Thai **J**ournal of **M**athematics : (2020) 117–125
Special Issue: The 14^{th} IMT-GT ICMSA 2018
http://thaijmath.in.cmu.ac.th
Online ISSN 1686-0209

Minimal and Maximal Ordered n-ideals in Ordered n-ary Semigroups

Anusorn Simuen[†,1], **Pattarawan Petchkaew**[§] **and Ronnason Chinram**[†,‡]

[†]Department of Mathematics and Statistics, Faculty of Science
Prince of Songkla University, Hat Yai, Songkhla 90110, Thailand
e-mail : `asimuen96@gmail.com`
[§] Mathematics and Statistics Program, Faculty of Science and Technology,
Songkhla Rajabhat University, Songkhla 90000, Thailand
e-mail : `pattarawan.pe@gmail.com`
[‡]Centre of Excellence in Mathematics,
CHE, Si Ayuthaya Road, Bangkok 10400, Thailand
e-mail : `ronnason.c@psu.ac.th`

Abstract : The notion of ordered n-ary semigroups is a generalization of ordered semigroups and n-ary semigroups. The concepts of minimal and maximal left ideals and right ideals play an important role in semigroups and ordered semigroups. In this paper, we extend these concepts to consider in ordered n-ary semigroups. The remarkable results concerning minimal and maximal ordered n-ideals in ordered n-ary semigroups are given.

Keywords : ordered i-ideals; minimal ordered n-ideals; maximal ordered n-ideals; ordered n-ary semigroups.
2010 Mathematics Subject Classification : 20N15; 20M12; 06F05.

1 Introduction and Preliminaries

The generalization of classical algebraic structures to n-ary structures was first initiated by Kasner [1] in 1904. The notion of ordered n-ary semigroups is a generalization of ordered semigroups and n-ary semigroups. In 2000, Cao and Xu studied minimal and maximal left ideals in ordered semigroups and characterized them in [2]. After that, Arslanov and Kehayopulu characterized minimal and maximal ideals in ordered semigroups in [3]. Next, Iampan [4] investigated some characterizations of minimal and maximal left ideals and right ideals in ternary semigroups in 2010. Recently, Petchkaew and Chinram studied minimal and maximal n-ideals in n-ary semigroups and gave some characterizations of minimal and maximal n-ideals in n-ary semigroups in [5]. Those are our motivations to do this paper. In this paper, we extend those results. The remarkable results concerning minimal and maximal n-ideals in ordered n-ary semigroups are given. First, we would like to recall the definition of n-ary semigroup which

[0]This research was supported by the Algebra and Applications Research Unit, Prince of Songkla University.
[1]`Corresponding author.`

was stated in [6], a nonempty set S together with an n-ary operation given by $f : S^n \to S$, where $n \geq 2$, is called an *n-ary groupoid* and is denoted by (S, f). The sequence of elements $x_i, x_{i+1}, \ldots, x_j$ is denoted by x_i^j. Note that in the case $j < i$, this is the empty symbol. If $x_{i+1} = x_{i+2} = \cdots = x_{i+t} = x$, then we write x^t instead of x_{i+1}^{i+t}. In this convention,

$$f(x_1, x_2, \ldots, x_n) = f(x_1^n)$$

and

$$f(x_1, \ldots, x_i, \underbrace{x \ldots, x}_{t}, x_{i+t+1}, \ldots, x_n) = f(x_1^i, x^t, x_{i+t+1}^n).$$

An n-ary groupoid (S, f) is called (i, j)-*associative* if

$$f(x_1^{i-1}, f(x_i^{n+i-1}), x_{n+i}^{2n-1}) = f(x_1^{j-1}, f(x_j^{n+j-1}), x_{n+j}^{2n-1})$$

hold for all $x_1, x_2, \ldots, x_{2n-1} \in S$. The operation f is *associative* if the above identity holds for all $1 \leq i \leq j \leq n$, and (S, f) is called an *n-ary semigroup*.

Example 1.1. Let $S = \{2, 2^n, 2^{n+1}, 2^{n+2}, \ldots\}$. For all $n \in \mathbb{N} \setminus \{1\}$, define $f : S^n \to S$ by $f(x_1^n) = x_1 \cdot x_2 \cdot \ldots \cdot x_n$ for all $x_1, x_2, \ldots, x_n \in S$ where $\cdot$ is a usual multiplication. Then (S, f) is an n-ary semigroup but not an m-ary semigroup for all positive integer m such that $1 < m < n$.

A partially ordered n-ary semigroup S is called an *ordered n-ary semigroup* if for all $x, y, a_1, a_2, \ldots, a_n \in S, x \leq y \Rightarrow f(a_1^{i-1}, x, a_{i+1}^n) \leq f(a_1^{i-1}, y, a_{i+1}^n)$ for all $i = 1, 2, \ldots, n$.

Example 1.2. Let (S, f) be an n-ary semigroup. We have that (S, f, id_S) is an ordered n-ary semigroup where $id_S := \{(a, a) \mid a \in S\}$ is an identity relation on S.

Let S be an ordered n-ary semigroup. For a subset H of S, let $(H] := \{t \in S \mid t \leq h \text{ for some } h \in H\}$. For $H = \{a\}$, we write $(a]$ instead of $(\{a\}]$.

Proposition 1.1. *Let A and B be subsets of an ordered n-ary semigroup S. The following statements are true.*

(1) $A \subseteq (A]$.

(2) $(A \cup B] = (A] \cup (B]$.

(3) *If* $A \subseteq B$, *then* $(A] \subseteq (B]$.

(4) $(A] = ((A]]$.

Proof. The proof is straightforward. □

For all subsets $A_1, A_2, \ldots, A_n$ of $T, f(A_1^n) := \{f(a_1^n) \mid a_i \in A_i\}$. If $A_1 = \{a_1\}$, then we write $f(\{a_1\}, A_2^n)$ as $f(a_1, A_2^n)$, and similarly in another case such as we write $f(\{a_1\}, A_2^{n-1}, \{a_n\})$ as $f(a_1, A_2^{n-1}, a_n)$ and so on.

Proposition 1.2. *Let $A_1, A_2, \ldots, A_n$ be subsets of an ordered n-ary semigroup S. Then*

$$f((A_1], (A_2], \ldots, (A_n]) \subseteq (f(A_1^n)].$$

Proof. The proof is straightforward. □

A nonempty subset H of an ordered n-ary semigroup (S, f) is called an *ordered n-ary subsemigroup* of S if $(H] \subseteq H$ and $f(a_1^n) \in H$ for all $a_1, a_2, \ldots, a_n \in H$.

A nonempty subset I of S is called an *ordered i-ideal* of S if $(I] \subseteq I$ and for every $x_1, ..., x_{i-1}, x_{i+1}, ..., x_n \in S$ with $a \in I$, then $f(x_1^{i-1}, a, x_{i+1}^n) \in I$. A nonempty subset I of S is called an *ordered ideal* of S if I is an i-ideal for every $1 \leq i \leq n$.

The intersection of all i-ideals of an ordered n-ary subsemigroup H of an ordered n-ary semigroup S containing a nonempty subset A of H is the *ordered i-ideal of H generated by A*. For $A = \{a\}$, we denote $I_{i,H}(a)$ to be the ordered i-ideal of H generate by $\{a\}$. If $H = S$, then we write $I_{i,S}(a)$ as $I_i(a)$. The intersection of all ordered i-ideals of an ordered n-ary semigroup S containing a nonempty subset A is the *ordered i-ideal of S generated by A* denoted by $I_i(A)$ and if $A = \{a\}$, we denote it by $I_i(a)$.

An element a of an ordered n-ary semigroup S with at least two elements is called a *zero element* of S if $f(x_1^{i-1}, a, x_{i+1}^n) = a$ with $x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n \in S$ for all $i = 1, 2, \ldots, n$ and denote it by 0. If an ordered n-ary semigroup S contains a zero element, then every i-ideal of S also contains a zero element.

An ordered n-ary semigroup S without zero is called *i-simple* if it has no proper ordered i-ideals. An ordered n-ary semigroup S with zero is called *0-i-simple* if it has no nonzero proper ordered i-ideals and $f(S^n) \neq \{0\}$.

An ordered i-ideal I of an ordered n-ary semigroup S without zero is called a *minimal ordered i-ideal* of S if there is no ordered i-ideal J of S such that $J \subsetneq I$. This implies that if there is an ordered i-ideal J of S such that $J \subseteq I$, we gain that $J = I$. A nonzero ordered i-ideal I of an ordered n-ary semigroup S with zero is called a *0-minimal ordered i-ideal* of S if there is no nonzero ordered i-ideal J of S such that $J \subsetneq I$. Equivalently, if S has an ordered i-ideal J such that $J \subsetneq I$, we obtain that $J = \{0\}$. A proper ordered i-ideal I of an ordered n-ary semigroup S is called a *maximal ordered i-ideal* of S if for any ordered i-ideal J of S such that $I \subsetneq J$, we have $J = S$. Equivalently, if J is a proper ordered i-ideal of S such that $I \subseteq J$, we acquire that $J = I$.

2 Basic properties

Throughout this paper, S is assumed to be an ordered n-ary semigroup. In this section, we provide some ideas, elementary properties and some our fundamental results which relate to ordered n-ideals, n-simples, and 0-n-simples.

Lemma 2.1. *Let A be any nonempty subset of S. Then $(f(S^{n-1}, A) \cup A]$ is the smallest ordered n-ideal of S containing A.*

Proof. First, we show that $(f(S^{n-1}, A) \cup A]$ is an ordered n-ideal of S. By Proposition 1.1(4), $((f(S^{n-1}, A) \cup A]] = (f(S^{n-1}, A) \cup A]$. Next, let $x_1, x_2, \ldots, x_{n-1} \in S$ and $y \in (f(S^{n-1}, A) \cup A]$. Then there exists $z \in f(S^{n-1}, A) \cup A$ such that $y \leq z$. So $z \in f(S^{n-1}, A)$ or $z \in A$.

Case 1: $z \in f(S^{n-1}, A)$. Then $z = f(s_1^{n-1}, a)$ for some $s_1, s_2, \ldots, s_{n-1} \in S$ and for some $a \in A$. Then $f(x_1^{n-1}, y) \leq f(x_1^{n-1}, z) = f(x_1^{n-1}, f(s_1^{n-1}, a)) = f(f(x_1^{n-1}, s_1), s_2^{n-1}, a) \in f(S^{n-1}, A) \subseteq f(S^{n-1}, A) \cup A$. This implies that

$$f(x_1^{n-1}, y) \in (f(S^{n-1}, A) \cup A].$$

Case 2: $z \in A$. Then $f(x_1^{n-1}, y) \leq f(x_1^{n-1}, z) \in f(S^{n-1}, A) \subseteq f(S^{n-1}, A) \cup A$. This implies that $f(x_1^{n-1}, y) \in (f(S^{n-1}, A) \cup A]$.

From Case 1 and Case 2, we can conclude that $(f(S^{n-1}, A) \cup A]$ is an ordered n-ideal of S.

Next, we show that $(f(S^{n-1}, A) \cup A]$ is a smallest ordered n-ideal of S containing A. Let I be any ordered n-ideal of S containing A. Let $y \in (f(S^{n-1}, A) \cup A]$. Then there exists $z \in f(S^{n-1}, A) \cup A$ such that $y \leq z$. If $z \in A$, then $z \in I$ because $A \subseteq I$. This implies that $y \in I$. If $z \in f(S^{n-1}, A)$, then $z = f(s_1^{n-1}, a)$ for some $s_1, s_2, \ldots, s_{n-1} \in S$ and for some $a \in A$. Thus $a \in I$ because $A \subseteq I$. Hence $z = f(s_1^{n-1}, a) \in I$ since I is an ordered n-ideal of S. Therefore, $y \in I$. We obtain $(f(S^{n-1}, A) \cup A] \subseteq I$. Hence $(f(S^{n-1}, A) \cup A]$ is the smallest ordered n-ideal of S containing A. □

Corollary 2.2. *For any an element a of S, $I_n(a) = (f(S^{n-1}, a) \cup \{a\}]$.*

Proof. This follows from Lemma 2.1. □

Lemma 2.3. *Let A be a nonempty subset of an n-ideal I of S. Then $(f(I^{n-1}, A)]$ is an ordered n-ideal of S.*

Proof. Let $s_1, s_2, \ldots, s_{n-1} \in S$ and let $y \in (f(I^{n-1}, A)]$. Then there exists $z \in f(I^{n-1}, A)$ such that $y \leq z$. Thus $z = f(x_1^{n-1}, a)$ for some $x_1, x_2, \ldots, x_{n-1} \in I$ and for some $a \in A$. Then $f(s_1^{n-1}, y) \leq f(s_1^{n-1}, z) = f(s_1^{n-1}, f(x^{n-1}, a)) = f(f(s_1^{n-1}, x_1), x_2^{n-1}, a) \in f(I^{n-1}, A)$ because I is an ordered n-ideal of S and $x_i \in I$ for all $i \in \{1, 2, \ldots, n-1\}$. So $f(s_1^{n-1}, y) \in (f(I^{n-1}, A)]$. This implies that $(f(I^{n-1}, A)]$ is an ordered n-ideal of S. □

Lemma 2.4. *Let A be any nonempty subset of S. Then $(f(S^{n-1}, A)]$ is an ordered n-ideal of S.*

Proof. This follows by Lemma 2.3 by using $I = S$. □

Lemma 2.5. *If S has no zero element, then the following statements are equivalent:*

(1) *S is n-simple.*

(2) *$(f(S^{n-1}, a)] = S$ for all $a \in S$.*

(3) *$I_n(a) = S$ for all $a \in S$.*

Proof. First, we show (1) ⇒ (2). Assume that S is n-simple. By Lemma 2.4, $(f(S^{n-1}, a)]$ is an ordered n-ideal of S for all $a \in S$. Thus $(f(S^{n-1}, a)] = S$ for all $a \in S$ because S is n-simple.

Next, we show (2) ⇒ (3). Assume that $(f(S^{n-1}, a)] = S$ for all $a \in S$. By Corollary 2.2, we obtain $I_n(a) = (f(S^{n-1}, a) \cup \{a\}] = S \cup (a] = S$. Therefore, $I_n(a) = S$ for all $a \in S$.

Finally, we show (3) ⇒ (1). Assume the statement (3) holds. Let I be any ordered n-ideal of S. Since I is nonempty, let $x \in I$. Thus $S = I_n(x) \subseteq I \subseteq S$. This implies that $I = S$. Hence, S is n-simple. □

Example 2.1. Consider $\mathbb{Z}_{50}$, let $S = \{\overline{5}, \overline{25}\}$ and $\leq := \{(\overline{5}, \overline{5}), (\overline{25}, \overline{25}), (\overline{5}, \overline{25})\}$. Define $f : S^n \to S$ by

$$f(x_1^n) = x_1 \cdot x_2 \cdot \ldots \cdot x_n$$

for all $x_1, x_2, \ldots, x_n \in S$ where $\cdot$ is the multiplication of $\mathbb{Z}_{50}$. Then $(S, f, \leq)$ is an ordered n-ary semigroup. It is easy to see that S is n-simple.

Lemma 2.6. *Let S be an ordered n-ary semigroup with a zero element 0. Then the following statements hold:*

(1) *If S is 0-n-simple, then $I_n(a) = S$ for all $a \in S \setminus \{0\}$.*

(2) *If $I_n(a) = S$ for all $a \in S \setminus \{0\}$, then either $(f(S^n)] = \{0\}$ or S is 0-n-simple.*

Proof. (1) Assume that S is 0-n-simple. Since $I_n(a)$ is a nonzero ordered n-ideal of S for all $a \in S \setminus \{0\}$, we obtain that $I_n(a) = S$ for all $a \in S \setminus \{0\}$.

(2) Assume that $I_n(a) = S$ for all $a \in S \setminus \{0\}$ and suppose that $(f(S^n)] \neq \{0\}$. Let I be a nonzero ordered n-ideal of S. Then there exists $x \in I \setminus \{0\}$. Hence $S = I_n(x) \subseteq I \subseteq S$, and so $I = S$. Therefore, S is 0-n-simple. □

Example 2.2. Consider $\mathbb{Z}_{50}$, let $S = \{\overline{0}, \overline{5}, \overline{25}\}$ and

$$\leq := \{(\overline{0},\overline{0}),(\overline{5},\overline{5}),(\overline{25},\overline{25}),(\overline{0},\overline{5}),(\overline{0},\overline{25}),(\overline{5},\overline{25})\}.$$

Define $f : S^n \to S$ by

$$f(x_1^n) = x_1 \cdot x_2 \cdot \ldots \cdot x_n$$

for all $x_1, x_2, \ldots, x_n \in S$ where $\cdot$ is the multiplication of $\mathbb{Z}_{50}$. Then $(S, f, \leq)$ is an ordered n-ary semigroup with a zero $\overline{0}$. It is easy to see that S is 0-n-simple.

Lemma 2.7. *Let $\{I_\gamma \mid \gamma \in \Gamma\}$ be a family of ordered n-ideals of S.*

(1) $\bigcup_{\gamma\in\Gamma} I_\gamma$ *is an ordered n-ideal of S.*

(2) *If* $\bigcap_{\gamma\in\Gamma} I_\gamma \neq \emptyset$, *then* $\bigcap_{\gamma\in\Gamma} I_\gamma$ *is also an ordered n-ideal of S.*

.

Proof. The proof is straightforward. □

Lemma 2.8. *Let I be an ordered n-ideal of S and H be an ordered n-ary subsemigroup of S, then the following statements hold:*

(1) *If H is n-simple such that $H \cap I \neq \emptyset$, then $H \subseteq I$.*

(2) *If H is 0-n-simple such that $(H \smallsetminus \{0\}) \cap I \neq \emptyset$, then $H \subseteq I$.*

Proof. (1) Assume that H is n-simple such that $H \cap I \neq \emptyset$. Then there exists $a \in H \cap I$. By Lemma 2.4 and Lemma 2.7(2), we obtain that $(f(H^{n-1}, a)] \cap H$ is an ordered n-ideal of H. Since H is n-simple, we gain $(f(H^{n-1}, a)] \cap H = H$. This implies that $H \subseteq (f(H^{n-1}, a)] \subseteq (f(S^{n-1}, I)] \subseteq I$. Therefore, $H \subseteq I$.

(2) Suppose that H is 0-n-simple such that $(H \smallsetminus \{0\}) \cap I \neq \emptyset$. Then there exists $a \in H \smallsetminus \{0\} \cap I$. By Lemma 2.6(1) and Corollary 2.2, we obtain $H = I_{n,H}(a) = (f(H^{n-1}, a) \cup \{a\}] \cap H \subseteq (f(S^{n-1}, a) \cup \{a\}] = I_n(a) \subseteq I$. Therefore, $H \subseteq I$. □

3 Minimal ordered n-ideals

In this section, we investigate the relationship between minimal ordered n-ideals and n-simple (0-n-simple) ordered n-ary semigroups.

Theorem 3.1. *Let S be an ordered n-ary semigroup without zero and I be an ordered n-ideal of S. Then I is a minimal ordered n-ideal of S if and only if I is n-simple.*

Proof. (1) Let I be a minimal ordered n-ideal of S and J be any ordered n-ideal of I. Therefore, $(f(I^{n-1}, J)] \subseteq J \subseteq I$. By Lemma 2.3, $(f(I^{n-1}, J)]$ is an ordered n-ideal of S. Since I is minimal, $I \subseteq (f(I^{n-1}, J)]$ and so $(f(I^{n-1}, J)] = I$. This implies that $J = I$. Therefore, I is n-simple. Conversely, suppose that I is n-simple. Let J be an ordered n-ideal of S such that $J \subseteq I$. So $I \cap J \neq \emptyset$, and hence $I \subseteq J$ by Lemma 2.8(1). This implies that $J = I$. Therefore, I is a minimal ordered n-ideal of S. □

Example 3.1. Consider $\mathbb{Z}_{50}$, let $S = \{\overline{1}, \overline{5}, \overline{25}\}$ and

$$\leq := \{(\overline{1},\overline{1}),(\overline{5},\overline{5}),(\overline{25},\overline{25}),(\overline{5},\overline{25}),(\overline{5},\overline{1}),(\overline{25},\overline{1})\}.$$

Define $f : S^n \to S$ by

$$f(x_1^n) = x_1 \cdot x_2 \cdot \ldots \cdot x_n$$

for all $x_1, x_2, \ldots, x_n \in S$ where $\cdot$ is the multiplication of $\mathbb{Z}_{50}$. Then $(S, f, \leq)$ is an ordered n-ary semigroup. It is easy to see that $I = \{\overline{5}, \overline{25}\}$ is a minimal ordered n-ideal of S.

Theorem 3.2. *If S has a zero element and I is a nonzero ordered n-ideal of S, then the following statement hold:*

(1) *If I is a 0-minimal ordered n-ideal of S, then either $(f(I^{n-1}, J)] = \{0\}$ for some nonzero ordered n-ideal J of I or I is 0-n-simple.*

(2) *If I is 0-n-simple, then I is a 0-minimal ordered n-ideal of S.*

Proof. (1) Assume that I is a 0-minimal ordered n-ideal of S and $(f(I^{n-1}, J)] \neq \{0\}$ for any nonzero ordered n-ideal J of I. Let J be a nonzero ordered n-ideal of I. Then $\{0\} \neq (f(I^{n-1}, J)] \subseteq J \subseteq I$. Moreover, we obtain that $(f(I^{n-1}, J)]$ is an ordered n-ideal of S by Lemma 2.3. Since I is 0-minimal, $I \subseteq (f(I^{n-1}, J)]$. This implies that $(f(I^{n-1}, J)] = J = I$. Therefore, I is 0-n-simple.

(2) Assume that I is 0-n-simple. Let J be a nonzero ordered n-ideal of S such that $J \subseteq I$. This implies that $I \setminus \{0\} \cap J \neq \emptyset$ and so $I \subseteq J$ by Lemma 2.8(2). Hence $J = I$. Therefore, I is a 0-minimal ordered n-ideal of S. □

Example 3.2. Consider $\mathbb{Z}_{50}$, let $S = \{\overline{0}, \overline{1}, \overline{5}, \overline{25}\}$ and

$$\leq := \{(\overline{0},\overline{0}), (\overline{1},\overline{1}), (\overline{5},\overline{5}), (\overline{25},\overline{25}), (\overline{0},\overline{1}), (\overline{0},\overline{5}), (\overline{0},\overline{25}), (\overline{5},\overline{25}), (\overline{5},\overline{1}), (\overline{25},\overline{1})\}.$$

Define $f : S^n \to S$ by

$$f(x_1^n) = x_1 \cdot x_2 \cdot \ldots \cdot x_n$$

for all $x_1, x_2, \ldots, x_n \in S$ where $\cdot$ is the multiplication of $\mathbb{Z}_{50}$. Then $(S, f, \leq)$ is an ordered n-ary semigroup with a zero $\overline{0}$. It is easy to see that $I = \{\overline{0}, \overline{5}, \overline{25}\}$ is a 0-minimal ordered n-ideal of S.

4 Maximal ordered n-ideals

In this section, we give the relationship between maximal n-ideals and the union $\mathcal{U}$ of all (nonzero) proper ordered n-ideals of ordered n-ary semigroups.

Theorem 4.1. *Let I be a proper n-ideal of S. Then I is a maximal ordered n-ideal if and only if*

(1) *$S \setminus I = \{a\}$ and $(f(a, S^{n-2}, a)] \subseteq I$ for some $a \in S$ or*

(2) *$S \setminus I \subseteq (f(S^{n-1}, a)]$ for all $a \in S \setminus I$.*

Proof. Let I be a maximal ordered n-ideal of S. We consider the following two cases:

Case 1: Suppose that there exists $a \in S \setminus I$ such that $(f(S^{n-1}, a)] \subseteq I$. Then $(f(a, S^{n-2}, a)] \subseteq (f(S^{n-1}, a)] \subseteq I$. By Corollary 2.2, we obtain $(I \cup \{a\}] = ((I \cup f(S^{n-1}, a)) \cup \{a\}] = (I \cup (f(S^{n-1}, a) \cup \{a\})] = (I \cup I_n(a)]$. This implies that $(I \cup \{a\}]$ is an ordered n-ideal of S because $(I \cup I_n(a)]$ is an n-ideal of S. Since I is a maximal ordered n-ideal of S and $I \subsetneq (I \cup \{a\}]$, we obtain that $(I \cup \{a\}] = S$. This implies that $S \setminus I = (a]$. Let $x \in S \setminus I$. Then $x \leq a$ and $(f(S^{n-1}, x)] \subseteq (f(S^{n-1}, a)] \subseteq I$. From $(f(S^{n-1}, x)] \subseteq I, x \in S \setminus I$, a similar argument shows that $S \setminus I \subseteq (x]$. Consequently, $a \in (x]$. Therefore $x = a$. Hence, we have that $S \setminus I = \{a\}$ and $(f(a, S^{n-2}, a)] \subseteq I$ for some $a \in S$ as desire. In this case, the statement (1) is satisfied.

Case 2: Suppose that $(f(S^{n-1}, a)] \nsubseteq I$ for all $a \in S \setminus I$. Let $a \in S \setminus I$. Then $(f(S^{n-1}, a)] \nsubseteq I$. Moreover, we obtain that $(f(S^{n-1}, a)]$ is an ordered n-ideal of S by Lemma 2.4. By Lemma 2.7, we gain that $(I \cup f(S^{n-1}, a)]$ is an ordered n-ideal of S. Since I is a maximal ordered n-ideal of S and $I \subsetneq (I \cup f(S^{n-1}, a)]$, we acquire that $(I \cup f(S^{n-1}, a)] = S$. Hence $a \in (f(S^{n-1}, a)]$ because $a \in S \setminus I$. This implies that $S \setminus I \subseteq (f(S^{n-1}, a)]$ for all $a \in S \setminus I$. Hence, this case satisfies the statement (2).

Conversely, suppose that J is an n-ideal of S such that $I \subsetneq J$. Then $J \setminus I \neq \emptyset$. If there exists $a \in S$ such that $S \setminus I = \{a\}$ and $(f(a, S^{n-2}, a)] \subseteq I$, then $J \setminus I \subseteq S \setminus I = \{a\}$, and hence $J \setminus I = \{a\}$. This implies that $J = I \cup \{a\} = S$. Hence we obtain that I is maximal. Next, if $S \setminus I \subseteq (f(S^{n-1}, a)]$ for all $a \in S \setminus I$, then $S \setminus I \subseteq (f(S^{n-1}, x)] \subseteq (f(S^{n-1}, J)] \subseteq J$ for all $x \in J \setminus I$. Hence $S = (S \setminus I) \cup I \subseteq J \cup J = J \subseteq S$, and so $J = S$. Therefore, I is a maximal ordered n-ideal of S.

Hence the proof of this theorem is completed. □

Example 4.1. (1) Let $S = \mathbb{N}$ and $\leq := \{(a, b) \mid a \geq b\}$. Define $f : S^n \to S$ by

$$f(x_1^n) = x_1 + x_2 + \ldots + x_n$$

for all $x_1, x_2, \ldots, x_n \in S$ where $+$ is the usual addition of $\mathbb{N}$. Then $(S, f, \leq)$ is an ordered n-ary semigroup. Let $I = \mathbb{N} \setminus \{1\}$. Thus $S \setminus I = \{1\}$ and $(f(1, S^{n-2}, 1)] \subseteq I$. By Theorem 4.1(1), I is a maximal ordered n-ideal of S.

(2) Let $S = \{0, -1, 1\}$ and $\leq := \{(0, 0), (1, 1), (-1, -1), (1, 0), (-1, 0)\}$. Define $f : S^n \to S$ by

$$f(x_1^n) = x_1 \cdot x_2 \cdot \ldots \cdot x_n$$

for all $x_1, x_2, \ldots, x_n \in S$ where $\cdot$ is the usual multiplication. Then $(S, f, \leq)$ is an ordered n-ary semigroup. Let $I = \{0\}$. Then $S \setminus I \subseteq (f(S^{n-1}, 1)]$ and $S \setminus I \subseteq (f(S^{n-1}, -1)]$. By Theorem 4.1(2), I is a maximal ordered n-ideal of S.

For an ordered n-ary semigroup S, the notation $\mathcal{U}$ is assumed to be the union of all nonzero proper ordered n-ideals of S if S has a zero element and the notation $\mathcal{U}$ is assumed to be the the union of all proper ordered n-ideals of S if S has no a zero element, from now on.

Lemma 4.2. *$\mathcal{U} = S$ if and only if $I_n(a) \neq S$ for all $a \in S$.*

Proof. Assume that $\mathcal{U} = S$. If $I_n(a) = S$ for some $a \in S$. Thus $a \notin I_\gamma$ for all proper ordered n-ideal I_γ of S. Hence $a \notin \mathcal{U} = S$, which is a contradiction. Therefore, $I_n(a) \neq S$ for all $a \in S$. Conversely, assume that $I_n(a) \neq S$ for all $a \in S$. This implies that $I_n(a)$ is a proper ordered n-ideal for all $a \in S$. Thus $S \subseteq \bigcup_{a \in S} I_n(a) \subseteq \mathcal{U} \subseteq S$. Hence $\mathcal{U} = S$. □

Theorem 4.3. *If S has no zero element, then the exactly one of the following statements is satisfied:*

(1) *S is n-simple.*

(2) *$I_n(a) \neq S$ for all $a \in S$.*

(3) *There exists $a \in S$ such that $I_n(a) = S, a \notin (f(S^{n-1}, a)], (f(a, S^{n-2}, a)] \subseteq \mathcal{U} = S \setminus \{a\}$ and $\mathcal{U}$ is the unique maximal ordered n-ideal of S.*

(4) *$S \setminus \mathcal{U} = \{a \in S \mid (f(S^{n-1}, a)] = S\}$ and $\mathcal{U}$ is the unique maximal ordered n-ideal of S.*

Proof. Assume that S is not n-simple. This implies that there exists a proper ordered n-ideal I of S. Hence $\mathcal{U}$ is an ordered n-ideal of S. We divide into two cases:

Case 1: If $\mathcal{U} = S$, then $I_n(a) \neq S$ for all $a \in S$ by Lemma 4.2. In this case, the statement (2) is satisfied.

Case 2: If $\mathcal{U} \neq S$, then $\mathcal{U}$ is a maximal ordered n-ideal of S. We would like to show that $\mathcal{U}$ is the unique maximal ordered n-ideal of S. Suppose that I is a maximal ordered n-ideal of S, and so I is a proper ordered n-ideal of S. Hence $I \subseteq \mathcal{U} \subsetneq S$. Since I is a maximal ordered n-ideal of S, $I = \mathcal{U}$. Therefore, $\mathcal{U}$ is the unique maximal ordered n-ideal of S as desire. Furthermore, by Theorem 4.1, we acquire

(1) $S \setminus \mathcal{U} = \{a\}$ and $(f(a, S^{n-2}, a)] \subseteq \mathcal{U}$ for some $a \in S$ or

(2) $S \setminus \mathcal{U} \subseteq (f(S^{n-1}, a)]$ for all $a \in S \setminus \mathcal{U}$.

First, we assume that $S \setminus \mathcal{U} = \{a\}$ and $(f(a, S^{n-2}, a)] \subseteq \mathcal{U}$ for some $a \in S$. Since $S \setminus \mathcal{U} = \{a\}$, $(f(a, S^{n-2}, a)] \subseteq \mathcal{U} = S \setminus \{a\}$. Since $a \notin \mathcal{U}$, $I_n(a) = S$. If $a \in (f(S^{n-1}, a)]$, then $\{a\} \subseteq (f(S^{n-1}, a)]$, and hence $S = I_n(a) = (f(S^{n-1}, a) \cup \{a\}] = (f(S^{n-1}, a)]$ by Corollary 2.2. This implies that $a = f(s_1^{n-1}, a)$ and $s_1 = f(s_n^{2n-2}, a)$ for some $s_1, s_2, \ldots, s_{2n-2} \in S$. Hence, $a = f(s_1^{n-1}, a) = f(s_1, s_2^{n-1}, a) = f(f(s_n^{2n-2}, a), s_2^{n-1}, a) = f(s_n^{2n-2}, f(a, s_2^{n-1}, a))$. Since $(f(a, S^{n-2}, a)] \subseteq \mathcal{U}$ and $\mathcal{U}$ is an ordered n-ideal of S, we have that $a = f(s_n^{2n-2}, f(a, s_2^{n-1}, a)) \in \mathcal{U}$, which is a contradiction. Therefore, $a \notin (f(S^{n-1}, a)]$. In this case, the statement (3) is satisfied.

Finally, assume that $S \setminus \mathcal{U} \subseteq (f(S^{n-1}, a)]$ for all $a \in S \setminus \mathcal{U}$. We would like to show that $S \setminus \mathcal{U} = \{a \in S \mid (f(S^{n-1}, a)] = S\}$. Let $a \in S \setminus \mathcal{U}$. By the hypothesis, we have that $a \in (f(S^{n-1}, a)]$, and so $\{a\} \subseteq (f(S^{n-1}, a)]$. Then $I_n(a) = (f(S^{n-1}, a) \cup \{a\}] = (f(S^{n-1}, a)]$ by Corollary 2.2. Since $a \notin \mathcal{U}$, $I_n(a) = S$. Hence $S = I_n(a) = (f(S^{n-1}, a)]$. Now, we get $S \setminus \mathcal{U} \subseteq \{a \in S \mid (f(S^{n-1}, a)] = S\}$. Conversely, let $a \in S$ be such that $S = (f(S^{n-1}, a)]$. If $a \in \mathcal{U}$, then $I_n(a) \subseteq \mathcal{U} \subsetneq S$. By Corollary 2.2, we have that $I_n(a) = (f(S^{n-1}, a) \cup \{a\}] = S \cup (\{a\}] = S$, which is a contradiction. This implies that $a \in S \setminus \mathcal{U}$. This implies that $\{a \in S \mid (f(S^{n-1}, a)] = S\} \subseteq S \setminus \mathcal{U}$. Therefore, $S \setminus \mathcal{U} = \{a \in S \mid (f(S^{n-1}, a)] = S\}$, as desired. In this case, the statement (4) is satisfied.

Hence the proof is completed. □

Example 4.2. (1) Let $S = \{-1, 1\}$ and $\leq := \{(-1, -1), (1, 1)\}$. Define $f : S^n \to S$ by

$$f(x_1^n) = x_1 \cdot x_2 \cdot \ldots \cdot x_n$$

for all $x_1, x_2, \ldots, x_n \in S$ where $\cdot$ is the usual multiplication. Then $(S, f, \leq)$ is an n-simple ordered n-ary semigroup, this implies that $\mathcal{U} = \emptyset$. So, S satisfies the condition (1) of Theorem 4.3.

(2) Let $S = \mathbb{N} \setminus \{1\}$ and and $\leq := id_S$. Define $f : S^n \to S$ by

$$f(x_1^n) = x_1 \cdot x_2 \cdot \ldots \cdot x_n$$

for all $x_1, x_2, \ldots, x_n \in S$ where $\cdot$ is the usual multiplication. It is easy to verify that $I_n(a) \neq S$ for all $a \in S$. Hence S satisfies the condition (2) of Theorem 4.3.

(3) Consider $\mathbb{Z}_{2^{n+1}}$, let $S = \{\overline{0}, \overline{2}, \overline{2^n}\}$ and $\leq := \{(\overline{0}, \overline{0}), (\overline{2}, \overline{2}), (\overline{2^n}, \overline{2^n}), (\overline{0}, \overline{2}), (\overline{0}, \overline{2^n})\}$. Define $f : S^n \to S$ by

$$f(x_1^n) = x_1 \cdot x_2 \cdot \ldots \cdot x_n$$

for all $x_1, x_2, \ldots, x_n \in S$ where $\cdot$ is the multiplication on $\mathbb{Z}_{2^{n+1}}$. Thus $\mathcal{U} = \{\overline{0}, \overline{2^n}\}$. It is easy to verify that S satisfies the condition (3) of Theorem 4.3 by using $a = \overline{2}$.

(4) Let $S = \mathbb{N}$ and $\leq := \{(a, b) \mid a \geq b\}$. Define $f : S^n \to S$ by

$$f(x_1^n) = x_1 \cdot x_2 \cdot \ldots \cdot x_n$$

for all $x_1, x_2, \ldots, x_n \in S$ where $\cdot$ is the usual multiplication. Then $\mathcal{U} = S \setminus \{1\}$. It is easy to verify that S satisfies the condition (4) of Theorem 4.3.

Theorem 4.4. *If S has a zero element and $f(S^n) \neq \{0\}$, then the exactly one of the following statements is satisfied:*

(1) *S is 0-n-simple.*

(2) *$I_n(a) \neq S$ for all $a \in S$.*

(3) *There exists $a \in S$ such that $I_n(a) = S, a \notin (f(S^{n-1}, a)], (f(a, S^{n-2}, a)] \subseteq \mathcal{U} = S \setminus \{a\}$ and $\mathcal{U}$ is the unique maximal ordered n-ideal of S.*

(4) *$S \setminus \mathcal{U} = \{a \in S \mid (f(S^{n-1}, a)] = S\}$ and $\mathcal{U}$ is the unique maximal ordered n-ideal of S.*

Proof. This proof is similar to the proof of Theorem 4.3. □

5 Remark

1. If we consider $n = 2$, an ordered n-ary semigroup is an ordered semigroup. In this case, an ordered n-ideal of an ordered n-ary semigroups is a left ideal of an ordered semigroup.
2. If we consider $\leq = id_S$, an ordered n-ideal of an ordered n-ary semigroup is an n-ideal of an n-ary semigroup.

The results of this paper generalize some results in [2] and [5].

Acknowledgements: The authors would like to thank the referees for valuable suggestions and comments.

References

[1] R. Kasner. An extension of the group concepts, Bull. American Math. Soc. 10 (1904) 290-291.

[2] Y. Cao and X. Xu, On minimal and maximal left ideal in ordered semigroups, Semigroup Forum 60 (2000) 202-207.

[3] M Arslanov and N Kehayopulu. A note on minimal and maximal ideals of ordered semigroups, Lobachevskii J. Math. 11 (2002) 3-6.

[4] A Iampan, The minimality and maximality of left (right) ideals in ternary semigroups, Int. J. Contemporary Math. Sci. 5 (2010) 2409-2417.

[5] P. Petchkaew and R. Chinram, The minimality and maximality of n-ideals in n-ary semigroups, European J. Pure Appl. 11 (2018) 762-773.

[6] Q. Wang, X. Zhou and J. Zhan, A novel study of soft sets over n-ary semigroups, Italian J. Pure Appl. Math. 37 (2017) 583-594.

(Received 4 December 2018)
(Accepted 25 June 2019)