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Right Magnifying Elements in E-preserving Transformation Semigroups with Restricted Range

Thananya Kaewnoi[†,1], **Montakarn Petapirak**[‡] **and Ronnason Chinram**[‡,§]

[†] Department of Mathematics and Statistics, Faculty of Science,
Prince of Songkla University, Hat Yai, Songkhla 90110, Thailand
e-mail : `thananya.k2538@gmail.com`
[‡]Algebra and Applications Research Unit,
Department of Mathematics and Statistics, Faculty of Science,
Prince of Songkla University, Hat Yai, Songkhla 90110, Thailand
e-mail : `montakarn.p@psu.ac.th`
[§]Centre of Excellence in Mathematics,
CHE, Si Ayuthaya Road, Bangkok 10400, Thailand
e-mail : `ronnason.c@psu.ac.th`

Abstract : An element a of a semigroup S is called a right magnifying element if there exists a proper subset M of S such that $S = Ma$. Given Y as a nonempty subset of a set X, the subsemigroup of the full transformation semigroup $T(X)$ on X, consisting of all transformations on X whose range is contained in Y, is denoted by $T(X,Y)$. Let E be an equivalence relation on X and $T_E(X,Y) = \{\alpha \in T(X,Y) \mid (x,y) \in E \text{ implies } ((x)\alpha,(y)\alpha) \in E\}$. In this paper, we provide the necessary and sufficient conditions for elements in $T_E(X,Y)$ to be right magnifying.

Keywords : transformation semigroups; right magnifying elements; equivalence relations.
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1 Introduction

Let X be a nonempty set and let $T(X)$ denote the semigroup of transformations from X into itself. Recall that an element a of a semigroup S is called a right (left) magnifying element if $Ma = S$ ($aM = S$) for some proper subset M of S. This notion was first introduced by Ljapin in [1]. Then the necessary and sufficient conditions for simple, bisimple, and regular semigroups with left identity to contain left magnifying elements had been established by Catino and Migliorini in [2]. In 1994, Magill, Jr. investigated conditions of elements in transformation semigroup to be magnifying elements and applied this result in specific transformation semigroups such as the semigroup of all linear transformations of a vector space and the semigroup of all continuous self-maps of a topological space (see [3]). Afterward, the result that

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[1]`Corresponding author.`

every semigroup containing magnifying elements is factorizable was shown by Gutan in [4]. Recently, magnifying elements have been studied by many authors (see [5], [6], [7], [8] and [9]) such as Chinram and Baupradist who gave necessary and sufficient conditions for elements in subsemigroups of $T(X)$, namely,

$$S(X,Y) = \{f \in T(X) \mid (Y)f \subseteq Y\} \text{ and } T(X,Y) = \{f \in T(X) \mid ran\, f \subseteq Y\}$$

to be left or right magnifying. Likewise, Luangchaisri, Changphas and Phanlert provided conditions for elements of partial transformation semigroups $P(X)$ to be left or right magnifying in [9].

Many authors have extensively studied semigroups preserving equivalence relations. This is a fruitful area of research in transformation semigroups which plays an important role in semigroup theory. In 2004, Araujo and Konieczny investigated a subsemigroup of $T(X)$, namely,

$$T(X,\rho,R) = \{a \in T(X) \mid Ra \subseteq R \text{ and } (x,y) \in \rho \text{ implies } (xa,ya) \in \rho\}$$

consisting of all transformations preserving both an equivalence relation ρ on X and a cross-section R of the partition X/ρ induced by ρ and characterized the equivalence relation ρ on X for this semigroup to be regular. In addition, they described the equivalence relation ρ on X which leads to $\mathcal{D} = \mathcal{J}$ in the semigroup $T(X,\rho,R)$ where $\mathcal{D}$ and $\mathcal{J}$ are Green's relations (see [10]). In 2009, Pei and Deng focused on the subsemigroup $T(X,\rho)$ of $T(X)$ defined as

$$T(X,\rho) = \{f \in T(X) \mid \forall (x,y) \in \rho, (f(x), f(y)) \in \rho\}$$

and showed that if X is an infinite set, then $\rho = X \times X$ if and only if $T(X,\rho)$ is regular. Furthermore, they provided the result that for an infinite set X, for any Green's relations $\mathcal{D}$ and $\mathcal{J}$ in the semigroup $T(X,\rho)$, $\mathcal{D} = \mathcal{J}$ if and only if $T(X,\rho)$ is regular (see [11]). In [12], Pei, Sun, and Zhai characterized the regular elements in $T_E(X;\theta)$ which is a variant semigroup of $T_E(X)$ with the sandwich function $\theta \in T_E(X)$ under the new operation $f \circ g = f\theta g$ in usual composition. Not only did they investigate some properties for regular elements in that semigroup but also they illustrated suitable conditions for its elements to be regular; moreover, they described Green's equivalences for all elements in $T_E(X,\theta)$. In [13], Sun and Lei gave necessary and sufficient conditions of $f \in T_{FE}(X) = T_F(X) \cap T_E(X)$ such that $T_F(X) = \{f \in T(X) \mid \forall (x,y) \in F, (f(x), f(y)) \in F\}$ for equivalence relations F and E on X such that $E \subseteq F$ under the assumption that E and F are comparable to be regular. They also characterized Green's relations on $T_{FE}(X)$. Recently, Kaewnoi, Petapirak, and Chinram have established necessary and sufficient conditions for elements of partial transformation semigroups preserving an equivalence relation to be left or right magnifying elements in [14].

Given a nonempty set X and $Y \subseteq X$, $T(X,Y)$ denote the subsemigroup of the full transformation semigroup $T(X)$ on X consisting of all transformations whose range is contained in Y, i.e.,

$$T(X,Y) = \{\alpha \in T(X) \mid ran\ \alpha \subseteq Y\}.$$

From now on, we write functions from the right, $(x)\alpha$ rather than $\alpha(x)$ and compose from the left to the right, $(x)(\alpha\beta)$ rather than $(\beta \circ \alpha)(x)$, for $\alpha, \beta \in T(X,Y)$ and $x \in X$. Let E be an equivalence relation of X. In this paper, we consider

$$T_E(X,Y) = \{\alpha \in T(X,Y) \mid (x,y) \in E \text{ implies } ((x)\alpha, (y)\alpha) \in E\}$$

which is a subsemigroup of $T(X,Y)$, and provide the necessary and sufficient conditions for elements in $T_E(X,Y)$ to be right magnifying.

2 Main Results

Lemma 2.1. *If α is a right magnifying element in $T_E(X,Y)$, then α is onto.*

Proof. Assume that α is a right magnifying element in $T_E(X,Y)$. According to the definition of right magnifying element, there exists a proper subset M of $T_E(X,Y)$ such that $M\alpha = T_E(X,Y)$. Let $y_0 \in Y$ and define the function $\gamma : X \to Y$ by

$$(x)\gamma = \begin{cases} x & \text{if } x \in Y, \\ x' & \text{if } x \notin Y \text{ and } (x,x') \in E \text{ for some } x' \in Y, \\ y_0 & \text{otherwise.} \end{cases}$$

Clearly, γ is an onto function in $T_E(X,Y)$. Then there exists $\beta \in M$ such that $\beta\alpha = \gamma$. This implies α is onto. □

Lemma 2.2. *Let α be a right magnifying element in $T_E(X,Y)$. For any $x,y \in Y$ such that $(x,y) \in E$, there exists $(a,b) \in E$ such that $x = (a)\alpha, y = (b)\alpha$.*

Proof. Assume that α is a right magnifying element in $T_E(X,Y)$. Then there exists a proper subset M of $T_E(X,Y)$ such that $M\alpha = T_E(X,Y)$. Let $y_0 \in Y$ and define $\gamma : X \to Y$ by

$$(x)\gamma = \begin{cases} x & \text{if } x \in Y, \\ x' & \text{if } x \notin Y \text{ and } (x,x') \in E \text{ for some } x' \in Y, \\ y_0 & \text{otherwise.} \end{cases}$$

Obviously, $\gamma \in T_E(X,Y)$. By assumption, there exists $\beta \in M$ such that $\beta\alpha = \gamma$. Let $x,y \in Y$ such that $(x,y) \in E$. Then $(x)\beta\alpha = (x)\gamma$ and $(y)\beta\alpha = (y)\gamma$. Since $\beta \in T_E(X,Y), ((x)\beta,(y)\beta) \in E$. Choose $a = (x)\beta$ and $b = (y)\beta$. Therefore, there exists $(a,b) \in E$ such that $x = (a)\alpha, y = (b)\alpha$ as desired. □

Lemma 2.3. *If $\alpha \in T_E(X,Y)$ is bijective, then α is not right magnifying.*

Proof. Assume that $\alpha \in T_E(X,Y)$ is bijective. Suppose that α is right magnifying. By the definition of right magnifying element, there exists a proper subset M of $T_E(X,Y)$ such that $M\alpha = T_E(X,Y)$.

Case 1 : $Y \neq X$. Let $\beta \in T_E(X,Y)$ such that β is onto. Then $\beta \in M\alpha$. Since α is bijective and $Y \neq X$, $ran\,\beta \neq Y$, which contradicts that β must be onto. Therefore, α is not right magnifying.

Case 2 : $Y = X$. By Lemma 2.2, this implies that $\alpha^{-1} \in T_E(X,Y)$. Since $M\alpha = T_E(X,Y)$, $M\alpha = T_E(X,Y)\alpha$. Hence,

$$M = M\alpha\alpha^{-1} = T_E(X,Y)\alpha\alpha^{-1} = T_E(X,Y),$$

which is a contradiction. □

Lemma 2.4. *Let $\alpha \in T_E(X, Y)$ be onto but not one-to-one.*

(1) If $(y)\alpha^{-1} \cap Y = \emptyset$ for some $y \in Y$, then α is not right magnifying.

(2) If $|(y)\alpha^{-1} \cap Y| = 1$ for all $y \in Y$, then α is not right magnifying.

Proof. Let $\alpha \in T_E(X, Y)$ be onto but not one-to-one.

(1) Assume that $(y)\alpha^{-1} \cap Y = \emptyset$ for some $y \in Y$. Let $y_0 \in Y$ such that $(y_0)\alpha^{-1} \cap Y = \emptyset$ and define $\gamma : X \to Y$ by $(x)\gamma = y_0$ for all $x \in X$. Clearly, $\gamma \in T_E(X, Y)$. Suppose that α is right magnifying. By definition, there exists a proper subset M of $T_E(X, Y)$ such that $M\alpha = T_E(X, Y)$. Let $\beta \in M$ such that $\beta\alpha = \gamma$ and $x \in X$. Then $(x)\beta\alpha = (x)\gamma = y_0$, which is impossible since $(y_0)\alpha^{-1} \cap Y = \emptyset$.

(2) Assume $|(y)\alpha^{-1} \cap Y| = 1$ for all $y \in Y$. Then $\alpha|_Y$ is bijective and $(\alpha|_Y)^{-1}$ exists. Suppose that α is a right magnifying element in $T_E(X, Y)$. Then there exists a proper subset M of $T_E(X, Y)$ such that $M\alpha = T_E(X, Y)$. Hence $M\alpha = T_E(X, Y)\alpha$. Clearly, $(\alpha(\alpha|_Y)^{-1})|_Y = id_Y$. Since $M\alpha = T_E(X, Y)\alpha$, $M = M\alpha(\alpha|_Y)^{-1} = T_E(X, Y)\alpha(\alpha|_Y)^{-1} = T_E(X, Y)$, a contradiction. □

Lemma 2.5. *Let $\alpha \in T_E(X, Y)$ be onto but not one-to-one. If for any $x, y \in Y$ such that $(x, y) \in E$, there exists $(a, b) \in E$ such that $x = (a)\alpha$, $y = (b)\alpha$, $(y)\alpha^{-1} \cap Y \neq \emptyset$ for all $y \in Y$ and $|(y)\alpha^{-1} \cap Y| > 1$ for some $y \in Y$, then α is right magnifying.*

Proof. Let $\alpha \in T_E(X, Y)$ be onto but not one-to-one. Assume that for any $x, y \in Y$ such that $(x, y) \in E$, there exists $(a, b) \in E$ such that $x = (a)\alpha$, $y = (b)\alpha$, $(y)\alpha^{-1} \cap Y \neq \emptyset$ for all $y \in Y$ and $|(y)\alpha^{-1} \cap Y| > 1$ for some $y \in Y$. Let $M = \{\beta \in T_E(X, Y) \mid \beta \text{ is not onto}\}$. Then $M \neq T_E(X, Y)$. Let $\gamma \in T_E(X, Y)$ be arbitrary. Since α is onto and $(y)\alpha^{-1} \cap Y \neq \emptyset$ for all $y \in Y$, there exists an element $y_x \in Y$ such that $(y_x)\alpha = (x)\gamma$ for all $x \in X$ (if $(x_1)\gamma = (x_2)\gamma$, we must choose $y_{x_1} = y_{x_2}$). Define $\beta \in T(X, Y)$ by $(x)\beta = y_x$ for all $x \in X$. We claim that $\beta \in M$. Firstly, we will show that $\beta \in T_E(X, Y)$. Let $(a, b) \in E$. Then $((a)\gamma, (b)\gamma) \in E$. By assumption, there exists $(y_a, y_b) \in E$ such that $(a)\gamma = (y_a)\alpha$ and $(b)\gamma = (y_b)\alpha$ which implies that $((a)\beta, (b)\beta) \in E$. Next, we will show that β is not onto. Since $|(y)\alpha^{-1} \cap Y| > 1$ for some $y \in Y$, there exists an element $y_0 \in Y$ and distinct elements $y_1, y_2 \in Y$ such that $(y_1)\alpha = (y_2)\alpha = y_0$. If $y_0 \notin ran\,\gamma$, we have $y_1, y_2 \notin ran\,\beta$. If $y_0 \in ran\,\gamma$, then at most one of $\{y_1, y_2\}$ can be in $ran\,\beta$. Then β is not onto. Hence $\beta \in M$ and for all $x \in X$, we have $(x)\beta\alpha = (y_x)\alpha = (x)\gamma$. Then $\beta\alpha = \gamma$, hence $M\alpha = T_E(X, Y)$. Therefore α is right magnifying. □

Theorem 2.6. *A function α in $T_E(X, Y)$ is right magnifying if and only if α satisfies the following properties:*

(1) α is onto but not one-to-one,

(2) $(y)\alpha^{-1} \cap Y \neq \emptyset$ for all $y \in Y$,

(3) $|(y)\alpha^{-1} \cap Y| > 1$ for some $y \in Y$ and

(4) for any $x, y \in Y$ such that $(x, y) \in E$, there exists $(a, b) \in E$ such that $x = (a)\alpha, y = (b)\alpha$.

Proof. Assume that α is right magnifying in $T_E(X, Y)$. By Lemma 2.1 and Lemma 2.3, α is onto but not one-to-one. Furthermore, we have $(y)\alpha^{-1} \cap Y \neq \emptyset$ for all $y \in Y$, $|(y)\alpha^{-1} \cap Y| > 1$ for some $y \in Y$ by using Lemma 2.4. Finally, we apply Lemma 2.2 to claim that for any $(x, y) \in E$, there exists $(a, b) \in E$ such that $x = (a)\alpha, y = (b)\alpha$. Conversely, it follows directly from Lemma 2.5. □

Example 2.7. Let $X = \mathbb{N}$ and $Y = \mathbb{N} \setminus \{1, 2\}$. Define a relation E on X by $(x, y) \in E$ if and only if $x \equiv y \pmod 2$. Clearly, E is an equivalence relation on X and $X/E = \{\{x \in X \mid x \text{ is odd}\}, \{x \in X \mid x \text{ is even}\}\}$. Let $\alpha \in T_E(X, Y)$ be defined by $(1)\alpha = (3)\alpha = 3, (2)\alpha = (4)\alpha = 4$, and $(x)\alpha = x - 2$ for all $x \geqslant 5$, that is,

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 3 & 4 & 3 & 4 & 3 & 4 & 5 & 6 & \cdots \end{pmatrix}.$$

Obviously, α is onto but not one-to-one, $(y)\alpha^{-1} \cap Y \neq \emptyset$ for all $y \in Y$, $|(3)\alpha^{-1} \cap Y| > 1$ and for any $x, y \in Y$ such that $(x, y) \in E$, there exists $(a, b) \in E$ such that $x = (a)\alpha, y = (b)\alpha$. Lemma 2.5 claims that α is right magnifying. Let $M = \{\beta \in T_E(X, Y) \mid \beta \text{ is not onto}\}$ and $\gamma \in T_E(X, Y)$ be any function. Then there exists $\beta \in M$ such that $\beta\alpha = \gamma$. For example, let $\gamma \in T_E(X, Y)$ be defined by $(x)\gamma = x + 2$ for all $x \in X$, that is,

$$\gamma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & \cdots \end{pmatrix}.$$

Define a function $\beta : X \longrightarrow Y$ by $(x)\beta = x + 4$ for all $x \in X$, that is,

$$\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & \cdots \end{pmatrix}.$$

Clearly, $\beta \in M$ and we have

$$\begin{aligned} \beta\alpha &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & \cdots \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 3 & 4 & 3 & 4 & 3 & 4 & 5 & 6 & \cdots \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & \cdots \end{pmatrix} = \gamma. \end{aligned}$$

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