Thai **J**ournal of **M**athematics : (2020) 83–91
Special Issue: The 14^{th} IMT-GT ICMSA 2018

http://thaijmath.in.cmu.ac.th
Online ISSN 1686-0209

The Order of Linear Relational Hypersubstitutions for Algebraic Systems

Dara Phusanga[†,1] **and Nareupanat Lekkoksung** [‡]

[†]Division of Mathematics, Faculty of Science
Maejo University, Chiang mai 50290, Thailand
e-mail : darapu@mju.ac.th
[‡]Department of Mathematics, Faculty of Science
Khon Kaen University, Khon Kaen 40002, Thailand
e-mail : n.lekkoksung@kkumail.com

Abstract : We introduce the concept of a linear relational hypersubstitution for algebraic systems which extends the concept of linear hypersubstitutions of universal algebras. The set of all linear relational hypersubstitutions together with an operation introduced in [1] form a monoid. For this monoid and a particular type $\tau = ((m); (n))$, we determine the order of its elements.

Keywords : order of element; term; formula; algebraic system; hypersubstitution.
2010 Mathematics Subject Classification : 20M07.

1 Introduction

In [2], Denecke, Lau, Pöschel, and Schweigert introduced the concept of a hypersubstitution of a given type τ for universal algebras. The authors use the concept of hypersubstitutions of arbitrary type τ for the characterization of so-called solid varieties of type τ. A solid variety is a variety which is closed under the following operation: taking an algebra $(A; (f_i^{\mathcal{A}})_{i\in I})$ of type $\tau = (n_i)_{i\in I}$ with the universe A and a family $(f_i^{\mathcal{A}})_{i\in I}$ of operations defined on A, where f_i is m_i-ary for $i \in I$. Then we replace the operation $f_i^{\mathcal{A}}$ by any m_i-ary term operation $t_i^{\mathcal{A}}$, for $i \in I$, and obtain a new algebra $(A; (t_i^{\mathcal{A}})_{i\in I})$, which has also to belong to the variety. So, a hypersubstitution of a given type $\tau = (m_i)_{i\in I}$ is a mapping which maps each operation symbol f_i to an m_i-ary term, for $i \in I$. Further, a binary operation $\circ_{\mathrm{h}}$ defined on the set $\mathrm{Hyp}(\tau)$ of all hypersubstitutions of type τ was introduced such that $(\mathrm{Hyp}(\tau); \circ_{\mathrm{h}}, \sigma_{\mathrm{id}})$ is a monoid (see [3]). This monoid was studied intensively for both arbitrary type and for some fixed type τ. The semigroup properties of the monoid of hypersubstitutions of type (2) was studied by Denecke and Wismath (see [4]). In [5], Wismath generalized these results for the monoid of hypersubstitutions of type (n). All idempotent hypersubstitutions of type $(2, 2)$ are determined by Changphas and Denecke [6]. The order of all hypersubstitutions of type $(2, 2)$ was studied by the same authors in [7]. Later in [8], Changphas and Hemvong determined the order of hypersubstitutions of type $(2, 1)$. In 2012, the same authors considered the order of hypersubstitutions of type (n) (see [9]).

[1]Corresponding author.

In this present paper, we focus on the notion of algebraic systems in the sense of Mal'cev [10]. An *algebraic system of type* (τ, τ') is a triple $(A; (f_i^{\mathcal{A}})_{i\in I}, (\gamma_j^{\mathcal{A}})_{j\in J})$ consisting of a universe set A, a family $(f_i^{\mathcal{A}})_{i\in I}$ of operations defined on A, and a family $(\gamma_j^{\mathcal{A}})_{j\in J}$ of relations on A, where $\tau = (m_i)_{i\in I}$ is a sequence of the arity of each operation $f_i^{\mathcal{A}}$ and $\tau' = (n_j)_{j\in J}$ is a sequence of the arity of each relation $\gamma_j^{\mathcal{A}}$. There were first attempts to define a concept of a hypersubstitution for algebraic systems. The concept of such hypersubstitutions, introduced in [1], does not be practicable enough. Another attempt to define a hypersubstitution for algebraic systems was done by the second author in her Ph.D. Thesis. But this concept has not proven to be impractical (see [11]).

Therefore, we will introduce a new concept that generalizes the idea of a hypersubstitution (for a universal algebra) of type τ in a canonical way. An operation $f_i^{\mathcal{A}}$ on a set A corresponds to the m_i-ary term operation defined on A and a relation $\gamma_j^{\mathcal{A}}$ corresponds to the "n_j*-ary relation*" on A. So, it seems quite natural to say that hypersubstitutions (for algebraic systems) of type (τ, τ') that we want to consider assign an operation symbol f_i to an m_i-ary term, for $i \in I$, and assign a relation symbol γ_j to an n_j-ary relational term, for $j \in J$. We call such mappings the *relational hypersubstitutions* of type (τ, τ'), and denote the set of all relational hypersubstitutions of type (τ, τ') by $\mathrm{Relhyp}(\tau, \tau')$. Hence, this concept of a relational hypersubstitution of type (τ, τ') is a canonical extension of the concept of a hypersubstitution of type τ. It was proven in [11] that a structure $(\mathrm{Hyp}(\tau, \tau'); \circ_{\mathrm{h}}, \sigma_{\mathrm{id}})$ forms a monoid, and we can see immediately in the proof that that $(\mathrm{Relhyp}(\tau, \tau'); \circ_{\mathrm{h}}, \sigma_{\mathrm{id}})$ is a submonoid of the monoid $(\mathrm{Hyp}(\tau, \tau'); \circ_{\mathrm{h}}, \sigma_{\mathrm{id}})$.

The aim of this paper is to determine the order of element of submonoid of relational hypersubstitutions of type $((m), (n))$, so-called, linear relational hypersubstitutions of type $((m), (n))$, where $m, n \geq 1$. That is, we have only one m-ary operation symbol and one n-ary relation symbol.

2 Preliminaries

For a natural number n, ($\mathbb{N}$ denotes the set of all natural numbers) let $X_n := \{x_1, x_2, \ldots, x_n\}$ be a finite set of variables and let $X := \{x_i : i \in \mathbb{N}\}$ be countable infinite set. We consider the indexed set $(f_i)_{i\in I}$ of operation symbols, where f_i is m_i-ary, for $i \in I$, and let $\tau = (n_i)_{i\in I}$. Further, we consider the indexed set $(\gamma_j)_{j\in J}$ of relation symbols, where γ_j is n_j-ary, for $j \in J$, and let $\tau' = (n_j)_{j\in J}$. The pair (τ, τ') is called the *type* of an algebraic system (see [10]). Let us note that τ is called the type of the (corresponding) universal algebra. The set $\mathrm{T}_\tau^{\mathrm{lin}}(X_n)$ of all *n-ary linear terms* of type τ is the smallest set containing X_n and is closed under the following operation: if $i \in I$ and $t_1, \ldots, t_{m_i} \in \mathrm{T}_\tau^{\mathrm{lin}}(X_n)$ with $\mathrm{var}(t_k) \cap \mathrm{var}(t_l) = \emptyset$ for all $1 \leq k < l \leq m_i$, then $f_i(t_1, \ldots, t_{m_i}) \in \mathrm{T}_\tau^{\mathrm{lin}}(X_n)$ (see [12]). Here $\mathrm{var}(t)$ is the set of all variables occurring in the term t.

Definition 2.1 ([1, 11, 13]). *For any $n \in \mathbb{N}$, we define an n-ary quantifier free linear formula of type (τ, τ') in the following inductive ways.*

(i) If $t_1, t_2 \in T_\tau^{lin}(X_n)$ with $var(t_1) \cap var(t_2) = \emptyset$, then the equation $t_1 \approx t_2$ is an n-ary quantifier free linear formula of type (τ, τ').

(ii) For any $j \in J$, if $t_1, \ldots, t_{n_j} \in T_\tau^{lin}(X_n)$ and $var(t_k) \cap var(t_l) = \emptyset$ for all $1 \leq k < l \leq n_j$, then $\gamma_j(t_1 \ldots, t_{n_j})$ is an n-ary quantifier free linear formula of type (τ, τ').

(iii) If F is an n-ary quantifier free linear formula of type (τ, τ'), then $\neg F$ is an n-ary quantifier free linear formula of type (τ, τ').

(iv) If F_1 and F_2 are n-ary quantifier free linear formulas of type (τ, τ'), then $F_1 \vee F_2$ is an n-ary quantifier free linear formula of type (τ, τ')..

We denote the set of all n-ary linear formulas and the set of all linear formulas of type (τ, τ') by $\mathrm{F}_{(\tau,\tau')}^{\mathrm{lin}}(X_n)$ and $\mathrm{F}_{(\tau,\tau')}^{\mathrm{lin}}(X)$, respectively. That is, $\mathrm{F}_{(\tau,\tau')}^{\mathrm{lin}}(X) := \bigcup_{n\in\mathbb{N}} \mathrm{F}_{(\tau,\tau')}^{\mathrm{lin}}(X_n)$. We let $\mathrm{rF}_{(\tau,\tau')}^{\mathrm{lin}}(X)$ be the set of all linear formulas of type (τ, τ') of the form (ii) in Definition 2.1. Remark that $\mathrm{F}_{(\tau,\tau')}^{\mathrm{lin}}(X_n) = \emptyset$

if and only if $n < n_j$ for all $j \in J$. We note here that for any $n \in \mathbb{N}$, we can define an *n-ary quantifier free formula* of type (τ, τ') by using the usual terms and omitting the conditions $\mathrm{var}(t_1) \cap \mathrm{var}(t_2) = \emptyset$ and $\mathrm{var}(t_k) \cap \mathrm{var}(t_l) = \emptyset$ for all $1 \leq k < l \leq m_j$ (see [1, 11]). In the same manner, the set of all n-ary quantifier free formulas and the set of all quantifier free formulas of type (τ, τ') are denoted by $\mathrm{F}_{(\tau,\tau')}(X_n)$ and $\mathrm{F}_{(\tau,\tau')}(X)$, respectively.

For any $m, n \in \mathbb{N}$, $t_1, \ldots, t_{n_j} \in \mathrm{T}^{\mathrm{lin}}_\tau(X_m)$, $s_1, \ldots, s_m \in \mathrm{T}^{\mathrm{lin}}_\tau(X_n)$, and $F_1, F_2 \in \mathrm{F}^{\mathrm{lin}}_{(\tau,\tau')}(X_n)$ with $\mathrm{var}(s_k) \cap \mathrm{var}(s_l) = \emptyset$ for all $1 \leq k < l \leq m$. Then we define the superposition partial operation

$$R^m_n \colon (\mathrm{T}^{\mathrm{lin}}_\tau(X_m) \cup \mathrm{F}^{\mathrm{lin}}_{(\tau,\tau')}(X_m)) \times (\mathrm{T}^{\mathrm{lin}}_\tau(X_n))^m \multimap\!\!\rightarrow \mathrm{T}^{\mathrm{lin}}_\tau(X_n) \cup \mathrm{F}^{\mathrm{lin}}_{(\tau,\tau')}(X_n)$$

by the following steps.

(i) $R^m_n(t_1, s_1, \ldots, s_m) := S^m_n(t_1, s_1, \ldots, s_m)$.

(ii) $R^m_n(t_1 \approx t_2, s_1, \ldots, s_m) := R^m_n(t_1, s_1, \ldots, s_m) \approx R^m_n(t_2, s_1, \ldots, s_m)$.

(iii) $R^m_n(\gamma_j(t_1, \ldots, t_{n_j}), s_1, \ldots, s_m) := \gamma_j(R^m_n(t_1, s_1, \ldots, s_m), \ldots, R^m_n(t_{n_j}, s_1, \ldots, s_m))$.

(iv) $R^m_n(\neg F_1, s_1, \ldots, s_m) := \neg R^m_n(F_1, s_1, \ldots, s_m)$.

(v) $R^m_n(F_1 \vee F_2, s_1, \ldots, s_m) := R^m_n(F_1, s_1, \ldots, s_m) \vee R^m_n(F_2, s_1, \ldots, s_m)$.

This operation define a partial many-sorted algebra

$$\mathbf{formclone}^{\mathrm{lin}}(\tau, \tau') := (\mathrm{T}^{\mathrm{lin}}_\tau(X_n) \cup \mathrm{F}^{\mathrm{lin}}_{(\tau,\tau')}(X_n)_{n \geq 1}; (R^m_n)_{m,n \geq 1}, (x_k)_{1 \leq k \leq n, n \in \mathbb{N}}),$$

moreover, this algebra satisfies the superassociative law (see [14]).

A *linear relational hypersubstitution (for algebraic systems)* of type (τ, τ') is a mapping

$$\sigma \colon \{f_i : i \in I\} \cup \{\gamma_j : j \in J\} \to \mathrm{T}^{\mathrm{lin}}_\tau(X) \cup \mathrm{rF}^{\mathrm{lin}}_{(\tau,\tau')}(X)$$

with $\sigma(f_i) \in \mathrm{T}^{\mathrm{lin}}_\tau(X_{m_i})$, for $i \in I$, and $\sigma(\gamma_j) \in \mathrm{rF}^{\mathrm{lin}}_{(\tau,\tau')}(X_{n_j})$, for $j \in J$. Denoted by $\mathrm{Relhyp}^{\mathrm{lin}}(\tau, \tau')$ the set of all linear relational hypersubstitutions of type (τ, τ'). We define a binary operation $\circ_{\mathrm{h}}$ on the set $\mathrm{Relhyp}(\tau, \tau')$. In order to do this, we introduce the extension of a mapping $\sigma \in \mathrm{Relhyp}^{\mathrm{lin}}(\tau, \tau')$ to a transformation $\widehat{\sigma}$ on $\mathrm{T}^{\mathrm{lin}}_\tau(X) \cup \mathrm{F}^{\mathrm{lin}}_{(\tau,\tau')}(X)$ in the following way:

(i) $\widehat{\sigma}[x_i] := x_i$ for $i \in \mathbb{N}$;

(ii) $\widehat{\sigma}[f_i(t_1, \ldots, t_{m_i})]$ is the term $R^{m_i}_m(\sigma(f_i), \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_{m_i}])$, where $i \in I$ and $t_1, \ldots, t_{m_i} \in \mathrm{T}^{\mathrm{lin}}_\tau(X_m)$, i.e., we replace any occurrence of the variable x_k in $\sigma(f_i)$ by the term $\widehat{\sigma}[t_k]$, for $k \in \{1, \ldots, m_i\}$;

(iii) $\widehat{\sigma}[\gamma_j(t_1, \ldots, t_{n_j})]$ is the expression $R^{n_j}_n(\sigma(\gamma_j), \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_{n_j}])$, where $j \in J$ and $t_1, \ldots, t_{n_j} \in \mathrm{T}^{\mathrm{lin}}_\tau(X_n)$, i.e., we replace any occurrence of the variable x_k in the expression $\sigma(\gamma_j)$ by the term $\widehat{\sigma}[t_k]$, for $k \in \{1, \ldots, n_j\}$.

We observe that $\widehat{\sigma}$ restricted to $\mathrm{T}^{\mathrm{lin}}_\tau(X)$ is the extension of the corresponding linear hypersubstitution of type τ to terms (see [15, 16]). Let us now consider the binary operation $\circ_{\mathrm{h}}$ on $\mathrm{Relhyp}^{\mathrm{lin}}(\tau, \tau')$ defined by $\sigma_1 \circ_{\mathrm{h}} \sigma_2 := \widehat{\sigma}_1 \circ \sigma_2$ for all linear relational hypersubstitutions σ_1, σ_2 of type (τ, τ'). In [14], Denecke showed that the structure $(\mathrm{Hyp}^{\mathrm{lin}}(\tau, \tau'); \circ_{\mathrm{h}}, \sigma_{\mathrm{id}})$ forms a monoid. Then the following proposition can be obtained immediately.

Proposition 2.2. *The structure* $(Relhyp^{lin}(\tau, \tau'); \circ_h, \sigma_{id})$ *is a submonoid of* $(Hyp^{lin}(\tau, \tau'); \circ_h, \sigma_{id})$.

3 Substructure of Linear Relational Hypersubstitutions

We introduce here two substructures of the structure $(\mathrm{Relhyp}^{\mathrm{lin}}(\tau,\tau');\circ_{\mathrm{h}},\sigma_{\mathrm{id}})$, moreover, we focus on a particular type when $\tau=(m)$ and $\tau'=(n)$ with $m,n\geq 1$. We denote a linear relational hypersubstitution σ of type $((m),(n))$ by $\sigma_{t,\gamma(s_1,\ldots,s_n)}$ if $\sigma(f)=t$ and $\sigma(\gamma)=\gamma(s_1,\ldots,s_n)$ where t is an m-ary linear term of type (m) and $\gamma(s_1,\ldots,s_n)$ is an n-ary linear formula of type $((m),(n))$.

Definition 3.1. *A linear relational hypersubstitution of type $((m),(n))$ is called a* linear projection relational hypersubstitution *of type $((m),(n))$ if it maps operation symbol f to a variable which preserve arity. The set of all linear projection relational hypersubstitutions of type $((m),(n))$ is denoted by p-Relhyp$^{lin}((m),(n))$.*

Definition 3.2. *A linear relational hypersubstitution of type $((m),(n))$ is called a* linear relational pre-hypersubstitution *of type $((m),(n))$ if it maps operation symbol f to a non-variable which preserve arity. The set of all linear relational pre-hypersubstitutions of type $((m),(n))$ is denoted by pre-Relhyp$^{lin}((m),(n))$.*

Lemma 3.1 ([15]). *The extension of any linear hypersubstitution maps a linear terms to linear terms.*

Lemma 3.2 ([14]). *The extension of any linear hypersubstitution of type (τ,τ') maps a linear formula of the form $\gamma(s_1,\ldots,s_{n_j})$ to a linear formula of the form $\gamma(t_1,\ldots,t_{n_j})$.*

By Lemma 3.1 and 3.2, we have the followings proposition.

Proposition 3.3. *The algebras*

$$\textbf{\textit{p-Relhyp}}^{lin}((m),(n)) := (\textit{p-Relhyp}^{lin}((m),(n));\circ_h)$$

and the algebra

$$\textbf{\textit{pre-Relhyp}}^{lin}((m),(n)) := (\textit{pre-Relhyp}^{lin}((m),(n));\circ_h,\sigma_{id})$$

is a subsemigroup and a submonoid of the monoid $\textbf{\textit{Relhyp}}^{lin}((m),(n))$, respectively.

In this paper, we will determine the order of elements in the monoid of linear relational hypersubstitutions of a particular type $((m),(n))$. Let S be a semigroup and $a\in S$. The order of a is defined as the cardinality of $\langle a\rangle$ the subsemigroup generated by a.

4 The Order of Linear Projection Relational Hypersubstitutions

In this section, we will focus on the order of elements in $\mathbf{p\text{-}Relhyp}^{\mathrm{lin}}((m),(n))$ the semigroup of linear projection relational hypersubstitutions of type $((m),(n))$ by separating our consideration into two cases; (i) $m=1$ and $n\geq 1$, and (ii) $m\geq 2$ and $n\geq 1$.

4.1 Case (i): $m=1$ and $n\geq 1$

We denote an m-ary term $f(\cdots f(x_1))$ of type (1) with k occurrence of the operation symbol f by $f^{(k)}(x_1)$. Note that $f^{(0)}(x_1)=x_1$. Let $\mathcal{K}:=(\mathbb{N})^n\setminus\{(0,\ldots,0)\}$ and $\mathcal{K}':=(\mathbb{N})^n$. For every $(k_1,\ldots,k_n)\in\mathcal{K}'$, we let

$$\text{p-Relhyp}^{\mathrm{lin},(k_1,\ldots,k_n)}((1),(n)) := \{\sigma_{x_1,\gamma(f^{(k_1)}(x_{\alpha(1)}),\ldots,f^{(k_n)}(x_{\alpha(n)}))} : \alpha\in S_n\},$$

where S_n is a symmetric group on n. Then we can see that

$$\text{p-Relhyp}^{\mathrm{lin}}((1),(n)) := \bigcup_{(k_1,\ldots,k_n)\in\mathcal{K}'} \text{p-Relhyp}^{\mathrm{lin},(k_1,\ldots,k_n)}((1),(n)).$$

Proposition 4.1. *Let $(k_1,\ldots,k_n)\in\mathcal{K}'$. The structure*

$$\textbf{\textit{p-Relhyp}}^{lin,(k_1,\ldots,k_n)}((1),(n)) := (\textit{p-Relhyp}^{lin,(k_1,\ldots,k_n)}((1),(n));\circ_h)$$

is a subsemigroup of $\textbf{\textit{p-Relhyp}}^{lin}((1),(n))$.

Proof. Clearly, $\text{p-Relhyp}^{\text{lin},(k_1,\ldots,k_n)}((1),(n))\subseteq\text{p-Relhyp}^{\text{lin}}((1),(n))$. Let $\sigma_1,\sigma_2\in\text{p-Hyp}^{\text{lin}}_{(k_1,\ldots,k_n)}((1),(n))$. Then

$$\sigma_1=\sigma_{x_1,\gamma(f^{(k_1)}(x_{\alpha(1)}),\ldots,f^{(k_n)}(x_{\alpha(n)}))}\quad\text{and}\quad\sigma_2=\sigma_{x_1,\gamma(f^{(k_1)}(x_{\beta(1)}),\ldots,f^{(k_n)}(x_{\beta(n)}))}$$

for some $\alpha,\beta\in S_n$. It is clear that $(\sigma_{x_1,\gamma(f^{(k_1)}(x_{\alpha(1)}),\ldots,f^{(k_n)}(x_{\alpha(n)}))}\circ_{\text{h}}\sigma_{x_1,\gamma(f^{(k_1)}(x_{\beta(1)}),\ldots,f^{(k_n)}(x_{\beta(n)}))})(f)=x_1$. Now, we consider

$$\begin{aligned}
&(\sigma_{x_1,\gamma(f^{(k_1)}(x_{\alpha(1)}),\ldots,f^{(k_n)}(x_{\alpha(n)}))}\circ_{\text{h}}\sigma_{x_1,\gamma(f^{(k_1)}(x_{\beta(1)}),\ldots,f^{(k_n)}(x_{\beta(n)}))})(\gamma)\\
&=\widehat{\sigma}_{x_1,\gamma(f^{(k_1)}(x_{\alpha(1)}),\ldots,f^{(k_n)}(x_{\alpha(n)}))}[\sigma_{x_1,\gamma(f^{(k_1)}(x_{\beta(1)}),\ldots,f^{(k_n)}(x_{\beta(n)}))}(\gamma)]\\
&=\widehat{\sigma}_{x_1,\gamma(f^{(k_1)}(x_{\alpha(1)}),\ldots,f^{(k_n)}(x_{\alpha(n)}))}[\gamma(f^{(k_1)}(x_{\beta(1)}),\ldots,f^{(k_n)}(x_{\beta(n)}))]\\
&=R^1_n(\sigma_{x_1,\gamma(f^{(k_1)}(x_{\alpha(1)}),\ldots,f^{(k_n)}(x_{\alpha(n)}))}(\gamma),\widehat{\sigma}_{x_1,\gamma(f^{(k_1)}(x_{\alpha(1)}),\ldots,f^{(k_n)}(x_{\alpha(n)}))}[f^{(k_1)}(x_{\beta(1)})],\\
&\qquad\ldots,\widehat{\sigma}_{x_1,\gamma(f^{(k_1)}(x_{\alpha(1)}),\ldots,f^{(k_n)}(x_{\alpha(n)}))}[f^{(k_n)}(x_{\beta(n)})])\\
&=R^1_n(\gamma(f^{(k_1)}(x_{\alpha(1)}),\ldots,f^{(k_n)}(x_{\alpha(n)})),x_{\beta(1)},\ldots,x_{\beta(n)})\\
&=\gamma(R^1_n(f^{(k_1)}(x_{\alpha(1)}),x_{\beta(1)},\ldots,x_{\beta(n)}),\ldots,R^1_n(f^{(k_n)}(x_{\alpha(n)}),x_{\beta(1)},\ldots,x_{\beta(n)})\\
&=\gamma(f^{(k_1)}(x_{\beta\alpha(1)}),\ldots,f^{(k_n)}(x_{\beta\alpha(n)})).
\end{aligned}$$

Since $\alpha,\beta\in S_n$, $\beta\alpha\in S_n$. This implies that $\sigma_1\circ_{\text{h}}\sigma_2\in\text{p-Relhyp}^{\text{lin},(k_1,\ldots,k_n)}((1),(n))$. □

Proposition 4.2. *Let $(k_1,\ldots,k_n)\in\mathcal{K}$. Then we have*

$$\textbf{\textit{p-Relhyp}}^{lin,(0,\ldots,0)}((1),(n)) := (\textit{p-Relhyp}^{lin,(0,\ldots,0)}((1),(n));\circ_h)$$

and

$$\textbf{\textit{p-Relhyp}}^{lin,(k_1,\ldots,k_n)}((1),(n)) := (\textit{p-Relhyp}^{lin,(k_1,\ldots,k_n)}((1),(n));\circ_h)$$

are isomorphic.

Proof. It is clear that the mapping

$$\varphi\colon\text{p-Relhyp}^{\text{lin},(0,\ldots,0)}((1),(n))\to\text{p-Relhyp}^{\text{lin},(k_1,\ldots,k_n)}((1),(n))$$

defined by $\varphi(\sigma_{x_1,\gamma(x_{\alpha(1)},\ldots,x_{\alpha(n)})})=\sigma_{x_1,\gamma(f^{(k_1)}(x_{\alpha(1)}),\ldots,f^{(k_n)}(x_{\alpha(n)}))}$ for all $\alpha\in S_n$ is an isomorphism. □

Theorem 4.3. *The symmetric group $(S_n;\circ)$ and $\textbf{\textit{p-Relhyp}}^{lin,(0,\ldots,0)}((1),(n))$ are anti-isomorphic.*

Proof. We define the mapping

$$\varphi\colon\text{p-Relhyp}^{\text{lin},(0,\ldots,0)}((1),(n))\to S_n$$

by $\varphi(\sigma_{x_1,\gamma(x_{\alpha(1)},\ldots,x_{\alpha(n)})})=\alpha$. It is not difficult to see that φ is bijective. Next, we show that φ is an anti-homomorphism. Let $\sigma_{x_1,\gamma(x_{\alpha(1)},\ldots,x_{\alpha(n)})},\sigma_{x_1,\gamma(x_{\alpha'(1)},\ldots,x_{\alpha'(n)})}\in\text{p-Relhyp}^{\text{lin},(0,\ldots,0)}((1),(n))$. Then

$$\begin{aligned}
&\varphi(\sigma_{x_1,\gamma(x_{\alpha(1)},\ldots,x_{\alpha(n)})}\circ_{\text{h}}\sigma_{x_1,\gamma(x_{\alpha'(1)},\ldots,x_{\alpha'(n)})})\\
&=\varphi(\sigma_{x_1,\gamma(x_{(\alpha'\alpha)(1)},\ldots,x_{(\alpha'\alpha)(n)})})\\
&=\alpha'\circ\alpha\\
&=\varphi(\sigma_{x_1,\gamma(x_{\alpha'(1)},\ldots,x_{\alpha'(n)})})\circ_{\text{h}}\varphi(\sigma_{x_1,\gamma(x_{\alpha(1)},\ldots,x_{\alpha(n)})}).
\end{aligned}$$

Therefore, we obtain as desire. □

Corollary 4.4. *Let* $\sigma_{x_1,\gamma(x_{\alpha(1)},\ldots,x_{\alpha(n)})} \in$ *p-Hyp*$^{lin,(0,\ldots,0)}((1),(n))$*. Then the order of* $\sigma_{x_1,\gamma(x_{\alpha(1)},\ldots,x_{\alpha(n)})}$ *is equal to the order of* α *in the symmetric group on* n*.*

By Proposition 4.2 and Corollary 4.4, we conclude the following result.

Theorem 4.1. *The order of a linear projection relational hypersubstitution of type* $((1),(n))$ *is equal to the order of a permutation on* n*.*

4.2 Case (ii): $m \geq 2$ and $n \geq 1$

For any $i \in \{1,\ldots,n\}$, we let $B_i := \{\sigma_{x_i,\gamma(x_{\alpha(1)},\ldots,x_{\alpha(n)})} : \alpha \in S_n\}$. Then we can see that

$$\text{p-Relhyp}^{\text{lin}}((m),(n)) := \bigcup_{i=1}^{n} B_i.$$

It is clear that B_i is closed under $\circ_{\text{h}}$. Therefore, we obtain the following proposition.

Proposition 4.5. *The algebra* $\boldsymbol{B}_i := (B_i; \circ_r)$ *is a subsemigroup of* ***p-Relhyp***$^{lin}((m),(n))$*.*

Theorem 4.6. *Let* $i \in \{1,\ldots,n\}$*. The symmetric group* $(S_n; \circ)$ *and* $\boldsymbol{B}_i$ *are anti-isomorphic.*

Proof. By the same arguments of Theorem 4.3, we can show that the mapping

$$\varphi\colon B_i \to S_n$$

defined by $\varphi(\sigma_{x_i,\gamma(x_{\alpha(1)},\ldots,x_{\alpha(n)})}) = \alpha$ is an anti-isomorphism. □

By the above theorem, we have the following result.

Theorem 4.2. *The order of a linear projection relational hypersubstitution of type* $((m),(n))$ *is equal to the order of a permutation on* n*.*

5 The Order of Linear Relational Pre-Hypersubstitutions

Now, we will consider the order of linear relational pre-hypersubstitutions of type $((m),(n))$. We will separate our consideration into two cases; $m = 1$ and $n \geq 1$, and $m \geq 2$ and $n \geq 1$.

5.1 Case (i): $m = 1$ and $n \geq 1$

We define the sets

$$B_1 := \{\sigma_{f(x_1),\gamma(f(x_{\alpha(1)}),\ldots,f(x_{\alpha(n)}))} : \alpha \in S_n\}.$$

and

$$B_2 := \{\sigma_{f^{(l)}(x_1),\gamma(f^{(k_1)}(x_{\alpha(1)}),\ldots,f^{(k_n)}(x_{\alpha(n)}))} : l > 1, (k_1,\ldots,k_n) \in \mathcal{K}, \alpha \in S_n\}.$$

Then it is clear that pre-Relhyp$^{\text{lin}}((1),(n)) = B_1 \cup B_2$. We can see that B_1 is closed under $\circ_{\text{h}}$. Hence, we have the following proposition.

Proposition 5.1. *The structure*

$$\boldsymbol{B}_1 := (B_1; \circ_h, \sigma_{id})$$

is a submonoid of ***pre-Relhyp***$^{lin}((1),(n))$*.*

By the similar argument of Theorem 4.3, we obtain the following results.

Proposition 5.2. *The monoid* $\boldsymbol{B}_1$ *and the symmetric group* $(S_n;\circ)$ *are anti-isomorphic.*

Corollary 5.3. *The order of any element in* B_1 *is equal to the order of a permutation on* n.

Next, we will first show that $\mathbf{B}_2 := (B_2;\circ_{\mathrm{h}},\sigma_{\mathrm{id}})$ is a submonoid of $\textbf{pre-Relhyp}^{\mathrm{lin}}((1),(n))$.

Lemma 5.4. *Let* $\sigma\in B_2$*. Then for each* $1\le i\le n$*, we have that* $\widehat{\sigma}[f^{(k)}(x_i)] = f^{(k)}(x_i)$ *for all* $k\ge 1$.

Proof. This is obvious by induction on k. □

Lemma 5.5. *Let* $k,l>1$ *and* $\sigma_{f^{(k)}(x_1),\gamma(s_1,\ldots,s_n)},\sigma_{f^{(l)}(x_1),\gamma(t_1,\ldots,t_n)}\in B_2$*. Then*

$$(\sigma_{f^{(k)}(x_1),\gamma(s_1,\ldots,s_n)}\circ_h\sigma_{f^{(l)}(x_1),\gamma(t_1,\ldots,t_n)})(f) = f^{(kl)}(x_1).$$

Proof. We will give a proof by induction on l. If $l=2$, then

$$\begin{aligned}
&(\sigma_{f^{(k)}(x_1),\gamma(s_1,\ldots,s_n)}\circ_{\mathrm{h}}\sigma_{f^{(l)}(x_1),\gamma(t_1,\ldots,t_n)})(f)\\
&=\widehat{\sigma}_{f^{(k)}(x_1),\gamma(s_1,\ldots,s_n)}[\sigma_{f^{(l)}(x_1),\gamma(t_1,\ldots,t_n)}(f)]\\
&=\widehat{\sigma}_{f^{(k)}(x_1),\gamma(s_1,\ldots,s_n)}[f^{(l)}(x_1)]\\
&=R_1^1(\sigma_{f^{(k)}(x_1),\gamma(s_1,\ldots,s_n)}(f),\widehat{\sigma}_{f^{(k)}(x_1),\gamma(s_1,\ldots,s_n)}[f^{(l-1)}(x_1)])\\
&=R_1^1(f^{(k)}(x_1),R_1^1(\sigma_{f^{(k)}(x_1),\gamma(s_1,\ldots,s_n)}(f),\widehat{\sigma}_{f^{(k)}(x_1),\gamma(s_1,\ldots,s_n)}[x_1]))\\
&=R_1^1(f^{(k)}(x_1),R_1^1(f^{(k)}(x_1),x_1))\\
&=R_1^1(f^{(k)}(x_1),f^{(k)}(x_1))\\
&=f^{(2k)}(x_1)=f^{(kl)}(x_1).
\end{aligned}$$

Now, we assume that

$$(\sigma_{f^{(k)}(x_1),\gamma(s_1,\ldots,s_n)}\circ_{\mathrm{h}}\sigma_{f^{(l-1)}(x_1),\gamma(t_1,\ldots,t_n)})(f) = f^{(k(l-1))}(x_1).$$

Then

$$\begin{aligned}
&(\sigma_{f^{(k)}(x_1),\gamma(s_1,\ldots,s_n)}\circ_{\mathrm{h}}\sigma_{f^{(l)}(x_1),\gamma(t_1,\ldots,t_n)})(f)\\
&=\widehat{\sigma}_{f^{(k)}(x_1),\gamma(s_1,\ldots,s_n)}[\sigma_{f^{(l)}(x_1),\gamma(t_1,\ldots,t_n)}(f)]\\
&=\widehat{\sigma}_{f^{(k)}(x_1),\gamma(s_1,\ldots,s_n)}[f^{(l)}(x_1)]\\
&=R_1^1(\sigma_{f^{(k)}(x_1),\gamma(s_1,\ldots,s_n)}(f),\widehat{\sigma}_{f^{(k)}(x_1),\gamma(s_1,\ldots,s_n)}[f^{(l-1)}(x_1)])\\
&=R_1^1(f^{(k)}(x_1),f^{(k(l-1))}(x_1))\\
&=f^{(kl)}(x_1).
\end{aligned}$$

Thus, we obtain as desire. □

Lemma 5.6. *Let* $(k_1,\ldots,k_n),(l_1,\ldots,l_n)\in\mathcal{K}$*,* $k,l>1$ *and*

$$\sigma_{f^{(k)}(x_1),\gamma(f^{(k_1)}(x_{\alpha(1)}),\ldots,f^{(k_n)}(x_{\alpha(n)}))},\sigma_{f^{(l)}(x_1),\gamma(f^{(l_1)}(x_{\beta(1)}),\ldots,f^{(l_n)}(x_{\beta(n)}))}\in B_2$$

Then

$$\begin{aligned}
&(\sigma_{f^{(k)}(x_1),\gamma(f^{(k_1)}(x_{\alpha(1)}),\ldots,f^{(k_n)}(x_{\alpha(n)}))}\circ_h\sigma_{f^{(l)}(x_1),\gamma(f^{(l_1)}(x_{\beta(1)}),\ldots,f^{(l_n)}(x_{\beta(n)}))})(\gamma)\\
&=\gamma(f^{(k_1+l_{\alpha(1)})}(x_{\beta\alpha(1)}),\ldots,f^{(k_n+l_{\alpha(n)})}(x_{\beta\alpha(n)})).
\end{aligned}$$

Proof. By Lemma 5.4, we have that

$$\begin{aligned}
&(\sigma_{f^{(k)}(x_1),\gamma(f^{(k_1)}(x_{\alpha(1)}),\ldots,f^{(k_n)}(x_{\alpha(n)}))} \circ_{\mathrm{h}} \sigma_{f^{(l)}(x_1),\gamma(f^{(l_1)}(x_{\beta(1)}),\ldots,f^{(l_n)}(x_{\beta(n)}))})(\gamma)\\
&= \widehat{\sigma}_{f^{(k)}(x_1),\gamma(f^{(k_1)}(x_{\alpha(1)}),\ldots,f^{(k_n)}(x_{\alpha(n)}))}[\sigma_{f^{(l)}(x_1),\gamma(f^{(l_1)}(x_{\beta(1)}),\ldots,f^{(l_n)}(x_{\beta(n)}))}(\gamma)]\\
&= \widehat{\sigma}_{f^{(k)}(x_1),\gamma(f^{(k_1)}(x_{\alpha(1)}),\ldots,f^{(k_n)}(x_{\alpha(n)}))}[\gamma(f^{(l_1)}(x_{\beta(1)}),\ldots,f^{(l_n)}(x_{\beta(n)}))]\\
&= R_n^n(\gamma(f^{(k_1)}(x_{\alpha(1)}),\ldots,f^{(k_n)}(x_{\alpha(n)})),\\
&\qquad \widehat{\sigma}_{f^{(k)}(x_1),\gamma(f^{(k_1)}(x_{\alpha(1)}),\ldots f^{(k_n)}(x_{\alpha(n)}))}[f^{(l_1)}(x_{\beta(1)})],\\
&\qquad \ldots \widehat{\sigma}_{f^{(k)}(x_1),\gamma(f^{(k_1)}(x_{\alpha(1)}),\ldots f^{(k_n)}(x_{\alpha(n)}))}[f^{(l_1)}(x_{\beta(n)})])\\
&= R_n^n(\gamma(f^{(k_1)}(x_{\alpha(1)}),\ldots,f^{(k_n)}(x_{\alpha(n)})),f^{(l_1)}(x_{\beta(1)}),\ldots,f^{(l_n)}(x_{\beta(n)}))\\
&= \gamma(f^{(k_1+l_{\alpha(1)})}(x_{\beta\alpha(1)}),\ldots,f^{(k_n+l_{\alpha(n)})}(x_{\beta\alpha(n)})).
\end{aligned}$$

□

By Lemmas 5.5 and 5.6, we conclude that:

Proposition 5.1. *The structure* $\boldsymbol{B}_2 := (B_2; \circ_h, \sigma_{id})$ *is a submonoid of* $\textbf{\textit{pre-Relhyp}}^{lin}((1),(n))$.

Corollary 5.2. *Let* $\alpha \in B_2$. *Then the order of* α *is infinite.*

Therefore, we obtain the following theorem.

Theorem 5.3. *The order of a linear relational pre-hypersubstitution of type* $((1),(n))$ *is either equal to the order of a permutation on* n*, or infinite.*

5.2 Case (ii): $m \geq 2$ and $n \geq 1$

We observe that

$$\text{pre-Relhyp}^{\text{lin}}((m),(n)) := \{\sigma_{f(x_{\alpha(1)},\ldots,x_{\alpha(m)}),\gamma(x_{\alpha'(1)},\ldots,x_{\alpha'(n)})} : \alpha \in S_m, \alpha' \in S_n\}.$$

Proposition 5.7. *The algebra*

$$\textbf{\textit{pre-Relhyp}}^{lin}((m),(n)) := (\textit{pre-Relhyp}^{lin}((m),(n)); \circ_h, \sigma_{id})$$

anti-isomorphics to the direct product $S_m \times S_n$*, where* S_m *and* S_n *is a symmetric group on* m *and* n*, respectively.*

Proof. Let $\varphi : \text{pre-Relhyp}^{\text{lin}}((m),(n)) \to S_m \times S_n$ be a mapping defined by

$$\varphi(\sigma_{f(x_{\alpha(1)},\ldots,x_{\alpha(m)}),\gamma(x_{\alpha'(1)},\ldots,x_{\alpha'(n)})}) = (\alpha, \alpha').$$

It is not difficult to see that φ is a bijection. To show that φ is an anti-isomorphism. Let $\sigma_1, \sigma_2 \in \text{pre-Relhyp}^{\text{lin}}((m),(n))$. Then

$$\sigma_1 = \sigma_{f(x_{\alpha(1)},\ldots,x_{\alpha(m)}),\gamma(x_{\alpha'(1)},\ldots,x_{\alpha'(n)})} \quad \text{and} \quad \sigma_2 = \sigma_{f(x_{\beta(1)},\ldots,x_{\beta(m)}),\gamma(x_{\beta'(1)},\ldots,x_{\beta'(n)})}$$

for some $\alpha \in S_m$ and $\alpha' \in S_n$. Thus,

$$\begin{aligned}
&\varphi(\sigma_{f(x_{\alpha(1)},\ldots,x_{\alpha(m)}),\gamma(x_{\alpha'(1)},\ldots,x_{\alpha'(n)})} \circ_{\mathrm{h}} \sigma_{f(x_{\beta(1)},\ldots,x_{\beta(m)}),\gamma(x_{\beta'(1)},\ldots,x_{\beta'(n)})})\\
&= \varphi(\sigma_{f(x_{\beta\alpha(1)},\ldots,x_{\beta\alpha(m)}),\gamma(x_{\beta'\alpha'(1)},\ldots,x_{\beta'\alpha'(n)})})\\
&= (\beta\alpha, \beta'\alpha')\\
&= (\beta, \beta') \circ (\alpha, \alpha')\\
&= \varphi(\sigma_{f(x_{\beta(1)},\ldots,x_{\beta(m)}),\gamma(x_{\beta'(1)},\ldots,x_{\beta'(n)})}) \circ \varphi(\sigma_{f(x_{\alpha(1)},\ldots,x_{\alpha(m)}),\gamma(x_{\alpha'(1)},\ldots,x_{\alpha'(n)})}).
\end{aligned}$$

Thus, we obtain as desire. □

Therefore, we obtain the following theorem.

Theorem 5.4. *The order of a linear relational pre-hypersubstitution of type* $((m),(n))$ *is the order of an element of direct product of symmetric group on* m *and* n.

Acknowledgement(s) : We would like to thank the referee(s) for his comments and suggestions on the manuscript. The first author is supported by Development and Promotion of Science and Technology Talents Project (DPST), and Department of Mathematics, Faculty of Science, Khon Kaen University.

References

[1] K. Denecke and D. Phusanga, Hyperformulas and Solid Algebraic Systems, Studia Logica, 90(2008) 263–286.

[2] K. Denecke D. Lau, R. Pöschel, and D. Schweigert, Hyperidentities, Hyperequational Classes and Clone Congruences, Contributions to General Algebra 7, Verlag Hölder-Pichler-Tempsky (1991), 97–118.

[3] J. Koppitz and K. Denecke, M-Solid Varieties of Algebras, Springer Science & Business Media, 2006.

[4] K. Denecke and S.L. Wismath, The monoid of hypersubstitutions of type $\tau = (2)$, Contributions to General Algebra 10, Klagenfurth (1998), 109-127.

[5] S. Wismath, The Monoid of Hypersubstitutions of Type (n), Southeast Asian Bull. Math., 24(2000) 115–128.

[6] T. Changphas, K. Denecke, All idempotent hypersubstitutions of type $(2,2)$, Semigroup Forum, 76(2008) 525–539.

[7] T. Changphas, K. Denecke, The Order of Hypersubstitutions of Type $(2,2)$, Acta Math. Sin. (Engl. Ser.), 23(2007) 659–670.

[8] T. Changphas and W. Hemvong, The Order of Hypersubstitutions of Type $(2,1)$, Int. J. Math. Math. Sci., Volume 2011, Article ID 615014, 18 pages.

[9] T. Changphas and W. Hemvong, The Order of Hypersubstitutions of Type $\tau = (n)$, Appl. Math. Sci. (Ruse), 6(2012) 2019–2027.

[10] A. I. Mal'cev. Algebraic systems. (English translation). Springer-Verlag, New York, Heidelberg, and Berlin, 1973.

[11] D. Phusanga, Derived Algebraic Systems, Ph.D. Theses, Potsdam 2013.

[12] M. Couceiro and E. Lehtonen, Galois theory for sets of operations closed under permutation, cylindrification and composition, Algebra Universalis, 67(2012) 273–297.

[13] D. Phusanga, Some varieties of algebraic systems of type $((n),(m))$, Asian-Eur. J. Math., (2017) doi.org/10.1142/S1793557119500050

[14] K. Denecke, The partial clone of linear formulas, preprint (2018), Potsdam.

[15] T. Changphas, K. Denecke and B. Pibaljomme, Linear terms and linear hypersubstitutions, preprint (2014), Khon Kaen.

[16] K. Denecke, The partial clone of linear terms, Sib. Math. J., 57(2016) 589–598.

(Received 21 November 2018)
(Accepted 20 June 2019)