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Linear-Hypersubstitutions for Algebraic Systems of Type $((n);(n))$ and Characterization of Their Idempotent Elements

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Abstract : A formula in which each variable occurs at most once is said to be a linear-formula ([1, 2]). A linear-hypersubstitution for algebraic systems of type $((n);(n))$ is a mapping $\sigma_{_{t,F}}$ which maps n-ary operation symbols f to n-ary linear-terms $\sigma_{_{t,F}}(f)$ and n-ary relational symbols γ to n-ary linear-formulas $\sigma_{_{t,F}}(\gamma)$. Any linear-hypersubstitution $\sigma_{t,F}$ can be extended to a mapping $\widehat{\sigma}_{_{t,F}}$ on the set of all linear-terms of type (n) and linear-formulas of type $((n);(n))$. A binary operation "$\circ_{_{lin}}$" on $Hyp^{lin}((n);(n))$ the set of all linear-hypersubstitutions for algebraic systems of type $((n);(n))$ can be defined by using this extension. The set $Hyp^{lin}((n);(n))$ together with the identity linear-hypersubstitution $(\sigma_{_{t,F}})_{_{id}}$ which maps $(\sigma_{_{t,F}})_{_{id}}(f) := f(x_1,\ldots,x_n)$ and $(\sigma_{_{t,F}})_{_{id}}(\gamma) := \gamma(x_1,\ldots,x_n)$ forms a monoid. The concept of an idempotent element plays an important role in semigroup theory [3]. In this paper, we characterize the idempotent of the monoid of linear-hypersubstitutions for algebraic systems of type $((n);(n))$.

Keywords : algebraic system; linear-formula; linear-hypersubstitution; idempotent.
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1 Introduction

Algebraic systems are understood in the sence of Mal'cev(see [4]). An *algebraic system* of type (τ,τ') is a triple $\mathcal{A} := (A;(f_i^{\mathcal{A}})_{i\in I},(\gamma_j^{\mathcal{A}})_{j\in J})$ consisting of a non-empty set A, an indexed set $(f_i^{\mathcal{A}})_{i\in I}$ of operations defined on A where $f_i^{\mathcal{A}} : A^{n_i} \to A$ is n_i-ary and an indexed set of relations $\gamma_j^{\mathcal{A}} \subseteq A^{n_j}$ is an n_j-ary . The pair (τ,τ') with $\tau = (n_i)_{i\in I},\ \tau' = (n_j)_{j\in J}$ of sequences of positive integers n_i, n_j is called the *type* of $\mathcal{A}$.

The concept of a term and a formula are one of the fundamental concepts of algebraic system. To be independent, first we repeat the most important definitions and results on hypersubstitutions for algebraic systems (see [5]). Using for $n \geq 1$, an n-ary alphabet $X_n = \{x_1, x_2, \ldots, x_n\}$ of individual variables and the alphabet $(f_i)_{i\in I}$ of operation symbols in the usual way one defines terms of type τ by the following steps :

(i) Every $x_l \in X_n$ is an n-ary term of type τ.

(ii) If $t_1,\ldots,t_{n_i}$ are n-ary terms of type τ and if f_i is an n_i-ary operation symbol of type τ, then $f_i(t_1,\ldots,t_{n_i})$ is an n-ary term of type τ.

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Let $W_\tau(X_n)$ be the set of all n-ary terms of type τ. If $X = \{x_1, x_2, \ldots\}$ is a countably infinite alphabet, then $W_\tau(X) := \bigcup\limits_{n\geq 1} W_\tau(X_n)$ denote the set of all terms of type τ (see [6, 7, 8]).

To define quantifier free formulas of type (τ, τ'), we need the logical connectives $\neg$ (for negation), $\vee$ (for disjunction) and the equation symbol $\approx$.

Definition 1.1. *Let $n \in \mathbb{N}^+$. An n-ary quantifier free formula of type (τ, τ') (for short, formula of type (τ, τ')) is defined in the following inductive way* :

(i) *If t_1, t_2 are n-ary terms of type τ, then the equation $t_1 \approx t_2$ is an n-ary quantifier free formula of type (τ, τ').*

(ii) *If $j \in J$ and $t_1, \ldots, t_{n_j}$ are n-ary terms of type τ, then $\gamma_j(t_1, \ldots, t_{n_j})$ is an n-ary quantifier free formula of type (τ, τ').*

(iii) *If F is an n-ary quantifier free formula of type (τ, τ'), then $\neg F$ is an n-ary quantifier free formula of type (τ, τ').*

(iv) *If F_1 and F_2 are n-ary quantifier free formulas of type (τ, τ'), then $F_1 \vee F_2$ is an n-ary quantifier free formula of type (τ, τ').*

Let $\mathcal{F}_{(\tau,\tau')}(X_n)$ be the set of all n-ary quantifier free formulas of type (τ, τ') and let $\mathcal{F}_{(\tau,\tau')}(X)$ $:= \bigcup\limits_{n\geq 1} \mathcal{F}_{(\tau,\tau')}(X_n)$ be the set of all quantifier free formulas of type (τ, τ').

2 Linear-Terms of Type τ and Linear-Formulas of Type $(\tau, \tau^{'})$

A term in which each variable occurs at most once, is said to be a linear. For a formal definition of n-ary linear-term, we replace (ii) in the definition of terms by a slightly different condition. Let $var(t)$ is the set of all variables occuring in a term t and $var(F)$ is the set of all variables occuring in a formula F.

Definition 2.1. *Let $n \in \mathbb{N}^+$. An n-ary linear-term of type τ is defined in the following inductive way* :

(i) *Every $x_j \in X_n$ is an n-ary linear-term of type τ.*

(ii) *If $t_1, \ldots, t_{n_i}$ are n-ary linear-terms of type τ and $var(t_l) \cap var(t_k) = \emptyset$ for all $1 \leq l < k \leq n_i$, then $f_i(t_1, \ldots, t_{n_i})$ is an n-ary linear-term of type τ.*

(iii) *The set $W_\tau^{lin}(X_n)$ of all n-ary linear-terms of type τ is the smallest set which contains $x_1, \ldots, x_n$ and closed under finite applications of (ii)*

The set of all linear-terms of type τ over the countably infinite alphabet X is defined by $W_\tau^{lin}(X) := \bigcup\limits_{n\geq 1} W_\tau^{lin}(X_n)$.

Definition 2.2. *Let $n \in \mathbb{N}^+$. An n-ary linear-formula of type (τ, τ') is defined by the following inductive way* :

(i) *If t_1, t_2 are n-ary linear-terms of type τ and $var(t_1) \cap var(t_2) = \emptyset$, then the equation $t_1 \approx t_2$ is an n-ary linear-formula of type (τ, τ').*

(ii) *If $t_1, \ldots, t_{n_j}$ are n-ary linear-terms of type τ, $var(t_l) \cap var(t_k) = \emptyset$; $l, k \in \{1, 2, \ldots, n_j\}$ and γ_j is an n_j-ary relational symbol, then $\gamma_j(t_1, \ldots, t_{n_j})$ is an n-ary linear-formula of type (τ, τ').*

(iii) *If F is an n-ary linear-formula of type (τ, τ'), then $\neg F$ is an n-ary linear-formula of type (τ, τ').*

(iv) *If F_1 , F_2 are n-ary linear-formulas of type (τ, τ') and $var(F_1) \cap var(F_2) = \emptyset$, then $F_1 \vee F_2$ is an n-ary linear-formula of type (τ, τ').*

Let $\mathcal{F}^{lin}_{(\tau,\tau')}(X_n)$ be the set of all n-ary linear-formulas of type (τ,τ') and let $\mathcal{F}^{lin}_{(\tau,\tau')}(X) := \bigcup_{n\geq 1} \mathcal{F}^{lin}_{(\tau,\tau')}(X_n)$ be the set of all linear-formulas of type (τ,τ').

For this paper, we consider the type $(\tau,\tau') := ((n);(n))$, then $f(t_1,\ldots,t_n)$ can not be a linear-term, where $t_1,\ldots,t_n \in W_n(X_n)\setminus X_n$ and $F_1 \vee F_2$ can not be a linear-formula, because $var(F_1)\cap vae(F_2) \neq \emptyset$ as the following the example:

Example 2.3. *Let $(\tau,\tau') := ((2);(2))$ with a binary operation symbol f and a binary relational symbol γ and let $X_2 = \{x_1,x_2\}$. Then $W^{lin}_{(2)}(X_2) = \{x_1,x_2,f(x_1,x_2),f(x_2,x_1)\}$ and $\mathcal{F}^{lin}_{((2);(2))}(X_2) = \{x_1 \approx x_2, x_2 \approx x_1, \gamma(x_1,x_2), \gamma(x_2,x_1), \neg(x_1 \approx x_2), \neg(x_2\approx x_1), \neg(\gamma(x_1,x_2)), \neg(\gamma(x_2,x_1)), \neg(\neg(x_1\approx x_2)),\ldots\}$.*

3 Superposition of Linear-Terms and Linear-Formulas of Type $((n);(n))$

Substituting the variables occuring in a linear-term by other linear-terms one obtains a new linear-term. This can be described by the superposition operation $S^n_{lin}, n \geq 1$ for linear-terms which is inductively defined as follows :

Definition 3.1. *Let $n \in \mathbb{N}^+$ and $t,t_1,\ldots,t_n \in W^{lin}_n(X_n)$ such that $var(t_l)\cap var(t_k) = \emptyset$, for $l,k \in \{1,\ldots,n\}$. The operation*

$$S^n_{lin} : W^{lin}_n(X_n) \times (W^{lin}_n(X_n))^n \to W^{lin}_n(X_n)$$

is defined in the following inductive way :

(i) *If $t = x_i$, then $S^n_{lin}(x_i,t_1,\ldots,t_n) := t_i$; $1 \leq i \leq n$,*

(ii) *If $t = f(s_1,\ldots,s_n)$ and assume that, $S^n_{lin}(s_l,t_1,\ldots,t_n)$ is a linear-term already, for $l \in \{1,\ldots,n\}$ such that $var(S^n_{lin}(s_l,t_1,\ldots,t_n)) \cap var(S^n_{lin}(s_k,t_1,\ldots,t_n)) = \emptyset$; $1 \leq l,k \leq n$, then $S^n_{lin}(f(s_1,\ldots,s_n),t_1,\ldots,t_n) := f(S^n_{lin}(s_1,t_1,\ldots,t_n),\ldots,S^n_{lin}(s_n,t_1,\ldots,t_n))$.*

Now, we will extend this superposition of linear-terms of type (n) to a superposition of linear-formulas of type $((n);(n))$ as follows :

Definition 3.2. *Let $n \in \mathbb{N}^+$ and $t,t_1,\ldots,t_n \in W^{lin}_n(X_n)$ such that $var(t_l)\cap var(t_k) = \emptyset$; $l,k \in \{1,\ldots,n\}$ and S^n_{lin} be the superposition of linear-terms which have defined above. The operation*

$$R^n_{lin} : W^{lin}_n(X_n) \cup \mathcal{F}^{lin}_{((n);(n))}(X_n) \times (W^{lin}_n(X_n))^n \to W^{lin}_n(X_n) \cup \mathcal{F}^{lin}_{((n);(n))}(X_n)$$

is defined in the following inductive way :

(i) *If $t \in W^{lin}_n(X_n)$, then $R^n_{lin}(t,t_1,\ldots,t_n) := S^n_{lin}(t,t_1,\ldots,t_n)$.*

(ii) *If F has the form $s_1 \approx s_2$ and $var(S^n_{lin}(s_1,t_1,\ldots,t_n)) \cap var(S^n_{lin}(s_2,t_1,\ldots,t_n)) = \emptyset$, then $R^n_{lin}(s_1\approx s_2,t_1,\ldots,t_n) := S^n_{lin}(s_1,t_1,\ldots,t_n) \approx S^n_{lin}(s_2,t_1,\ldots,t_n)$.*

(iii) *If F has the form $\gamma(s_1,\ldots,s_n)$, and assume that $S^n_{lin}(s_l,t_1,\ldots,t_n)$ is already a linear-term ; $l \in \{1,\ldots,n\}$ such that $var(S^n_{lin}(s_l,t_1,\ldots,t_n)) \cap var(S^n_{lin}(s_k,t_1,\ldots,t_n)) = \emptyset$; $1 \leq l,k \leq n$, then $R^n_{lin}(\gamma(s_1,\ldots,s_n),t_1,\ldots,t_n) := \gamma(S^n_{lin}(s_1,t_1,\ldots,t_n),\ldots,S^n_{lin}(s_n,t_1,\ldots,t_n))$.*

(iv) *If F has the form $\neg F$, and assume that $R^n_{lin}(F,t_1,\ldots,t_n)$ is already a linear-formula , then $R^n_{lin}(\neg F,t_1,\ldots,t_n) := \neg(R^n_{lin}(F,t_1,\ldots,t_n))$.*

Theorem 3.3. *Let $\beta \in W_n^{lin}(X_n) \cup \mathcal{F}^{lin}_{((n);(n))}(X_n)$. The operation R^n_{lin} satisfies :*

(LFC1)$R^n_{lin}(R^n_{lin}(\beta, t_1, \ldots, t_n), s_1, \ldots, s_n) = R^n_{lin}(\beta, R^n_{lin}(t_1, s_1, \ldots, s_n), \ldots, R^n_{lin}(t_n, s_1, \ldots, s_n))$ *whenever* $t_1, \ldots, t_n, s_1, \ldots, s_n \in W_n^{lin}(X_n)$ *and* $var(t_l) \cap var(t_k) = \emptyset$, $var(s_l) \cap var(s_k) = \emptyset$; $l, k \in \{1, \ldots, n\}$.

(LFC2) $R^n_{lin}(x_i, t_1, \ldots, t_n) = t_i$ *whenever* $t_1, \ldots, t_n \in W_n^{lin}(X_n)$ *and* $var(t_l) \cap var(t_k) = \emptyset$; $l, k \in \{1, \ldots, n\}$.

(LFC3) $R^n_{lin}(\beta, x_1, \ldots, x_n) = \beta$.

Proof. Let π be a permutation on the set $\{1, 2, \ldots, n\}$. For $\beta = t \in W_n^{lin}(X_n)$, we will give a proof of (LFC1) by induction on the complexity of a linear-term t.

(i) If $t = x_i$; $1 \leq i \leq n$, then

$$\begin{aligned}
&R^n_{lin}(R^n_{lin}(x_i, t_1, \ldots, t_n), s_1, \ldots, s_n)\\
&= R^n_{lin}(S^n_{lin}(x_i, t_1, \ldots, t_n), s_1, \ldots, s_n)\\
&= S^n_{lin}(t_i, s_1, \ldots, s_n)\\
&= S^n_{lin}(x_i, S^n_{lin}(t_1, s_1, \ldots, s_n), \ldots, S^n_{lin}(t_n, s_1, \ldots, s_n))\\
&= R^n_{lin}(x_i, R^n_{lin}(t_1, s_1, \ldots, s_n), \ldots, R^n_{lin}(t_n, s_1, \ldots, s_n)).
\end{aligned}$$

(ii) If $t = f(x_{\pi(1)}, \ldots, x_{\pi(n)})$ and assume that $S^n_{lin}(S^n_{lin}(x_{\pi(l)}, t_1, \ldots, t_n), s_1, \ldots, s_n)$ $= S^n_{lin}(x_{\pi(l)}, S^n_{lin}(t_1, s_1, \ldots, s_n), \ldots, S^n_{lin}(t_n, s_1, \ldots, s_n)); 1 \leq l \leq n$, then

$$\begin{aligned}
&R^n_{lin}(R^n_{lin}(f(x_{\pi(1)}, \ldots, x_{\pi(n)}), t_1, \ldots, t_n), s_1, \ldots, s_n)\\
&= R^n_{lin}(S^n_{lin}(f(x_{\pi(1)}, \ldots, x_{\pi(n)}), t_1, \ldots, t_n), s_1, \ldots, s_n)\\
&= R^n_{lin}(f(S^n_{lin}(x_{\pi(1)}, t_1, \ldots, t_n), \ldots, S^n_{lin}(x_{\pi(n)}, t_1, \ldots, t_n)), s_1, \ldots, s_n)\\
&= f(S^n_{lin}(S^n_{lin}(x_{\pi(1)}, t_1, \ldots, t_n), s_1, \ldots, s_n), \ldots, S^n_{lin}(S^n_{lin}(x_{\pi(n)}, t_1, \ldots, t_n), s_1, \ldots, s_n))\\
&= f(S^n_{lin}(x_{\pi(1)}, S^n_{lin}(t_1, s_1, \ldots, s_n), \ldots, S^n_{lin}(t_n, s_1, \ldots, s_n)), \ldots,\\
&\quad S^n_{lin}(x_{\pi(n)}, S^n_{lin}(t_1, s_1, \ldots, s_n), \ldots, S^n_{lin}(t_n, s_1, \ldots, s_n)))\\
&= S^n_{lin}(f(x_{\pi(1)}, \ldots, x_{\pi(n)}), S^n_{lin}(t_1, s_1, \ldots, s_n), \ldots, S^n_{lin}(t_n, s_1, \ldots, s_n))\\
&= R^n_{lin}(f(x_{\pi(1)}, \ldots, x_{\pi(n)}), R^n_{lin}(t_1, s_1, \ldots, s_n), \ldots, R^n_{lin}(t_n, s_1, \ldots, s_n)).
\end{aligned}$$

For $\beta = F \in \mathcal{F}^{lin}_{((n);(n))}(X_n)$, we will give a proof of (LFC1) by induction on the complexity of a linear-formula F.

(i) If F has the form $x_i \approx x_j$ for $i \neq j \in \{1, \ldots, n\}$, then

$$\begin{aligned}
&R^n_{lin}(R^n_{lin}(x_i \approx x_j, t_1, \ldots, t_n), s_1, \ldots, s_n)\\
&= R^n_{lin}(S^n_{lin}(x_i, t_1, \ldots, t_n) \approx S^n_{lin}(x_j, t_1, \ldots, t_n), s_1, \ldots, s_n)\\
&= S^n_{lin}(S^n_{lin}(x_i, t_1, \ldots, t_n), s_1, \ldots, s_n) \approx S^n_{lin}(S^n_{lin}(x_j, t_1, \ldots, t_n), s_1, \ldots, s_n)\\
&= S^n_{lin}(x_i, S^n_{lin}(t_1, s_1, \ldots, s_n), \ldots, S^n_{lin}(t_n, s_1, \ldots, s_n)) \approx\\
&\quad S^n_{lin}(x_j, S^n_{lin}(t_1, s_1, \ldots, s_n), \ldots, S^n_{lin}(t_n, s_1, \ldots, s_n))\\
&= R^n_{lin}(x_i \approx x_j, R^n_{lin}(t_1, s_1, \ldots, s_n), \ldots, R^n_{lin}(t_n, s_1, \ldots, s_n)).
\end{aligned}$$

(ii) If F has the form $\gamma(x_{\pi(1)}, \ldots, x_{\pi(n)})$, then

$$\begin{aligned}
&R^n_{lin}(R^n_{lin}(\gamma(x_{\pi(1)}, \ldots, x_{\pi(n)}), t_1, \ldots, t_n), s_1, \ldots, s_n)\\
&= R^n_{lin}(\gamma(S^n_{lin}(x_{\pi(1)}, t_1, \ldots, t_n), \ldots, S^n_{lin}(x_{\pi(n)}, t_1, \ldots, t_n)), s_1, \ldots, s_n)\\
&= \gamma(S^n_{lin}(S^n_{lin}(x_{\pi(1)}, t_1, \ldots, t_n), s_1, \ldots, s_n), \ldots, S^n_{lin}(S^n_{lin}(x_{\pi(n)}, t_1, \ldots, t_n), s_1, \ldots, s_n))\\
&= \gamma(S^n_{lin}(x_{\pi(1)}, S^n_{lin}(t_1, s_1, \ldots, s_n), \ldots, S^n_{lin}(t_n, s_1, \ldots, s_n)), \ldots,\\
&\quad S^n_{lin}(x_{\pi(n)}, S^n_{lin}(t_1, s_1, \ldots, s_n), \ldots, S^n_{lin}(t_n, s_1, \ldots, s_n)))\\
&= R^n_{lin}(\gamma(x_{\pi(1)}, \ldots, x_{\pi(n)}), R^n_{lin}(t_1, s_1, \ldots, s_n), \ldots, R^n_{lin}(t_n, s_1, \ldots, s_n)).
\end{aligned}$$

(iii) If F has the form $\neg F$ and assume that $R^n_{lin}(R^n_{lin}(F, t_1, \ldots, t_n), s_1, \ldots, s_n)$ $= R^n_{lin}(F, R^n_{lin}(t_1, s_1, \ldots, s_n), \ldots, R^n_{lin}(t_n, s_1, \ldots, s_n))$, then

$$\begin{aligned}
&R^n_{lin}(R^n_{lin}(\neg F, t_1, \ldots, t_n), s_1, \ldots, s_n)\\
&= R^n_{lin}(\neg(R^n_{lin}(F, t_1, \ldots, t_n)), s_1, \ldots, s_n)\\
&= \neg(R^n_{lin}(R^n_{lin}(F, t_1, \ldots, t_n), s_1, \ldots, s_n))\\
&= R^n_{lin}(\neg F, R^n_{lin}(t_1, s_1, \ldots, s_n), \ldots, R^n_{lin}(t_n, s_1, \ldots, s_n)).
\end{aligned}$$

For (LFC2) is clearly by Definition 3.1(i).
The proof of (LFC3), we will proceed in a similar way considering the completely of a linear-term t.

(i) If $t = x_i$; $1 \leq i \leq n$, then
$R^n_{lin}(x_i, x_1, \ldots, x_n) \; = \; S^n_{lin}(x_i, x_1, \ldots, x_n) \; = \; x_i.$

(ii) If $t = f(x_{\pi(1)}, \ldots, x_{\pi(n)})$ and assume that $R^n_{lin}(x_{\pi(l)}, x_1, \ldots, x_n) = x_{\pi(l)}$; $1 \leq l \leq n$, then
$$\begin{aligned} & R^n_{lin}(f(x_{\pi(1)}, \ldots, x_{\pi(n)}), x_1, \ldots, x_n) \\ &= S^n_{lin}(f(x_{\pi(1)}, \ldots, x_{\pi(n)}), x_1, \ldots, x_n) \\ &= f(S^n_{lin}(x_{\pi(1)}, x_1 \ldots, x_n), \ldots, S^n_{lin}(x_{\pi(n)}, x_1 \ldots, x_n)) \\ &= f(x_{\pi(1)}, \ldots, x_{\pi(n)}). \end{aligned}$$

Next, we will proceed in a similar way considering the completely of a linear-formula F.

(i) If F has the form $x_i \approx x_j$ for $i \neq j \in \{1, \ldots, n\}$, then
$R^n_{lin}(x_i \approx x_j, x_1, \ldots, x_n) = S^n_{lin}(x_i, x_1, \ldots, x_n) \approx S^n_{lin}(x_j, x_1, \ldots, x_n) = x_i \approx x_j.$

(ii) If F has the form $\gamma(x_{\pi(1)}, \ldots, x_{\pi(n)})$, then
$$\begin{aligned} & R^n_{lin}(\gamma(x_{\pi(1)}, \ldots, x_{\pi(n)}), x_1, \ldots, x_n) \\ &= \gamma(S^n_{lin}(x_{\pi(1)}, x_1 \ldots, x_n), \ldots, S^n_{lin}(x_{\pi(n)}, x_1 \ldots, x_n)) = \gamma(x_{\pi(1)}, \ldots, x_{\pi(n)}). \end{aligned}$$

(iii) If F has the form $\neg F$ and assume that $R^n_{lin}(F, x_1, \ldots, x_n) = F$, then
$R^n_{lin}(\neg F, x_1, \ldots, x_n) = \neg(R^n_{lin}(F, x_1, \ldots, x_n)) = \neg F.$

□

4 Linear-Hypersubstitutions for Algebraic Systems of Type ((n);(n))

The concept of linear-hypersubstitutions for universal algebras was introduced by Changphas, Denecke and Pibaljomme [9]. We are going to extend this concept to algebraic systems of type $((n);(n))$ as the following:

Definition 4.1. *Any mapping*

$$\sigma : \{f\} \cup \{\gamma\} \to W^{lin}_n(X_n) \cup \mathcal{F}^{lin}_{((n);(n))}(X_n)$$

which maps operation symbols f to linear-terms and relational symbols γ to linear-formulas preserving arities is called a linear-hypersubstitution for algebraic systems (of type $((n);(n))$).

Let $Hyp^{lin}((n);(n))$ be the set of all linear-hypersubstitutions for algebraic systems of type $((n);(n))$.

We define the extension of linear-hypersubstitutions for algebraic systems of type $((n);(n))$ as follows:

$$\widehat{\sigma} : W^{lin}_n(X_n) \cup \mathcal{F}^{lin}_{((n);(n))}(X_n) \to W^{lin}_n(X_n) \cup \mathcal{F}^{lin}_{((n);(n))}(X_n)$$

inductively defined as follows:

(i) $\widehat{\sigma}[x] := x$ for any variable $x \in X_n$,

(ii) $\widehat{\sigma}[f(x_{\pi(1)}, \ldots, x_{\pi(n)})] := S^n_{lin}(\sigma(f), \widehat{\sigma}[x_{\pi(1)}], \ldots, \widehat{\sigma}[x_{\pi(n)}])$,

(iii) $\widehat{\sigma}[x_i \approx x_j] := \widehat{\sigma}[x_i] \approx \widehat{\sigma}[x_j]$ for $i \neq j \in \{1, \ldots, n\}$,

(iv) $\widehat{\sigma}[\gamma(x_{\pi(1)}, \ldots, x_{\pi(n)})] := R^n_{lin}(\sigma(\gamma), \widehat{\sigma}[x_{\pi(1)}], \ldots, \widehat{\sigma}[x_{\pi(n)}])$,

(v) $\widehat{\sigma}[\neg F] := \neg\widehat{\sigma}[F]$ for $F \in \mathcal{F}^{lin}_{((n);(n))}(X_n)$.

Then, $\widehat{\sigma}$ is called the extension of a linear-hypersubstitution for algebraic system σ .

Next, we defined a binary operation " $\circ_{lin}$ " on $Hyp^{lin}((n);(n))$ by $\sigma_1 \circ_{lin} \sigma_2 := \widehat{\sigma}_1 \circ \sigma_2$ where $\circ$ denotes the usual composition of mapping and $\sigma_1, \sigma_2 \in Hyp^{lin}((n);(n))$. The purpose of this paper, the structure $(Hyp^{lin}((n);(n)), \circ_{lin}, \sigma_{id})$ becomes a monoid. An importent property for extension is proved as follows:

Lemma 4.2. *For $n \in \mathbb{N}$, let $\sigma \in Hyp^{lin}((n);(n))$, and let $t_1, ..., t_n \in W_n^{lin}(X_n)$ and $var(t_l) \cap var(t_k) = \emptyset; 1 \leq l, k \leq n$. Then*

$$\widehat{\sigma}[R^n_{lin}(\beta, t_1, ..., t_n)] = R^n_{lin}(\widehat{\sigma}[\beta], \widehat{\sigma}[t_1], ..., \widehat{\sigma}[t_n]),$$

for any $\beta \in W_n^{lin}(X_n) \cup \mathcal{F}^{lin}_{((n);(n))}(X_n)$.

Proof. For $\beta = t \in W_n^{lin}(X_n)$, we will give a proof by induction on the complexity of the definition of a linear-term t.

(i) If $t = x_i; 1 \leq i \leq n$, then
$\widehat{\sigma}\,[S^n_{lin}\,(x_i, t_1, ..., t_n)] \;=\; \widehat{\sigma}[t_i] \;=\; S^n_{lin}\,(x_i, \widehat{\sigma}[t_1], ..., \widehat{\sigma}[t_n]) \;=\; S^n_{lin}(\widehat{\sigma}[x_i], \widehat{\sigma}[t_1], ..., \widehat{\sigma}[t_n])$.

(ii) If $t = f(x_{\pi(1)}, \ldots, x_{\pi(n)})$, and assume that
$\widehat{\sigma}\left[S^n_{lin}\left(x_{\pi(l)}, t_1, ..., t_n\right)\right] = S^n_{lin}(\widehat{\sigma}[x_{\pi(l)}], \widehat{\sigma}[t_1], ..., \widehat{\sigma}[t_n]); 1 \leq l \leq n$, then

$$\begin{aligned}
&\widehat{\sigma}\left[S^n_{lin}\left(f(x_{\pi(1)}, \ldots, x_{\pi(n)}), t_1, ..., t_n\right)\right] \\
&= \widehat{\sigma}[f(S^n_{lin}(x_{\pi(1)}, t_1, \ldots, t_n), \ldots, S^n_{lin}(x_{\pi(n)}, t_1, \ldots, t_n))] \\
&= S^n_{lin}(\sigma(f), \widehat{\sigma}[S^n_{lin}(x_{\pi(1)}, t_1, \ldots, t_n)], \ldots, \widehat{\sigma}[S^n_{lin}(x_{\pi(n)}, t_1, \ldots, t_n)]) \\
&= S^n_{lin}(\sigma(f), S^n_{lin}(\widehat{\sigma}[x_{\pi(1)}], \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_n]), \ldots, S^n_{lin}(\widehat{\sigma}[x_{\pi(n)}], \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_n]) \\
&= S^n_{lin}(S^n_{lin}(\sigma(f), \widehat{\sigma}[x_{\pi(1)}], \ldots, \widehat{\sigma}[x_{\pi(n)}]), \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_n]) \\
&= S^n_{lin}(\widehat{\sigma}[f(x_{\pi(1)}, \ldots, x_{\pi(n)})], \widehat{\sigma}[t_1], ..., \widehat{\sigma}[t_n]).
\end{aligned}$$

For $\beta = F \in \mathcal{F}^{lin}_{((n);(n))}(X_n)$, we will give a proof by induction on the complexity of the definition of a linear-formula F.

(i) If F has the form $x_i \approx x_j$ for $i \neq j \in \{1, \ldots, n\}$, then

$$\begin{aligned}
&\widehat{\sigma}[R^n_{lin}(x_i \approx x_j, t_1, \ldots, t_n)] \\
&= \widehat{\sigma}[S^n_{lin}(x_i, t_1, \ldots, t_n) \approx S^n_{lin}(x_j, t_1, \ldots, t_n)] \\
&= S^n_{lin}(\widehat{\sigma}[x_i], \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_n]) \approx S^n_{lin}(\widehat{\sigma}[x_j], \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_n]) \\
&= R^n_{lin}(\widehat{\sigma}[x_i \approx x_j], \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_n]).
\end{aligned}$$

(ii) If F has the form $\gamma(x_{\pi(1)}, \ldots, x_{\pi(n)})$ and assume that
$\widehat{\sigma}[R^n_{lin}(x_{\pi(l)}, t_1, \ldots, t_n)] = R^n_{lin}(\widehat{\sigma}[x_{\pi(l)}], (\widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_n]); 1 \leq l \leq n$, then

$$\begin{aligned}
&\widehat{\sigma}[R^n_{lin}(\gamma(x_{\pi(1)}, \ldots, x_{\pi(n)}), t_1, \ldots, t_n)] \\
&= \widehat{\sigma}[\gamma(S^n_{lin}(x_{\pi(1)}, t_1, \ldots, t_n), \ldots, S^n_{lin}(x_{\pi(n)}, t_1, \ldots, t_n))] \\
&= R^n_{lin}(\sigma(\gamma), \widehat{\sigma}[S^n_{lin}(x_{\pi(1)}, t_1, \ldots, t_n)], \ldots, \widehat{\sigma}[S^n_{lin}(x_{\pi(n)}, t_1, \ldots, t_n)]) \\
&= R^n_{lin}(R^n_{lin}(\sigma(\gamma), \widehat{\sigma}[x_{\pi(1)}], \ldots, \widehat{\sigma}[x_{\pi(n)}]), \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_n]) \\
&= R^n_{lin}(\widehat{\sigma}[\gamma(x_{\pi(1)}, \ldots, x_{\pi(n)})], \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_n]).
\end{aligned}$$

(iii) If F has the form $\neg F$ and assume that
$\widehat{\sigma}[R^n_{lin}(F, t_1, \ldots, t_n)] = R^n_{lin}(\widehat{\sigma}[F], \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_n])$, then

$$\begin{aligned}
&\widehat{\sigma}[R^n_{lin}(\neg F, t_1, \ldots, t_n)] \\
&= \neg(\widehat{\sigma}[R^n_{lin}(F, t_1, \ldots, t_n)]) \\
&= \neg(R^n_{lin}(\widehat{\sigma}[F], \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_n])) \\
&= R^n_{lin}(\widehat{\sigma}[\neg F]), \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_n]).
\end{aligned}$$

□

Lemma 4.3. *For any $\sigma_1, \sigma_2 \in Hyp^{lin}((n);(n))$, we have*

$$(\sigma_1 \circ_{lin} \sigma_2)\widehat{\ } = \widehat{\sigma}_1 \circ \widehat{\sigma}_2.$$

Proof. For $t \in W_n^{lin}(X_n)$, we will give a proof by induction on the complexity of the definition of a linear-term t.

(i) If $t = x_i$; $1 \leq i \leq n$,then
$(\sigma_1 \circ_{lin} \sigma_2)\widehat{}\ [x_i] = x_i = \widehat{\sigma}_1[x_i] = \widehat{\sigma}_1[\widehat{\sigma}_2[x_i]] = (\widehat{\sigma}_1 \circ \widehat{\sigma}_2)[x_i]$.

(ii) If $t = f(x_{\pi(1)}, \ldots, x_{\pi(n)})$, then

$$\begin{aligned}
&(\sigma_1 \circ_{lin} \sigma_2)\widehat{}\ [f(x_{\pi(1)}, \ldots, x_{\pi(n)})] \\
&= S_{lin}^n((\sigma_1 \circ_{lin} \sigma_2)(f), (\sigma_1 \circ_{lin} \sigma_2)\widehat{}\ [x_{\pi(1)}], \ldots, (\sigma_1 \circ_{lin} \sigma_2)\widehat{}\ [x_{\pi(n)}]) \\
&= S_{lin}^n((\widehat{\sigma}_1 \circ \sigma_2)(f), (\widehat{\sigma}_1 \circ \widehat{\sigma}_2)[x_{\pi(1)}], \ldots, (\widehat{\sigma}_1 \circ \widehat{\sigma}_2)[x_{\pi(n)}]) \\
&= S_{lin}^n(\widehat{\sigma}_1[\sigma_2(f)], \widehat{\sigma}_1[\widehat{\sigma}_2[x_{\pi(1)}]], \ldots, \widehat{\sigma}_1[\widehat{\sigma}_2[x_{\pi(n)}]]) \\
&= \widehat{\sigma}_1[S_{lin}^n(\sigma_2(f), \widehat{\sigma}_2[x_{\pi(1)}], \ldots, \widehat{\sigma}_2[x_{\pi(n)}])] \\
&= \widehat{\sigma}_1[\widehat{\sigma}_2[f(x_{\pi(1)}, \ldots, x_{\pi(n)})]] \\
&= (\widehat{\sigma}_1 \circ \widehat{\sigma}_2)[f(x_{\pi(1)}, \ldots, x_{\pi(n)})].
\end{aligned}$$

For $F \in \mathcal{F}_{((n);(n))}^{lin}(X_n)$, we will give a proof by induction on the complexity of the definition of a linear-formula F.

(i) If F has the form $x_i \approx x_j$ for $i \neq j \in \{1, \ldots, n\}$,
then $(\sigma_1 \circ_{lin} \sigma_2)\widehat{}\,[x_i \approx x_j]$

$$\begin{aligned}
&= (\sigma_1 \circ_{lin} \sigma_2)\widehat{}\,[x_i] \approx (\sigma_1 \circ_{lin} \sigma_2)\widehat{}\,[x_j] \\
&= x_i \approx x_j. \\
&= \widehat{\sigma}_1[\widehat{\sigma}_2[x_i]] \approx \widehat{\sigma}_1[\widehat{\sigma}_2[x_j]] \\
&= (\widehat{\sigma}_1 \circ \widehat{\sigma}_2)[x_i] \approx (\widehat{\sigma}_1 \circ \widehat{\sigma}_2)[x_j] \\
&= (\widehat{\sigma}_1 \circ \widehat{\sigma}_2)[x_i \approx x_j].
\end{aligned}$$

(ii) If F has the form $\gamma(x_{\pi(1)}, \ldots, x_{\pi(n)})$, then

$$\begin{aligned}
&(\sigma_1 \circ_{lin} \sigma_2)\widehat{}\,[\gamma(x_{\pi(1)}, \ldots, x_{\pi(n)})] \\
&= R_{lin}^n((\sigma_1 \circ_{lin} \sigma_2)(\gamma), (\sigma_1 \circ_{lin} \sigma_2)\widehat{}\,[x_{\pi(1)}], \ldots, (\sigma_1 \circ_{lin} \sigma_2)\widehat{}\,[x_{\pi(n)}]) \\
&= R_{lin}^n((\widehat{\sigma}_1 \circ \sigma_2)(\gamma), x_{\pi(1)}, \ldots, x_{\pi(n)}) \\
&= R_{lin}^n(\widehat{\sigma}_1[\sigma_2(\gamma)], x_{\pi(1)}, \ldots, x_{\pi(n)}) \\
&= \widehat{\sigma}_1[\widehat{\sigma}_2[\gamma(x_{\pi(1)}, \ldots, x_{\pi(n)})]] \\
&= (\widehat{\sigma}_1 \circ \widehat{\sigma}_2)[\gamma(x_{\pi(1)}, \ldots, x_{\pi(n)})].
\end{aligned}$$

(iii) If F has the form $\neg F$ and assume that $(\sigma_1 \circ_{lin} \sigma_2)\widehat{}\,[F] = (\widehat{\sigma}_1 \circ \widehat{\sigma}_2)[F]$, then
$(\sigma_1 \circ_{lin} \sigma_2)\widehat{}\,[\neg F] = \neg((\sigma_1 \circ_{lin} \sigma_2)\widehat{}\,[F]) = \neg((\widehat{\sigma}_1 \circ \widehat{\sigma}_2)[F]) = \widehat{\sigma}_1[\neg(\widehat{\sigma}_2[F])] = \widehat{\sigma}_1[\widehat{\sigma}_2[\neg(F)]] = (\widehat{\sigma}_1 \circ \widehat{\sigma}_2)[\neg(F)]$. □

Let σ_{id} be a linear-hypersubstitution for algebraic systems of type $((n);(n))$ which maps the operation symbols f to the linear-term $f(x_1, \ldots, x_n)$, and the relational symbols γ to the linear-formula $\gamma(x_1, \ldots, x_n)$.

Lemma 4.4. *Let $n \in \mathbb{N}^+$. For any $t \in W_n^{lin}(X_n)$ and any $F \in \mathcal{F}_{((n);(n))}^{lin}(X_n)$. We have*

$$\widehat{\sigma}_{id}[t] = t \text{ and } \widehat{\sigma}_{id}[F] = F.$$

Proof. Let $t \in W_n^{lin}(X_n)$, we will give a proof by induction on the complexity of the definition of a linear-term t.

(i) If $t = x_i$; $i \in 1 \leq i \leq n$, then $\widehat{\sigma}_{id}[x_i] = x_i$.

(ii) If $t = f(x_{\pi(1)}, \ldots, x_{\pi(n)}), \pi \in P_n$, then $\widehat{\sigma}_{id}[f(x_{\pi(1)}, \ldots, x_{\pi(n)})]$

$$\begin{aligned}
&= S_{lin}^n(\sigma_{id}(f), \widehat{\sigma}_{id}[x_{\pi(1)}], \ldots, \widehat{\sigma}_{id}[x_{\pi(n)}]) \\
&= S_{lin}^n(f(x_1, \ldots, x_n), x_{\pi(1)}, \ldots, x_{\pi(n)})\ ) \\
&= f(S_{lin}^n(x_1, x_{\pi(1)}, \ldots, x_{\pi(n)}), \ldots, S_{lin}^n(x_n, x_{\pi(1)}, \ldots, x_{\pi(n)})) \\
&= f(x_{\pi(1)}, \ldots, x_{\pi(n)}).
\end{aligned}$$

For $F \in \mathcal{F}^{lin}_{((n);(n))}(X_n)$, we will give a proof by induction on the complexity of the definition of a linear-formula F.
(i) If F has the form $x_i \approx x_j$ for $i \neq j \in \{1,\ldots,n\}$, then $\widehat{\sigma}_{id}[x_i \approx x_j] = \widehat{\sigma}_{id}[x_i] \approx \widehat{\sigma}_{id}[x_j] = x_i \approx x_j$.
(ii) If F has the form $\gamma(x_{\pi(1)},\ldots,x_{\pi(n)})$, then

$$\begin{aligned}
&\widehat{\sigma}_{id}[\gamma(x_{\pi(1)},\ldots,x_{\pi(n)})]\\
&= R^n_{lin}(\sigma_{id}(\gamma),\widehat{\sigma}_{id}[x_{\pi(1)}],\ldots,\widehat{\sigma}_{id}[x_{\pi(n)}]\\
&= R^n_{lin}(\gamma(x_1,\ldots,x_n),x_{\pi(1)},\ldots,x_{\pi(n)})\\
&= \gamma(S^n_{lin}(x_1,x_{\pi(1)},\ldots,x_{\pi(n)}),\ldots,S^n_{lin}(x_n,x_{\pi(1)},\ldots,x_{\pi(n)}))\\
&= \gamma(x_{\pi(1)},\ldots,x_{\pi(n)}).
\end{aligned}$$

(iii) If F has the form $\neg F$ and assume that $\widehat{\sigma}_{id}[F] = F$, then $\widehat{\sigma}_{id}[\neg F] = \neg(\widehat{\sigma}_{id}[F]) = \neg F$. □

All together, we obtain a monoid.

Theorem 4.5. *$\mathcal{H}yp^{lin}((n);(n)) := (Hyp^{lin}((n);(n)); \circ_{lin}, \sigma_{id})$ is a monoid.*

Proof. Using Lemma 4.3 and using the fact that $\circ$ is associative, it can be shown that $\circ_{lin}$ is associative. In fact, for every $\sigma_1,\sigma_2,\sigma_3 \in Hyp^{lin}((n);(n))$ we have

$$\begin{aligned}
\sigma_1 \circ_{lin} (\sigma_2 \circ_{lin} \sigma_3) &= \widehat{\sigma}_1 \circ (\sigma_2 \circ_{lin} \sigma_3) = \widehat{\sigma}_1 \circ (\widehat{\sigma}_2 \circ \sigma_3) = (\widehat{\sigma}_1 \circ \widehat{\sigma}_2) \circ \sigma_3\\
&= (\sigma_1 \circ_{lin} \sigma_2)\widehat{\ } \circ \sigma_3 = (\sigma_1 \circ_{lin} \sigma_2) \circ_{lin} \sigma_3.
\end{aligned}$$

Using Lemma 4.4 shows that σ_{id} is an identity element with respect to $\circ_{lin}$. First, we will show that σ_{id} is left identity element. Let $\beta \in \{f\} \cup \{\gamma\}$, then $(\sigma_{id} \circ_{lin} \sigma)(\beta) = (\widehat{\sigma}_{id} \circ \sigma)(\beta) = \widehat{\sigma}_{id}[\sigma(\beta)] = \sigma(\beta)$.
Now, we will show that σ_{id} is a right identity element as follows:
If $\beta = f$, then

$$\begin{aligned}
(\sigma \circ_{lin} \sigma_{id})(f) &= (\widehat{\sigma} \circ \sigma_{id})(f) = \widehat{\sigma}[\sigma_{id}(f)] = \widehat{\sigma}[f(x_1,\ldots,x_n)]\\
&= S^n_{lin}(\sigma(f),\widehat{\sigma}[x_1],\ldots,\widehat{\sigma}[x_n]) = S^n_{lin}(\sigma(f),x_1,\ldots,x_n) = \sigma(f).
\end{aligned}$$

If $\beta = \gamma$, then

$$\begin{aligned}
(\sigma \circ_{lin} \sigma_{id})(\gamma) &= (\widehat{\sigma} \circ \sigma_{id})(\gamma) = \widehat{\sigma}[\sigma_{id}(\gamma)] = \widehat{\sigma}[\gamma(x_1,\ldots,x_n)]\\
&= R^n_{lin}(\sigma(\gamma),\widehat{\sigma}[x_1],\ldots,\widehat{\sigma}[x_n]) = \sigma(\gamma).
\end{aligned}$$

Therefore $\sigma_{id} \circ_{lin} \sigma = \sigma = \sigma \circ_{lin} \sigma_{id}$. □

5 All Idempotent Elements of Linear-Hypersubstitutions for Algebraic Systems of Type ((n);(n))

In this section, we will characterize all idempotent elements of linear-hypersubstitutions for algebraic systems of type $((n);(n))$. A linear-hypersubstitutions for algebraic systems σ which map f to a linear-term t and γ to a linear-formula F preserves arities is denoted by $\sigma := \sigma_{t,F}$ that means $\sigma_{t,F}(f) = t$ and $\sigma_{t,F}(\gamma) = F$. First, we will recall the definition of an idempotent element.

Definition 5.1. [3] *Let $(S;\cdot)$ be a semigroup and $a \in S$ is called idempotent element if $a \cdot a = a$. In general, we denote the set of all idempotent elements of S by $E(S)$.*

Proposition 5.2. *For any $t \in W^{lin}_n(X_n)$ and $F \in \mathcal{F}^{lin}_{((n);(n))}(X_n)$. The element $\sigma_{t,F} \in Hyp^{lin}((n);(n))$ is an idempotent if and only if $\widehat{\sigma}_{t,F}[t] = t$ and $\widehat{\sigma}_{t,F}[F] = F$.*

Proof. Assume that $\sigma_{t,F}$ is an idempotent, i.e. $(\sigma_{t,F} \circ_{lin} \sigma_{t,F})(f) = \sigma_{t,F}(f)$ and $(\sigma_{t,F} \circ_{lin} \sigma_{t,F})(\gamma) = \sigma_{t,F}(\gamma)$. Then $\widehat{\sigma}_{t,F}[t] = \widehat{\sigma}_{t,F}[\sigma_{t,F}(f)] = (\sigma_{t,F} \circ_{lin} \sigma_{t,F})(f) = \sigma_{t,F}(f) = t$ and $\widehat{\sigma}_{t,F}[F] = \widehat{\sigma}_{t,F}[\sigma_{t,F}(\gamma)] = (\sigma_{t,F} \circ_{lin} \sigma_{t,F})(\gamma) = \sigma_{t,F}(\gamma) = F$. Conversely, let $\widehat{\sigma}_{t,F}[t] = t$ and $\widehat{\sigma}_{t,F}[F] = F$, we have $(\sigma_{t,F} \circ_{lin} \sigma_{t,F})(f) = \widehat{\sigma}_{t,F}[\sigma_{t,F}(f)] = \widehat{\sigma}_{t,F}[t] = t = \sigma_{t,F}(f)$ and $(\sigma_{t,F} \circ_{lin} \sigma_{t,F})(\gamma) = \widehat{\sigma}_{t,F}[\sigma_{t,F}(\gamma)] = \widehat{\sigma}_{t,F}[F] = F = \sigma_{t,F}(\gamma)$.
This shows that $\sigma_{t,F}$ is an idempotent element. □

Proposition 5.3. *If $t =: x \in X_n$ and $F =: x_l \approx x_k$ for $l \neq k \in \{1,\ldots,n\}$, then $\sigma_{t,F} \in Hyp^{lin}((n);(n))$ is an idempotent element.*

Proof. For $n \in \mathbb{N}^+$.Let $\sigma_{t,F} \in Hyp^{lin}((n);(n))$, $t =: x \in X_n$ and $F =: x_l \approx x_k$ for $l \neq k \in \{1,\ldots,n\}$. We have $\widehat{\sigma}_{t,F}[x] = x = t$ and $\widehat{\sigma}_{t,F}[x_l \approx x_k] = \widehat{\sigma}_{t,F}[x_l] \approx \widehat{\sigma}_{t,F}[x_k] = x_l \approx x_k = F$. By Proposition 5.2, we get $\sigma_{t,F}$ is an idempotent element. □

Proposition 5.4. *For $n \in \mathbb{N}^+$. If $t = x \in X_n$ and $F = \gamma(x_1,\ldots,x_n)$, then $\sigma_{t,F} \in Hyp^{lin}((n);(n))$ is an idempotent element.*

Proof. Let $\sigma_{t,F} \in Hyp^{lin}((n);(n)), t =: x \in X_n$ and $F =: \gamma(x_1,\ldots,x_n)$. We have $\widehat{\sigma}_{t,F}[x] = x$ and

$$\begin{aligned}\widehat{\sigma}_{t,F}[\gamma(x_1,\ldots,x_n)] &= R^n_{lin}(\sigma_{t,F}(\gamma), \widehat{\sigma}_{t,F}[x_1],\ldots,\widehat{\sigma}_{t,F}[x_n]) \\ &= R^n_{lin}(\gamma(x_1,\ldots,x_n), x_1,\ldots,x_n) \\ &= \gamma(S^n_{lin}(x_1,x_1,\ldots,x_n),\ldots,S^n_{lin}(x_n,x_1,\ldots,x_n)) \\ &= \gamma(x_1,\ldots,x_n).\end{aligned}$$

By Proposition 5.2, $\sigma_{t,F}$ is an idempotent element. □

Proposition 5.5. *For $n \in \mathbb{N}^+$.If $t = f(x_1,\ldots,x_n)$ and $F = x_l \approx x_k$, for $l \neq k \in \{1,\ldots,n\}$, then $\sigma_{t,F} \in Hyp^{lin}((n);(n))$ is an idempotent element.*

Proof. Let $\sigma_{t,F} \in Hyp^{lin}((n);(n))$,$t =: f(x_1,\ldots,x_n)$ and $F =: x_l \approx x_k$, for $l \neq k \in \{1,\ldots,n\}$. We have

$$\begin{aligned}\widehat{\sigma}_{t,F}[f(x_1,\ldots,x_n)] &= S^n_{lin}(\sigma_{t,F}(f), \widehat{\sigma}_{t,F}[x_1],\ldots,\widehat{\sigma}_{t,F}[x_n]) \\ &= S^n_{lin}(f(x_1,\ldots,x_n), x_1,\ldots,x_n) \\ &= f(S^n_{lin}(x_1,x_1,\ldots,x_n),\ldots,S^n_{lin}(x_n,x_1,\ldots,x_n) \\ &= f(x_1,\ldots,x_n).\end{aligned}$$

By Proposition 5.3, we get $\widehat{\sigma}_{t,F}[x_l \approx x_k] = x_l \approx x_k$. Therefore $\sigma_{t,F}$ is an idempotent element. □

Proposition 5.6. *For $n \in \mathbb{N}^+$.If $t = f(x_1,\ldots,x_n)$ and $F = \gamma(x_1,\ldots,x_n)$, then $\sigma_{t,F} \in Hyp^{lin}((n);(n))$ is an idempotent element.*

Proof. In a similar way to the proof of Proposition 5.3 and Proposition 5.4, we proceed for $\widehat{\sigma}_{t,F}[f(x_1,\ldots,x_n)] = f(x_1,\ldots,x_n)$ and $\widehat{\sigma}_{t,F}[\gamma(x_1,\ldots,x_n)] = \gamma(x_1,\ldots,x_n)$, respectively. □

Proposition 5.7. *Let $\sigma_{t,F} \in Hyp^{lin}(((n);(n))$. If $\widehat{\sigma}_{t,F}[t] = t$ and $F = x_l \approx x_k$ for $l \neq k \in \{1,\ldots,n\}$, then $\sigma_{t,\neg F}$ is an idempotent element.*

Proof. Let $\sigma_{t,F} \in Hyp^{lin}((n);(n))$. For $\widehat{\sigma}_{t,F}[t] = t$ and $F = x_l \approx x_k$ for $l \neq k \in \{1,\ldots,n\}$, we get $(\sigma_{t,\neg F} \circ_{lin} \sigma_{t,\neg F})(f) = \widehat{\sigma}_{t,\neg F}[\sigma_{t,\neg F}(f)] = \widehat{\sigma}_{t,\neg F}[t] = t = \sigma_{t,\neg F}(f)$ and $(\sigma_{t,\neg F} \circ_{lin} \sigma_{t,\neg F})(\gamma) = \widehat{\sigma}_{t,\neg F}[\sigma_{t,\neg F}(\gamma)] = \widehat{\sigma}_{t,\neg F}[\neg F] = \neg(\widehat{\sigma}_{t,\neg F}[F]) = \neg F = \sigma_{t,\neg F}(\gamma)$. □

If ρ is a permutation on set $\{1,2,\ldots,n\}$ such that ρ replaces each element by the element itself, ρ is called the identity permutation on set $\{1,2,\ldots,n\}$. Thus

$$\rho = \begin{pmatrix} 1 & 2 & 3 & \ldots & n \\ 1 & 2 & 3 & \ldots & n \end{pmatrix}.$$

Proposition 5.8. *Let $n \in \mathbb{N}^+$ and ρ be an identity permutation on the set $\{1,2,\ldots,n\}$. If $t =: f(x_{\pi(1)},\ldots,x_{\pi(n)})$ or $F =: \gamma(x_{\pi(1)},\ldots,x_{\pi(n)})$ where π is a permutation such that $\pi \neq \rho$, then $\sigma_{t,F} \in Hyp^{lin}((n);(n))$ is not an idempotent element.*

Proof. If $t = f(x_{\pi(1)},\ldots,x_{\pi(n)})$, then

$$\begin{aligned}(\sigma_{t,F} \circ_{lin} \sigma_{t,F})(f) &= \widehat{\sigma}_{t,F}[\sigma_{t,F}(f)] = \widehat{\sigma}_{t,F}[f(x_{\pi(1)},\ldots,x_{\pi(n)})] \\ &= S^n_{lin}(\sigma_{t,F}(f), \widehat{\sigma}_{t,F}[x_{\pi(1)}],\ldots,\widehat{\sigma}_{t,F}[x_{\pi(n)}] \\ &= S^n_{lin}(f(x_{\pi(1)},\ldots,x_{\pi(n)}), x_{\pi(1)},\ldots,x_{\pi(n)}) \\ &= f(S^n_{lin}(x_{\pi(1)},x_{\pi(1)},\ldots,x_{\pi(n)}),\ldots,S^n_{lin}(x_{\pi(n)},x_{\pi(1)},\ldots,x_{\pi(n)}). \\ &\neq f(x_{\pi(1)},\ldots,x_{\pi(n)}) \qquad (\because \pi \neq \rho) \\ &\neq \sigma_{t,F})(f).\end{aligned}$$

If $F = \gamma(x_{\pi(1)}, \ldots, x_{\pi(n)})$, then

$$\begin{array}{rcl}(\sigma_{t,F} \circ_{lin} \sigma_{t,F})(\gamma) & = & \widehat{\sigma}_{t,F}[\sigma_{t,F}(\gamma)] = \widehat{\sigma}_{t,F}[\gamma(x_{\pi(1)}, \ldots, x_{\pi(n)})] \\ & = & R^n_{lin}(\sigma_{t,F}(\gamma), \widehat{\sigma}_{t,F}[x_{\pi(1)}], \ldots, \widehat{\sigma}_{t,F}[x_{\pi(n)}] \\ & = & R^n_{lin}(\gamma(x_{\pi(1)}, \ldots, x_{\pi(n)}), x_{\pi(1)}, \ldots, x_{\pi(n)}) \\ & = & \gamma(S^n_{lin}(x_{\pi(1)}, x_{\pi(1)}, \ldots, x_{\pi(n)}), \ldots, S^n_{lin}(x_{\pi(n)}, x_{\pi(1)}, \ldots, x_{\pi(n)}). \\ & \neq & \gamma(x_{\pi(1)}, \ldots, x_{\pi(n)}) \qquad (\because \pi \neq \rho) \\ & \neq & \sigma_{t,F})(\gamma).\end{array}$$

Therefore $\sigma_{t,F}$ is not an idempotent element. □

Proposition 5.9. *Let $t = x \in X_n$ and $F = \gamma(x_{\pi(1)}, \ldots, x_{\pi(n)})$ whenever $\pi \neq \rho$, then $\sigma_{t,F} \in Hyp^{lin}((n); (n))$ is not an idempotent element.*

Proof. It is an immediate consequence of Proposition 5.8. □

6 Conclusion

The main result of the paper is the characterization idempotent elements of linear-hypersubstitutions for algebraic systems of type $((n); (n))$. We investigated that all these linear-hypersubstitutions for algebraic systems of type $((n); (n))$ which satisfy the conditions are idempotent by using Proposition 5.3-5.7.

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