpha$, is defined by

$$\overline{\lambda}_\alpha(n) = (-\alpha)^{\Omega(n)} \quad (n \in \mathbb{N}), \quad \overline{\lambda}_0 := I,$$

where $\Omega(n)$ is the number of prime factors of n counting with multiplicity with $\Omega(1) = 0$.

The following facts are easily checked.

- The gu-Euler's totient, gu-Cohen's totient, gu-divisor function and gu-Liouville function are multiplicative functions.
- $\overline{\varphi}_\alpha(n) = \begin{cases} \sum_{d||n} d(-\alpha)^{\omega(n/d)} & \text{if } \alpha \in \mathbb{R} \setminus \{0\} \\ \zeta_1(n) & \text{if } \alpha = 0. \end{cases}$
- For $\alpha \in \mathbb{N}$, the function φ^*_α is identical with the unitary analogue of the Cohen's totient.
- $\overline{\sigma}^\alpha = \zeta_\alpha \sqcup \overline{\mu}_{-1}$.
- $\overline{\sigma}^1 = \overline{\sigma}$ and $\overline{\sigma}^0(n) = \overline{d}(n)$.
- $\overline{\lambda}_1$ is identical with the classical Liouville function, which is defined by $\lambda(n) = (-1)^{\Omega(n)}$ ([1]).
- For $\alpha \in \mathbb{N}$, we have $\overline{\lambda}_\alpha = \overline{\lambda} \sqcup \cdots \sqcup \overline{\lambda}$ (the unitary convolution of α factors).

2.2 Properties

In this sub-section, we derive a number of properties relating to the functions defined above some of which generalize those in (1.2). Note first that for $\alpha, \beta \in \mathbb{C} \setminus \{0\}$, $\overline{\mu}_\alpha \sqcup \overline{\mu}_\beta = \overline{\mu}_{\alpha+\beta}$ (see [6]).

Proposition 2.2.1. *For $\alpha, \beta \in \mathbb{R}$, $n \in \mathbb{N}$, p prime, we have*

1) $\overline{\sigma}^\alpha(n) = \prod_{p|n} \left(p^{\alpha\nu_p(n)} + 1\right)$

2) $\zeta_\alpha = \overline{\mu} \sqcup \overline{\sigma}^\alpha$

3) $\overline{\varphi}_\alpha(n) = \prod_{p|n} \left(p^{\nu_p(n)} - \alpha\right)$

4) $\sum_{d||n} \overline{\varphi}_\alpha(d) = \prod_{p|n} \left(p^{\nu_p(n)} + 1 - \alpha\right)$

5) $\overline{\varphi}_\alpha \sqcup \overline{\sigma}^\beta = \zeta_1 \sqcup \zeta_\beta \sqcup \overline{\mu}_{\alpha-1}$

6) $\varphi^*_\alpha(n) = \prod_{p|n} \left(p^{\alpha\nu_p(n)} - 1\right)$

7) $\sum_{d||n} \varphi^*_\alpha(d) = \zeta_\alpha(n)$

8) $\zeta_\alpha \sqcup \varphi^*_\beta = \zeta_\beta \sqcup \varphi^*_\alpha$

9) $\varphi^*_\alpha \sqcup \overline{d} = \overline{\sigma}^\alpha$

10) $\varphi^*_\alpha \sqcup \overline{\sigma}^\beta = \zeta_\alpha \sqcup \zeta_\beta$

11) $\sum_{d||n} \overline{\lambda}_\alpha(d) = \prod_{p|n} \left(1 + (-\alpha)^{\nu_p(n)}\right)$

12) $\left(\overline{\lambda}_\alpha \sqcup \overline{\mu}_\beta\right)(n) = \prod_{p|n} \left((-\alpha)^{\nu_p(n)} - \beta\right)$

13) $(\overline{\lambda}_\alpha \sqcup \overline{\sigma}^\beta)(n) = \prod_{p|n} \left(p^{\beta\nu_p(n)} + (-\alpha)^{\nu_p(n)} + 1\right)$

14) $(\overline{\lambda}_\alpha \sqcup \overline{\varphi}_\beta)(n) = \prod_{p|n} \left(p^{\nu_p(n)} + (-\alpha)^{\nu_p(n)} - \beta\right)$

15) $(\overline{\lambda}_\alpha \sqcup \varphi^*_\beta)(n) = \prod_{p|n} \left(p^{\beta\nu_p(n)} + (-\alpha)^{\nu_p(n)} - 1\right)$.

Proof. 1) Since $\overline{\sigma}^\alpha$ is multiplicative, we get $\overline{\sigma}^\alpha(1) = 1$. To prove the assertion, it suffices to evaluate $\overline{\sigma}^\alpha(p^a)$ for prime p and $a \in \mathbb{N}$, which is

$$\overline{\sigma}^\alpha(p^a) = \sum_{d||p^a} d^\alpha = p^{\alpha a} + 1.$$

Assertions 2), 5), 7), 8), 9) and 10) follow, respectively, from the identities

$$\begin{aligned}
\overline{\mu} \sqcup \overline{\sigma}^\alpha &= \overline{\mu} \sqcup \zeta_\alpha \sqcup \overline{\mu}_{-1} = \zeta_\alpha \\
\overline{\varphi}_\alpha \sqcup \overline{\sigma}^\beta &= \zeta_1 \sqcup \overline{\mu}_\alpha \sqcup \zeta_\beta \sqcup \overline{\mu}_{-1} = \zeta_1 \sqcup \zeta_\beta \sqcup \overline{\mu}_{\alpha-1} \\
\sum_{d||n} \varphi^*_\alpha(d) &= (\varphi^*_\alpha \sqcup U)(n) = (\zeta_\alpha \sqcup \overline{\mu} \sqcup \overline{\mu}_{-1})(n) = \zeta_\alpha(n) \\
\zeta_\alpha \sqcup \varphi^*_\beta &= \zeta_\alpha \sqcup \zeta_\beta \sqcup \overline{\mu} = \zeta_\beta \sqcup \zeta_\alpha \sqcup \overline{\mu} = \zeta_\beta \sqcup \varphi^*_\alpha \\
\varphi^*_\alpha \sqcup \overline{d} &= \zeta_\alpha \sqcup \overline{\mu} \sqcup \overline{\mu}_{-2} = \zeta_\alpha \sqcup \overline{\mu}_{-1} = \overline{\sigma}^\alpha \\
\varphi^*_\alpha \sqcup \overline{\sigma}^\beta &= \zeta_\alpha \sqcup \overline{\mu} \sqcup \zeta_\beta \sqcup \overline{\mu}_{-1} = \zeta_\alpha \sqcup \zeta_\beta \sqcup I = \zeta_\alpha \sqcup \zeta_\beta.
\end{aligned}$$

Since the functions $\overline{\varphi}_\alpha,\ \varphi^*_\alpha,\ \overline{\lambda}_\alpha \sqcup \overline{\mu}_\beta,\ \overline{\lambda}_\alpha \sqcup \overline{\sigma}^\beta,\ \overline{\lambda}_\alpha \sqcup \overline{\mu}_\beta,\ \overline{\lambda}_\alpha \sqcup \varphi^*_\beta$ are multiplicative, as in the proof of Assertion 1), Assertions 3), 6), 12), 13), 14) and 15) follow at once from the evaluation

$$\overline{\varphi}_\alpha(p^a) = (\zeta_1 \sqcup \overline{\mu}_\alpha)(p^a) = \sum_{d||p^a} d(-\alpha)^{\omega\left(\frac{p^a}{d}\right)} = p^a(-\alpha)^{\omega(1)} + (-\alpha)^{\omega(p^a)} = p^a - \alpha$$

$$\varphi^*_\alpha(p^a) = (\zeta_\alpha \sqcup \overline{\mu})(p^a) = \overline{\mu}(p^a) + \zeta_\alpha(p^a) = p^{a\alpha} - 1$$

$$(\overline{\lambda}_\alpha \sqcup \overline{\mu}_\beta)(p^a) = \sum_{d||p^a} \overline{\lambda}_\alpha(d)\overline{\mu}_\beta\left(\frac{p^a}{d}\right) = \overline{\lambda}_\alpha(p^a) + \overline{\mu}_\beta(p^a) = (-\alpha)^a - \beta$$

$$(\overline{\lambda}_\alpha \sqcup \overline{\sigma}^\beta)(p^a) = \sum_{d||p^a} \overline{\lambda}_\alpha(d)\overline{\sigma}^\beta\left(\frac{p^a}{d}\right) = \overline{\lambda}_\alpha(p^a) + \overline{\sigma}^\beta(p^a) = (-\alpha)^a + p^{\beta a} + 1$$

$$(\overline{\lambda}_\alpha \sqcup \overline{\varphi}_\beta)(p^a) = \sum_{d||p^a} \overline{\lambda}_\alpha(d)\overline{\varphi}_\beta\left(\frac{p^a}{d}\right) = \overline{\lambda}_\alpha(p^a) + \overline{\varphi}_\beta(p^a) = (-\alpha)^a + p^a - \beta$$

$$(\overline{\lambda}_\alpha \sqcup \varphi^*_\beta)(p^a) = \sum_{d||p^a} \overline{\lambda}_\alpha(d)\varphi^*_\beta\left(\frac{p^a}{d}\right) = \overline{\lambda}_\alpha(p^a) + \varphi^*_\beta(p^a) = (-\alpha)^a + p^{a\beta} - 1.$$

When $n = 1$, we have

$$\sum_{d||1} \overline{\varphi}_\alpha(d) = \overline{\varphi}_\alpha(1) = 1,\ \sum_{d||1} \overline{\lambda}_\alpha(d) = \overline{\lambda}_\alpha(1) = (-\alpha)^{\Omega(1)} = 1.$$

For $n \geq 2$, since the functions $\overline{\varphi}_\alpha \sqcup U,\ \overline{\lambda}_\alpha$ are multiplicative, as in the proof Assertion 1), Assertions 4) and 11) follow from

$$(\overline{\varphi}_\alpha \sqcup U)(p^a) = \sum_{d||p^a} \overline{\varphi}_\alpha(p^a) = \overline{\varphi}_\alpha(1) + \overline{\varphi}_\alpha(p^a) = p^a + 1 - \alpha$$

$$\sum_{d||p^a} \overline{\lambda}_\alpha(d) = \overline{\lambda}_\alpha(1) + \overline{\lambda}_\alpha(p^a) = 1 + (-\alpha)^{\Omega(p^a)} = 1 + (-\alpha)^a.$$

□

3 Results

Complementing the concept of exponentially odd numbers, an exponentially even number is defined to be a positive integer whose prime factorization contains only even prime powers, and let E_e be the set of all exponentially even numbers. Let $\beta(n)$ be the maximal unitary divisor of the exponentially even part of n, with $\beta(1) := 1$ and let $\alpha(n)$ be the maximal unitary divisor of the exponentially odd part of n, with $\alpha(1) := 1$. For any positive integer whose prime factorization is $n = p_1^{a_1} \cdots p_s^{a_s} q_1^{b_1} \cdots q_r^{b_r}$, where p_i, q_j are distinct primes, and $a_i, b_j \in \mathbb{N}$ are such that each a_i is odd, and each b_j is even, we clearly see that

$$\beta(n) = \begin{cases} q_1^{b_1} \cdots q_r^{b_r} & \text{if there is at least one } b_i \neq 0 \\ 1 & \text{if all } b_i = 0 \end{cases},\ \alpha(n) = \begin{cases} p_1^{a_1} \cdots p_s^{a_s} & \text{if there is at least one } a_i \neq 0 \\ 1 & \text{if all } a_i = 0. \end{cases}$$

Table 1 : Examples of $\alpha(n)$ and $\beta(n)$

n	$d;\ d\|\|n$	$d_1; d \in E_o$	$d_2; d \in E_e$	$\alpha(n)$	$\beta(n)$
2	1, 2	1, 2	1	2	1
8	1, 8	1, 8	1	8	1
12	1, 3, 4, 12	1, 3	1, 4	3	4
16	1, 16	1	1, 16	1	16
24	1, 3, 8, 24	1, 3, 8, 24	1	24	1
36	1, 4, 9, 36	1	1, 4, 9, 36	1	36

1. The **even-phi function**, $\widetilde{\tau}(n)$ is defined to be the number of integers $m \leq n$ which is semiprime to $\beta(n)$, i.e.,
$$\widetilde{\tau}(n) = \sum_{\substack{m\leq n\\(m,\beta(n))_*=1}} 1.$$

2. The **odd-phi function**, $\widetilde{\phi}(n)$ is defined to be the number of integers $m \leq n$ which is semiprime to $\alpha(n)$, i.e.,
$$\widetilde{\phi}(n) = \sum_{\substack{m\leq n\\(m,\alpha(n))_*=1}} 1.$$

Table 2 : Examples of the even-phi function and the odd-phi function

n	$\beta(n)$	$m; m \leq n$ and $(m,\beta(n))_* = 1$	$\widetilde{\tau}(n)$	$\alpha(n)$	$m; m \leq n$ and $(m,\alpha(n))_* = 1$	$\widetilde{\phi}(n)$
2	1	1, 2	1	2	1	1
8	1	1, 2, . . . , 8	7	8	1, 2, . . . , 7	7
12	4	1, 2, 3, 5, 6, 7, 9, 10, 11	8	3	1, 2, 4, 5, 7, 8, 10, 11	8
16	16	1, . . . , 15	16	1	1, . . . , 16	16
24	1	1, . . . , 24	14	24	1, 2, 4, 5, 7, 10, 11, 13 14, 17, 19, 20, 22, 23	14
36	36	1, 2, 3, 5, 6, 7, 10, 11, 13, 14, 15, 17, 19, 21, 22, 23, 25, 26, 29, 30, 31, 33, 34, 35	24	1	1, . . . , 36	36

Let
$$K(n) = \begin{cases} \overline{\mu}(\sqrt{n}) & \text{if } n \text{ is a perfect square} \\ 0 & \text{otherwise.} \end{cases}$$

In this part, we show that the even-phi function can be written as the unitary convolution of ζ_1 with K and the odd-phi function can be written as the unitary convolution of ζ_1 with T defined to be a multiplicative function such that for p prime,
$$T(p^a) = \begin{cases} -1 & \text{if } a \text{ is odd} \\ 0 & \text{if } a \text{ is even.} \end{cases}$$

We start with some auxiliary results.

Lemma 3.1. *For any prime p and $a \in \mathbb{N}$, we have*
$$\widetilde{\tau}(p^a) = \begin{cases} p^a & \text{if } a \text{ is odd} \\ p^a - 1 & \text{if } a \text{ is even} \end{cases}, \qquad \widetilde{\phi}(p^a) = \begin{cases} p^a - 1 & \text{if } a \text{ is odd} \\ p^a & \text{if } a \text{ is even.} \end{cases}$$

Proof. If a is odd, then $\beta(p^a) = 1, \alpha(p^a) = p^a$, and if a is even, then $\beta(p^a) = p^a, \alpha(p^a) = 1$. Thus,
$$\widetilde{\tau}(p^a) = \sum_{\substack{m\leq p^a\\(m,\beta(p^a))_*=1}} 1 = \begin{cases} p^a, & \text{if } a \text{ is odd} \\ p^a - 1, & \text{if } a \text{ is even} \end{cases}, \quad \widetilde{\phi}(p^a) = \sum_{\substack{m\leq p^a\\(m,\alpha(p^a))_*=1}} 1 = \begin{cases} p^a - 1, & \text{if } a \text{ is odd} \\ p^a, & \text{if } a \text{ is even.} \end{cases}$$

□

To state the next lemma more conveniently, we introduce the following terminology. An arithmetic function f is said to be multiplicative over E_o (respectively, over E_e) if $f(1) = 1$ and $f(mn) = f(m)f(n)$ for relatively prime $m, n \in E_o$ (respectively, E_e).

Lemma 3.2. *The functions $\widetilde{\tau}$ and $\widetilde{\phi}$ are both multiplicative over both E_o and E_e.*

Proof. Since $\beta(1) = 1$, we get

$$\widetilde{\tau}(1) = \sum_{\substack{m \le 1 \\ (m,\beta(1))_* = 1}} 1 = \sum_{\substack{m \le 1 \\ (m,1)_* = 1}} 1 = 1.$$

For $p_1^{a_1} \dots p_s^{a_s} (> 1) \in E_o$, with p_i being prime and a_i being odd, we have $\beta(p_1^{a_1} \cdots p_s^{a_s}) = 1$. Using part 2 of Lemma 3.1, we deduce that

$$\widetilde{\tau}(p_1^{a_1} \cdots p_s^{a_s}) = \sum_{\substack{m \le p_1^{a_1} \cdots p_s^{a_s} \\ (m,\beta(p_1^{a_1} \cdots p_s^{a_s}))_* = 1}} 1 = \sum_{\substack{m \le p_1^{a_1} \cdots p_s^{a_s} \\ (m,1)_* = 1}} 1 = p_1^{a_1} \cdots p_s^{a_s} = \widetilde{\tau}(p_1^{a_1}) \cdots \widetilde{\tau}(p_s^{a_s}),$$

which implies that $\widetilde{\tau}$ is multiplicative over E_o.

Similarly, for $q_1^{b_1} \cdots q_r^{b_r} (> 1) \in E_e$, with q_i being prime and b_i being even, from $\beta(q_1^{b_1} \cdots q_r^{b_r}) = q_1^{b_1} \cdots q_r^{b_r}$ using part 1 of Lemma 3.1, we have

$$\begin{aligned}
\widetilde{\tau}(q_1^{b_1} \cdots q_r^{b_r}) &= \sum_{\substack{m \le q_1^{b_1} \cdots q_r^{b_r} \\ (m,\beta(q_1^{b_1} \cdots q_r^{b_r}))_* = 1}} 1 = \sum_{\substack{m \le q_1^{b_1} \cdots q_r^{b_r} \\ (m,\, q_1^{b_1} \cdots q_r^{b_r})_* = 1}} 1 \\
&= q_1^{b_1} \cdots q_r^{b_r} - \sum_{i=1}^{r} \frac{q_1^{b_1} \cdots q_r^{b_r}}{q_i^{b_i}} + \sum_{i,j=1, i<j}^{r} \frac{q_1^{b_1} \cdots q_r^{b_r}}{q_i^{b_i} q_j^{b_j}} + \cdots + (-1)^{r-1} \sum_{i=1}^{r} q_i^{b_i} + (-1)^r \\
&= (q_1^{b_1} - 1)(q_2^{b_2} - 1) \cdots (q_r^{b_r} - 1) = \widetilde{\tau}(q_1^{b_1}) \cdots \widetilde{\tau}(q_r^{b_r}),
\end{aligned}$$

which shows that $\widetilde{\tau}$ is multiplicative over E_e. The proof for the function $\widetilde{\phi}$ is similar and is omitted. □

Next, we show that $\widetilde{\phi}$ and $\widetilde{\tau}$ are multiplicative functions.

Theorem 3.3. *Both $\widetilde{\tau}$ and $\widetilde{\phi}$ are multiplicative functions.*

Proof. For a prime factorization $p_1^{a_1} \cdots p_s^{a_s} q_1^{b_1} \cdots q_r^{b_r}$, where each a_i is odd(even) and each b_j is even(odd), we get $\beta(p_1^{a_1} \cdots p_s^{a_s} q_1^{b_1} \cdots q_r^{b_r}) = q_1^{b_1} \cdots q_r^{b_r}$. Putting $p = p_1^{a_1} \cdots p_s^{a_s}$ and using part 2 of Lemma 3.1, we have

$$\begin{aligned}
\widetilde{\tau}(p_1^{a_1} \cdots p_s^{a_s} q_1^{b_1} \cdots q_r^{b_r}) &= \widetilde{\tau}(p q_1^{b_1} \cdots q_r^{b_r}) = \sum_{\substack{m \le p q_1^{b_1} \cdots q_r^{b_r} \\ (m,\beta(p q_1^{b_1} \cdots q_r^{b_r}))_* = 1}} 1 = \sum_{\substack{m \le p q_1^{b_1} \cdots q_r^{b_r} \\ (m, q_1^{b_1} \cdots q_r^{b_r})_* = 1}} 1 \\
&= p q_1^{b_1} \dots q_r^{b_r} - \sum_{i=1}^{r} \frac{p q_1^{b_1} \cdots q_r^{b_r}}{q_i^{b_i}} + \sum_{i,j=1, i<j}^{r} \frac{p q_1^{b_1} \cdots q_r^{b_r}}{q_i^{b_i} q_j^{b_j}} + \cdots + (-1)^{r-1} \sum_{i=1}^{r} p q_i^{b_i} + (-1)^r p \\
&= p(q_1^{b_1} - 1) \cdots (q_r^{b_r} - 1) = p_1^{q_1} \cdots p_s^{a_s} (q_1^{b_1} - 1) \cdots (q_r^{b_r} - 1) \\
&= \widetilde{\tau}(p_1^{a_1}) \cdots \widetilde{\tau}(p_s^{a_s}) \widetilde{\tau}(q_1^{b_1}) \cdots \widetilde{\tau}(q_r^{b_r}).
\end{aligned}$$

Since $\widetilde{\tau}(1) = 1$, the multiplicativity of $\widetilde{\tau}$ is immediate. The proof for the function $\widetilde{\phi}$ is similar and is omitted. □

It is shown in [3] that $(K \sqcup U)(n) = \chi_o(n)$ where $\chi_o(n)$ is the characteristic function of n being exponentially odd, while [7], $(T \sqcup U)(n) = \chi_e(n)$ where $\chi_e(n)$ is the characteristic function of n being exponentially even. We end the paper with the promised result

Theorem 3.4. *We have*

$$\widetilde{\tau} = \zeta_1 \sqcup K, \quad \widetilde{\phi} = \zeta_1 \sqcup T.$$

Proof. Clearly,

$$(\zeta_1 \sqcup K)(1) = \zeta_1(1)K(1) = 1 = \widetilde{\tau}(1).$$

For $n > 1$, since ζ_1, K and $\widetilde{\tau}$ are multiplicative functions, it suffices to verify their values at prime power p^a, which is

$$(\zeta_1 \sqcup K)(p^a) = \sum_{d||p^a} \zeta_1(d) K\left(\frac{p^a}{d}\right) = K(p^a) + p^a K(1) = \begin{cases} p^a & \text{if } a \text{ is odd} \\ p^a - 1 & \text{if } a \text{ is even} \end{cases} = \widetilde{\tau}(p^a).$$

When $n = 1$, we have

$$(\zeta_1 \sqcup T)(1) = \zeta_1(1)T(1) = 1 = \widetilde{\phi}(1).$$

For $n > 1$, since ζ_1, T and $\widetilde{\phi}$ are multiplicative functions, it suffices to verify their values at prime power p^a, which is

$$(\zeta_1 \sqcup T)(p^a) = \sum_{d||p^a} \zeta_1(d) T\left(\frac{p^a}{d}\right) = T(p^a) + p^a T(1) = \begin{cases} -1 + p^a & \text{if } a \text{ is odd} \\ p^a & \text{if } a \text{ is even} \end{cases} = \widetilde{\phi}(p^a).$$

□

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