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Convergence and Fixed Point Sets of Generalized Homogeneous Maps

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Abstract : We introduce the notion of generalized homogeneous maps and show that, under suitable conditions, their convergence sets are star-convex and their fixed point sets are contractible. We also prove that the fixed point set of a virtually nonexpansive generalized homogeneous map is always star-convex.

Keywords : fixed point set, convergence set, virtually nonexpansive map.
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Introduction

The structures of fixed point sets have been studied by many authors for various types of nonexpansive maps and most of the results always immediately imply the convexity, and hence the contractibility, of the fixed point sets in appropriate settings (see for example, [2],[3],[5]). However, in real world problems, even in a finite dimensional normed linear space, a simple, yet geometrically interesting, map tends to increase distances in some directions and still has a contractible fixed point set. For example, the map $f : \mathbb{C} \to \mathbb{C}$ given by $f(x+iy) = x + i\left(\frac{y+|x|}{2}\right)$ is not nonexpansive, but $F(f) = \{x + i|x| : x \in \mathbb{R}\}$ is clearly nonconvex and yet contractible.

This paper is an attempt towards a theory describing the contractibility of the fixed point set in general. We first recall some basic properties of convergence sets and improve some results presented in [4]. Then we introduce a notion of a ϕ-homogeneous map, which is a generalisation of a well-known homogeneous map, and prove the star-convexity of the convergence set for such a ϕ-homogeneous map under a certain condition. By applying the technique introduced in [4], we immediately obtain the contractibility of the fixed point set for a virtually nonexpansive ϕ-homogeneous map. However, in the last section, we further explore a virtually nonexpansive ϕ-homogeneous map and discover a surprising, yet stronger, result on the star-convexity of its fixed point set.

1 Preliminaries

In this section, we recall definitions and basic properties of convergence sets for various kinds of maps. Let (X, d) be a nonempty metric space and $f : X \to X$ a continuous selfmap. From [4], the convergence set of f is defined to be the set

$$C(f) = \{x \in X : \text{ the sequence } (f^n(x)) \text{ converges}\},$$

where the symbol f^n denotes n-th iterate of f (f^0 is simply the identity map). It is clear from the definition that the fixed point set of f, $F(f)$, is contained in $C(f)$, and $C(f) \neq \emptyset$ if and only if $F(f) \neq \emptyset$. Since we are going to study structures of the convergence set, it is natural to assume that $F(f) \neq \emptyset$ throughout.

We call f

- *isometric* if $d(f(x), f(y)) = d(x, y)$ for all $x, y \in X$.
- *nonexpansive (NX)* if $d(f(x), f(y)) \leq d(x, y)$ for any $x, y \in X$.
- *quasi-nonexpansive (QNX)* if $d(f(x), p) \leq d(x, p)$ for any $x \in X$ and $p \in F(f)$.
- *asymptotically nonexpansive (ANX)* if there is a sequence $(k_n) \subseteq \mathbb{R}^+$ converging to 1 such that $d(f^n(x), f^n(y)) \leq k_n d(x, y)$ for any $x, y \in X$ and $n \in \mathbb{N}$.
- *asymptotically quasi-nonexpansive (AQNX)* if there is a sequence $(k_n) \subseteq \mathbb{R}^+$ converging to 1 such that $d(f^n(x), p) \leq k_n d(x, p)$ for any $x \in X$, $p \in F(f)$ and $n \in \mathbb{N}$.
- *virtually nonexpansive (VNX)* if

  $$C(f) \subseteq \{x : \{f^n\}_{n=0}^{\infty} \text{ is equicontinuous at x}\},$$

  or equivalently (see [4]),

  $$F(f) \subseteq \{x : \{f^n\}_{n=0}^{\infty} \text{ is equicontinuous at x}\}.$$

- *periodic* if there is $n \in \mathbb{N}$ such that $f^n = 1_X$.
- *almost periodic* if for each $\epsilon > 0$, there is a relatively dense subset (see [1]) $A \subseteq \mathbb{N}$ such that $d(f^n(x), x) < \epsilon$ for all $n \in A$ and all $x \in X$.
- *recurrent* if for each $\epsilon > 0$, there is $n \in \mathbb{N}$ such that $d(f^n(x), x) < \epsilon$ for all $x \in X$.
- *pointwise recurrent* if if for each $x \in X$ and each $\epsilon > 0$, there is $n \in \mathbb{N}$ such that $d(f^n(x), x) < \epsilon$.

From the definitions, we immediately obtain the following implications :

$$\begin{array}{ccccc} \text{isometric} & \Rightarrow & \text{NX} & \Rightarrow & \text{QNX} \\ & & \Downarrow & & \Downarrow \\ & & \text{ANX} & \Rightarrow & \text{AQNX} \end{array}$$

$$\text{periodic} \Rightarrow \text{almost periodic} \Rightarrow \text{recurrent} \Rightarrow \text{pointwise recurrent}$$

and from [4], we also have

$$\text{AQNX} \Rightarrow \text{VNX}.$$

The following two theorems describe the topological structure of the convergence sets of all maps defined above.

Theorem 1.1 *If f is isometric or pointwise recurrent, then $C(f) = F(f)$ and hence $C(f)$ is a closed subset of X.*

Proof If f is isometric, the proof is trivial. Otherwise, suppose f is pointwise recurrent. Let $x \in C(f)$ and suppose that $(f^n(x)) \to x_0 \in F(f)$. Let $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that $d(f^n(x), x_0) < \frac{\epsilon}{2}$ for all $n \geq N$. Now, from the definition of pointwise recurrent map, it is not difficult to see that there are infinitely many n's such that $d(f^n(x), x)) < \frac{\epsilon}{2}$. In particular, there exists $M \geq N$ such that $d(f^M(x), x)) < \frac{\epsilon}{2}$. Therefore,

$$d(x, x_0) \leq d(x, f^M(x))) + d(f^M(x), x_0)) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Since ϵ is arbitrary, we must have $x = x_0 \in F(f)$.

Theorem 1.2 *If X is complete and f is virtually nonexpansive, then $C(f)$ is a G_δ-set in X.*

Proof Since $C(f) \subseteq E(f)$, for each $x \in C(f)$ and $m \in \mathbb{N}$, there exists $\delta_{x,m} > 0$ such that $d(f^n(x), f^n(y)) < \frac{1}{m}$ for any $y \in B(x, \delta_{x,m})$ and $n \in \mathbb{N}$. Now, we claim that $C(f) = \bigcap_{m\in\mathbb{N}} \bigcup_{x\in C(f)} B(x, \delta_{x,m})$ which is clearly a G_δ-set in X. The containment is trivial in one direction ($\subseteq$). For the other ($\supseteq$), let $y \in \bigcap_{m\in\mathbb{N}} \bigcup_{x\in C(f)} B(x, \delta_{x,m})$ and $\epsilon > 0$. Choose $m \in \mathbb{N}$ such that $\frac{1}{m} \leq \frac{\epsilon}{4}$. Then, there are $x \in C(f)$ and $\delta_{x,m} > 0$ so that $d(x, y) < \delta_{x,m}$, and hence

$$d(f^n(x), f^n(y)) < \frac{1}{m} \leq \frac{\epsilon}{4}$$

for all $n \in \mathbb{N}$. Since $x \in C(f)$, there is $x_0 \in X$ such that, for some $N \in \mathbb{N}$, $d(f^n(x), x_0) < \frac{\epsilon}{4}$ for all $n \geq N$. Hence, for all $i, j \geq N$, we have

$$d(f^i(y), f^j(y)) \leq d(f^i(y), f^i(x)) + d(f^i(x), x_0) + d(f^j(y), f^j(x)) + d(f^j(x), x_0) < \epsilon.$$

It follows that there sequence $(f^n(y))$ is Cauchy and converges by the completeness of X. Therefore, $y \in C(f)$.

Remark 1.3 *The previous theorem improves Theorem 2.4 ([4]) by dropping the total boundedness of $F(f)$.*

Moreover, when f is virtually nonexpansive, it is also proved in [4] that $F(f)$ is a retract of $C(f)$.

2 ϕ-Homogeneous Maps and Their Convergence Sets

In this section, we will introduce generalized homogeneous maps and explore their convergence sets. For a nonempty subset of a topological $\mathbb{R}$-linear space X and $x_0 \in X$, recall that X is x_0-*star-convex* if for each $x \in X$, we have

$$\{tx + (1-t)x_0 : t \in [0,1]\} \subseteq X.$$

In this section, we let X be a 0-star-convex subset of a topological $\mathbb{R}$-linear space.

Definition 2.1 *For continuous selfmaps $f : X \to X$ and $\phi : [0,1] \to [0,1]$, we will call f a generalized homogeneous map, or simply a ϕ-homogeneous map, if*

$$f(tx) = \phi(t)f(x)$$

for all $(x,t) \in X \times [0,1]$.

Example 2.2 *For a topological $\mathbb{R}$-linear space V, a homogeneous degree-n map $f : V \to V$ (that is $f(tv) = t^n f(v)$ for all $(v,t) \in V \times \mathbb{R}$) is a clearly ϕ-homogeneous with $\phi(t) = t^n$. In particular, a homogeneous linear map is ϕ-homogeneous.*

The following example shows that a ϕ-homogeneous map is far from being a linear map.

Example 2.3 *Let D^2 and S^1 denote the closed unit disc and circle in $\mathbb{C}$, respectively. The following selfmaps of $\mathbb{C}$ are clearly ϕ-homogeneous :*

1. $f_1(z) = \overline{z}$ *(with $\phi(t) = t$)*
2. $f_2(z) = z^2$ *(with $\phi(t) = t^2$)*
3. $f_3(z) = |z|z$ *(with $\phi(t) = |t|t = t^2$)*
4. $f_4(x+iy) = x + i(2y - |x|)$ *(with $\phi(t) = t$)*
5. $f_5(x+iy) = x + i|x|$ *(with $\phi(t) = t$)*

Proposition 2.4 *If $f : X \to X$ is a non-constant ϕ-homogeneous map, we have the followings :*

1. *$\phi(st) = \phi(s)\phi(t)$ for all $s, t \in [0, 1]$,*
2. *$\{0, 1\} \subseteq F(\phi)$,*
3. *$0 \in F(f)$.*

Proof Since f is non-constant, there exist $x, y, z \in X$ such that $f(x) \neq 0$ and $f(y) \neq f(z)$. Then, for any $s, t \in [0, 1]$, we have

$$\phi(st)f(x) = f((st)x) = f(s(tx)) = \phi(s)f(tx) = \phi(s)\phi(t)f(x)$$

which implies $\phi(st) = \phi(s)\phi(t)$ and proves (1). For (2), since $f(x) = \phi(1)f(x)$ and $\phi(0)f(y) = f(0) = \phi(0)f(z)$, we must have $\phi(1) = 1$ and $\phi(0) = 0$. Finally, (3) follows directly from $f(0) = \phi(0)f(x) = 0$.

Theorem 2.5 *Let $f : X \to X$ be a ϕ-homogeneous map with $C(\phi) = [0, 1]$. Then $C(f)$ is 0-star-convex.*

Proof For $x \in C(f)$ and $t \in [0, 1]$, since X is 0-star-convex, we have $tx \in X$ and hence the limit $\lim_{n\to\infty} f^n(tx) = \lim_{n\to\infty} \phi^n(t)f^n(x)$ exists. Therefore, $tx \in C(f)$ which implies that $C(f)$ is 0-star-convex.

Example 2.6 *From Example 2.3, it is easy to see that $C(f_1) = \mathbb{R}, (D^2)^\circ \subseteq C(f_2) \subseteq D^2, C(f_3) = D^2, C(f_4) = \{x + iy : y = |x|\}$ and $C(f_5) = \mathbb{C}$ are all 0-star-convex. However, $F(f_2) = \{0, 1\}$ and $F(f_3) = \{0\} \cup S^1$ are not even path-connected, $F(f_1) = \mathbb{R}$ is convex and hence contractible, and $F(f_4) = \{x + iy : y = |x|\} = F(f_5)$ is contractible but not convex. In fact, it is not difficult to verify that only f_1 and f_5 are virtually nonexpansive.*

The next corollary shows that, by using the notion of virtual nonexpansiveness, we immediately obtain the contractibility of the fixed point set.

Corollary 2.7 *Assume further that X is metrizable. Let $f : X \to X$ be a virtually nonexpansive ϕ-homogeneous map with $C(\phi) = [0, 1]$. Then $F(f)$ is contractible.*

Proof By Theorem 2.5, $C(f)$ is 0-star-convex and hence contractible. Since f is virtually nonexpansive, $F(f)$ is a retract of $C(f)$. Therefore, $F(f)$ is also contractible.

Example 2.8 *Let $X = \{x + iy : |x| - 1 < y < |x| + 1, -1 < x < 1\} \subseteq \mathbb{C}$ and $f : X \to X$ defined by*

$$f(x + iy) = x + i\left(\frac{y + |x|}{2}\right).$$

Clearly, X is 0-star-convex and f is a virtually nonexpansive (but not quasi-nonexpansive) ϕ-homogeneous map with $\phi(t) = t$. Then, $C(\phi) = [0, 1]$. By Corollary 2.7, $F(f)$ is contractible which is obviously true since

$$F(f) = \{x + i|x| : -1 < x < 1\}.$$

Example 2.9 *For a more complicate virtually nonexpansive selfmap, consider $h : \mathbb{R}^3 \to \mathbb{R}^3$ given by*

$$h(x, y, z) = (h_1, h_2, h_3),$$

where

$$h_1 = \frac{1}{12}\left(9x + y + 2\sqrt{2}z - 3\sqrt{2}\left|\frac{x}{\sqrt{2}} - \frac{y}{\sqrt{2}} + z\right|\right),$$

$$h_2 = \frac{1}{36}\left(3x + 35y - 2\sqrt{2}z + 3\sqrt{2}\left|\frac{x}{\sqrt{2}} - \frac{y}{\sqrt{2}} + z\right|\right),$$

$$h_3 = \frac{1}{18}\left(3\sqrt{2}x - \sqrt{2}y + 14z + 6\left|\frac{x}{\sqrt{2}} - \frac{y}{\sqrt{2}} + z\right|\right).$$

Clearly, h is a ϕ-homogeneous map with $\phi(t) = t$. Then $C(\phi) = [0, 1]$ and it follows immediately that $F(h)$ is contractible.

To confirm this, we observe that $h = p^{-1} \circ f \circ p$, where

$$f(x, y, z) = (x, y, \frac{z + |x - y|}{2})$$

and $p : \mathbb{R}^3 \to \mathbb{R}^3$ is the rotation in $\mathbb{R}^3$ given by the transformation matrix

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{3\sqrt{2}} & \frac{2}{3} \\ 0 & \frac{4}{3\sqrt{2}} & -\frac{1}{3} \\ -\frac{1}{\sqrt{2}} & \frac{1}{3\sqrt{2}} & \frac{2}{3} \end{pmatrix}.$$

Therefore,

$$F(h) = p^{-1}(F(f)) = \{(\frac{x - |x - y|}{\sqrt{2}}, \frac{x + 4y + |x - y|}{3\sqrt{2}}, \frac{2x - y + 2|x - y|}{3}) : x, y \in \mathbb{R}\},$$

which can be seen to be contractible and nonconvex by a parametric 3D-graph plotting program.

Remark 2.10 *We can produce a similar example of t by replacing f with*

$$f(x, y, z) = (x, y, \frac{z + g(x, y)}{2}),$$

where $g : \mathbb{R}^2 \to \mathbb{R}^2$ is any continuous map satisfying $g(tx, ty) = tg(x, y)$ for all $t \in [0, 1]$. For example, when $f(x, y, z) = (x, y, \frac{z+\sqrt{|xy|}}{2})$, we have

$$h(x, y, z) = (h_1, h_2, h_3),$$

where

$$h_1 = \frac{1}{12}\left(9x + y + 2\sqrt{2}z - 3\sqrt{2}\sqrt{|xy|}\right),$$

$$h_2 = \frac{1}{36}\left(3x + 35y - 2\sqrt{2}z + 3\sqrt{2}\sqrt{|xy|}\right),$$

$$h_3 = \frac{1}{18}\left(3\sqrt{2}x - \sqrt{2}y + 14z + 6\sqrt{|xy|}\right).$$

3 Virtually Nonexpansive ϕ-Homogeneous Maps

According to Corollary 2.7 and the last example in the previous section, it is natural to ask whether there is a virtually nonexpansive ϕ-homogeneous map that is not homogeneous degree 1. The answer is clearly affirmative by considering the map $f(z) = z^n$ defined on the interior of the closed unit disc D^2 (in $\mathbb{C}$) for any fixed integer $n > 2$. Notice also that such a map has only one fixed point. In this section, we will prove a rather surprising result : if f is a virtually nonexpansive ϕ-homogeneous map that fixes more than one point, it must be homogeneous degree 1. Hence, this result assures the star-convexity of the fixed point set of a virtually nonexpansive ϕ-homogeneous map, which is stronger than Corollary 2.7.

In this section, we let X be a 0-star-convex subset of a normed linear space.

Lemma 3.1 *Let $f : X \to X$ be a ϕ-homogeneous map with more than one fixed point. If f is virtually nonexpansive, then ϕ is also virtually nonexpansive.*

Proof Since f has more than one fixed point, we fix $x_0 \in F(f) - \{0\}$. Let $t_0 \in F(\phi)$ and $\epsilon > 0$. Then $t_0x_0 \in F(f)$ because $f(t_0x_0) = \phi(t_0)f(x_0) = t_0x_0$. By virtual nonexpansiveness of f, there exists $\delta > 0$ such that

$$\|f^n(x) - t_0x_0\| = \|f^n(x) - f^n(t_0x_0)\| < \|x_0\|\epsilon$$

for all $n \in \mathbb{N}$ whenever $\|x - t_0x_0\| < \delta$. Then for each $t \in [0, 1]$ with $|t - t_0| < \frac{\delta}{\|x_0\|}$, we have $\|tx_0 - t_0x_0\| < \delta$ and hence

$$\begin{aligned}
|\phi^n(t) - \phi^n(t_0)| &= \frac{|\phi^n(t) - \phi^n(t_0)|\|x_0\|}{\|x_0\|} \\
&= \frac{\||\phi^n(t)x_0 - \phi^n(t_0)x_0\|}{\|x_0\|} \\
&= \frac{\|f^n(tx_0) - t_0x_0\|}{\|x_0\|} \\
&= \frac{\|f^n(tx_0) - f^n(t_0x_0)\|}{\|x_0\|} \\
&< \frac{\|x_0\|\epsilon}{\|x_0\|} = \epsilon.
\end{aligned}$$

Therefore, ϕ is virtually nonexpansive.

Lemma 3.2 *The only virtually nonexpansive selfmap of* $[0, 1]$ *that fixes both 0 and 1 is the identity map.*

Proof Let $\phi : [0, 1] \to [0, 1]$ be a virtually nonexpansive map that fixes 0 and 1. Suppose ϕ is not the identity map. WLOG, we may assume that there exists $t_0 \in [0, 1]$ such that $\phi(t_0) > t_0$. Since both $F(\phi)$ and $\phi^{-1}([0, t_0])$ are closed in $[0, 1]$ and hence compact, let

$$a = \max\{t \in F(\phi) : t \le t_0\} = \max(F(\phi) \cap \phi^{-1}([0, t_0])).$$

Clearly, $a < t_0$ and $F(\phi) \cap [a, t_0] = \{a\}$. By intermediate value theorem, we observe that $\phi(t) > t$ for all $t \in (a, t_0)$. For if there is $t_1 \in (a, t_0)$ such that $\phi(t_1) \leq t_1$, we will have $\phi(t_2) = t_2$ for some $t_2 \in [t_1, t_0)$ which contradicts to the maximality of a.

Now, we let $m_0 = t_0$. Since $\phi(a) = a < m_0 = t_0 < \phi(t_0) = \phi(m_0)$, by intermediate value theorem, there exists $m_1 \in (a, m_0)$ such that $\phi(m_1) = m_0$. By the above observation, we also have $a < m_1 < \phi(m_1)$. Inductively, for each $n \in \mathbb{N}$, there exists $m_n \in (a, m_{n-1})$ such that $a < m_n < \phi(m_n) = m_{n-1}$. Therefore, we obtain a strictly decreasing sequence (m_n) in $[a, t_0]$ satisfying $\phi^n(m_n) = m_0$. By sequential compactness of $[a, t_0]$, the sequence (m_n) converges, says to $b \in [a, t_0]$. Then, by continuity of ϕ, we have

$$\phi(b) = \phi(\lim_{n\to\infty} m_n) = \lim_{n\to\infty} \phi(m_n) = \lim_{n\to\infty} m_{n-1} = b,$$

that is $b \in F(\phi) \cap [a, t_0] = \{a\}$. It follows that the sequence (m_n) converges to a.

Now, for $\epsilon = |m_0 - a| > 0$ and any $\delta > 0$, we can always find a large enough $N \in \mathbb{N}$ such that $|m_N - a| < \delta$ and $|\phi^N(m_N) - \phi^N(a)| = |m_0 - a| = \epsilon$. This means the family $\{\phi^n\}$ of iterates of ϕ is not equicontinuous at a which contracdicts to the virtual nonexpansiveness of ϕ. Therefore, ϕ must be the identity map.

Theorem 3.3 *If $f : X \to X$ is a virtually ϕ-homogeneous map that fixes more than one point, then f is homogeneous degree 1; that is $f(tx) = tf(x)$ for all $(x, t) \in X \times [0, 1]$.*

Proof By Lemma 3.1, ϕ is virtually nonexpansive. Since ϕ fixes both 0 and 1 by Proposition 2.4, it must be the identity map by Lemma 3.2. Therefore, $f(tx) = tf(x) = tx$ for any $(x, t) \in X \times [0, 1]$.

Theorem 3.4 *If $f : X \to X$ is a virtually nonexpansive ϕ-homogeneous map, then $F(f)$ is 0-star-convex.*

Proof If f has only one fixed point, we are done. Otherwise, f is homogeneous degree 1 by the previous theorem. It follows that $f(tx) = tf(x) = tx$ for any $(x, t) \in F(f) \times [0, 1]$; that is $F(f)$ is 0-star-convex.

Remark 3.5 *It should be pointed out here that Lemma 3.2 does not have a higher dimension analog in the following sense : there exists a virtually nonexpansive nonidentity selfmap of the closed unit ball D^n $(n \geq 2)$ that fixes the boundary of D^n. For example, consider $f : D^2 \to D^2$ defined by*

$$f(re^{i\theta}) = A(r)e^{i(\theta + B(r))}$$

for all $re^{i\theta} \in D^2 \subseteq \mathbb{C}$, where

$$A(r) = \begin{cases} r & ; 0 \leq r \leq \frac{1}{2} \\ 1 - 2(r-1)^2 & ; \frac{1}{2} \leq r \leq 1, \end{cases}$$

and

$$B(r) = \begin{cases} r(\frac{3}{4} - r) & ; 0 \leq r \leq \frac{3}{4} \\ 0 & ; \frac{3}{4} \leq r \leq 1. \end{cases}$$

It is straightforward to verify that f is virtually nonexpansive and $F(f) = \{0\} \cup S^1$. This example also serves as an example of a virtually nonexpansive selfmap (with a contractible domain) whose fixed point set is not contractible.

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