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Modified Finite Integration Method Using Chebyshev Polynomial Expansion for Solving One-Dimensional Nonlinear Burgers' Equations with Shock Wave

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Abstract : Based on the recently modified finite integration method (FIM) for solving linear differential equations by using the Chebyshev polynomial expansion, in this paper, we improve the modified FIM to be able to handle nonlinear Burgers' equations with shock waves in one dimension. The main idea is to approximate the nonlinear term of the Burgers' equation and apply the modified FIM to construct the finite integration matrices on each computational grid points which are generated by the zeros of the Chebyshev polynomial of a certain degree. In addition, the term involving partial derivative with respect to time is approximated by the forward difference quotient. Illustrative numerical solutions obtained by the proposed modified FIM algorithm are compared with the traditional FIM, finite difference method (FDM), finite element method (FEM), other methods and their analytical solution from several examples. Evidently, the proposed modified FIM algorithm has made a significant improvement in terms of accuracy and computational time for small values of the viscosity.

Keywords : finite integration method; Chebyshev polynomial; Burgers' equation; kinematic viscosity.
2010 Mathematics Subject Classification : 65L05; 65L10; 65M70; 65N30.

1 Introduction

Most of the principles and behaviors of the natural phenomena can be described by the statements or relationships involving rates of change. These rates can be expressed in mathematical relations that are in terms of linear or nonlinear differential equations. Generally, most of the real incidents that appear in our daily life are inborn nonlinear. Consequently, nonlinear differential equations have been the matter of study and research interest in several branches of science and engineering. One of the interesting issues is the Burgers' equation. It was first introduced by Bateman in 1915 [1]. He mentioned that this kind of equation was worthy of study and he gave its steady solutions. In 1948, Burgers [2] studied a mathematical model for turbulence. This model is known as *Burgers' equation*. The Burgers' equation has some common features with the Navier-Stokes equations, i.e., same kind of nonlinearity and the presence of the viscosity term. Therefore, one can consider studying turbulence with a simple model of Burgers' equation as a test problem instead of the Navier-Stokes equations. Nowadays, the Burgers' equation, which is a fundamental partial differential equation (PDE), has been hired in a large variety

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of applications in applied mathematics, physics and engineering such as a simplified fluid dynamics model, modeling of transport with accumulation, advection and diffusion terms, gas dynamics, traffic flow, modeling of shock waves, heat conduction, acoustic waves, statistics of flow problems, mixing and turbulent diffusion and so on, see [2], [3] and [4] for details.

Under various boundary conditions and the real problem configuration, it is very unlikely that the Burgers' equation can be represented as its analytical solution. The numerical method plays an essential role in finding approximate solutions to the problems. There are many numerical methods available for solving the Burgers' equation such as the finite difference method (FDM), finite element method (FEM), etc., see [5]. Recently, in terms of a numerical method for solving PDEs, Wen et al. [6] and Li et al. [7] developed the finite integration method (FIM) with the trapezoidal rule and radial basis function for solving one- and multi-dimensional linear PDEs. In 2016, Li et al. [8] improved the FIM by using three numerical quadrature formulas, including Simpson's rule, Cotes integral and Lagrange interpolation to solve linear PDEs. They showed that their proposed FIMs, in which we refer as the traditional FIMs, were highly accurate compared with the FDM. After that in 2018, Boonklurb et al. [9] modified the FIM using Chebyshev polynomial for solving linear PDEs which gave higher accuracy than the FDM and traditional FIMs.

In this paper, we improve the modified FIM using the Chebyshev polynomial expansion to be able to deal with one-dimensional nonlinear Burgers' equations with a shock wave. In Section 2, for ease of reference, some preliminary facts and results concerning the Chebyshev polynomial and the method of constructing the finite integration matrices are presented. At the end of the section, our proposed modified FIM using Chebyshev polynomial for solving Burgers' equation is elaborated. We implement our proposed modified FIM algorithm on several examples to demonstrate its efficiency compare with the FDM, FEM, traditional FIM, other methods and their analytical solutions for small values of the viscosity in Section 3. Finally, conclusion and some discussion for the future work are given in Section 4.

2 Modified FIM by Using Chebyshev Polynomial Expansion for Solving Burgers' Equation

Let us consider definitions and some significant properties of the Chebyshev polynomial [10] that are used to construct the first and higher order integration matrices for solving a one-dimensional nonlinear Burgers' equation with shock wave as follow.

Definition 2.1. *The Chebyshev polynomial of degree $n \geq 0$ is defined by*

$$T_n(x) = \cos(n \cos^{-1} x) \text{ for } x \in [-1, 1]. \tag{2.1}$$

Lemma 2.1. (i) *For $n \in \mathbb{N}$, the zeros of Chebyshev polynomial $T_n(x)$ are*

$$x_k = \cos\left(\frac{2k-1}{2n}\pi\right), \; k \in \{1, 2, 3, ..., n\}. \tag{2.2}$$

(ii) *For $x \in [-1, 1]$, the single layer integrations of Chebyshev polynomial $T_n(x)$ are*

$$\begin{aligned}
\bar{T}_0(x) &= \int_{-1}^{x} T_0(\xi)\, d\xi = x + 1, \\
\bar{T}_1(x) &= \int_{-1}^{x} T_1(\xi)\, d\xi = \frac{x^2}{2} - \frac{1}{2}, \\
\bar{T}_n(x) &= \int_{-1}^{x} T_n(\xi)\, d\xi = \frac{1}{2}\left[\frac{T_{n+1}(x)}{n+1} - \frac{T_{n-1}(x)}{n-1}\right] - \frac{(-1)^n}{n^2-1}, \; n \in \{2, 3, 4, ...\}.
\end{aligned}$$

(iii) *Let* $\{x_k\}_{k=1}^{n}$ *be the zeros of Chebyshev polynomial* $T_n(x)$ *and define the Chebyshev matrix* $\boldsymbol{T}$ *by*

$$\boldsymbol{T} = \begin{bmatrix} T_0(x_1) & T_1(x_1) & \cdots & T_{n-1}(x_1) \\ T_0(x_2) & T_1(x_2) & \cdots & T_{n-1}(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ T_0(x_n) & T_1(x_n) & \cdots & T_{n-1}(x_n) \end{bmatrix}.$$

Then, it has the multiplicative inverse $\boldsymbol{T}^{-1} = \frac{1}{n}\text{diag}(1, 2, 2, ..., 2)\,\boldsymbol{T}^{T}$.

2.1 Finite Integration Matrices

Let N be a nonnegative integer. Define an approximate solution $u(x)$ of a certain PDE by a linear combination of the Chebyshev polynomials, i.e.,

$$u(x) = \sum_{n=0}^{N-1} c_n T_n(x) \text{ for } x \in [-1, 1]. \tag{2.3}$$

Let $\bar{x}_k$ for $k \in \{1, 2, 3, ..., N\}$ be nodal points discretized by the zeros of Chebyshev polynomial $T_N(x)$ defined in (2.2). Substituting each $\bar{x}_k$ into (2.3), it can be expressed in matrix form as

$$\begin{bmatrix} u(\bar{x}_1) \\ u(\bar{x}_2) \\ \vdots \\ u(\bar{x}_N) \end{bmatrix} = \begin{bmatrix} T_0(\bar{x}_1) & T_1(\bar{x}_1) & \cdots & T_{N-1}(\bar{x}_1) \\ T_0(\bar{x}_2) & T_1(\bar{x}_2) & \cdots & T_{N-1}(\bar{x}_2) \\ \vdots & \vdots & \ddots & \vdots \\ T_0(\bar{x}_N) & T_1(\bar{x}_N) & \cdots & T_{N-1}(\bar{x}_N) \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{N-1} \end{bmatrix},$$

which is denoted by $\mathbf{u} = \mathbf{Tc}$. The coefficients c_n for $n \in \{0, 1, 2, ..., N-1\}$ can be performed by $\mathbf{c} = \mathbf{T}^{-1}\mathbf{u}$. Let us consider the single layer integration of $u(x)$ from -1 to $\bar{x}_k$ which is denoted by $U(\bar{x}_k)$, we obtain

$$U(\bar{x}_k) = \int_{-1}^{\bar{x}_k} u(\xi)\, d\xi = \sum_{n=0}^{N-1} c_n \int_{-1}^{\bar{x}_k} T_n(\xi)\, d\xi = \sum_{n=0}^{N-1} c_n \bar{T}_n(\bar{x}_k)$$

for $k \in \{1, 2, 3, ..., N\}$ or in matrix form:

$$\begin{bmatrix} U(\bar{x}_1) \\ U(\bar{x}_2) \\ \vdots \\ U(\bar{x}_N) \end{bmatrix} = \begin{bmatrix} \bar{T}_0(\bar{x}_1) & \bar{T}_1(\bar{x}_1) & \cdots & \bar{T}_{N-1}(\bar{x}_1) \\ \bar{T}_0(\bar{x}_2) & \bar{T}_1(\bar{x}_2) & \cdots & \bar{T}_{N-1}(\bar{x}_2) \\ \vdots & \vdots & \ddots & \vdots \\ \bar{T}_0(\bar{x}_N) & \bar{T}_1(\bar{x}_N) & \cdots & \bar{T}_{N-1}(\bar{x}_N) \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{N-1} \end{bmatrix}.$$

We denote the above matrix by $\mathbf{U} = \bar{\mathbf{T}}\mathbf{c} = \bar{\mathbf{T}}\mathbf{T}^{-1}\mathbf{u} := \mathbf{A}\mathbf{u}$, where $\mathbf{A} = \bar{\mathbf{T}}\mathbf{T}^{-1} := [a_{ki}]_{N\times N}$ is called the *first order integration matrix* for the modified FIM in one dimension, i.e.,

$$U(\bar{x}_k) = \int_{-1}^{\bar{x}_k} u(\xi)\, d\xi = \sum_{i=1}^{N} a_{ki} u(\bar{x}_i)$$

for $k \in \{1, 2, 3, ..., N\}$. Next, let us consider the double layer integration of $u(x)$ from -1 to $\bar{x}_k$ which is denoted by $U^{(2)}(\bar{x}_k)$, we get

$$U^{(2)}(\bar{x}_k) = \int_{-1}^{\bar{x}_k} \int_{-1}^{\xi_2} u(\xi_1)\, d\xi_1 d\xi_2 = \sum_{i=1}^{N} a_{ki} \int_{-1}^{\bar{x}_i} u(\xi_1)\, d\xi_1 = \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ki} a_{ij} u(\bar{x}_j)$$

for $k \in \{1, 2, 3, ..., N\}$. It can be written in matrix form as $\mathbf{U}^{(2)} = \mathbf{A}^2\mathbf{u}$. Similarly, we can calculate the m^{th} layer integration of $u(x)$ from -1 to $\bar{x}_k$ which is denoted by $U^{(m)}(\bar{x}_k)$. Then, we have

$$U^{(m)}(\bar{x}_k) = \int_{-1}^{\bar{x}_k} \cdots \int_{-1}^{\xi_2} u(\xi_1)\, d\xi_1 \dots d\xi_m = \sum_{i_m=1}^{N} \cdots \sum_{j=1}^{N} a_{ki_m} \dots a_{i_1 j} u(\bar{x}_j)$$

for $k \in \{1, 2, 3, ..., N\}$, whose the matrix form can be expressed as $\mathbf{U}^{(m)} = \mathbf{A}^m\mathbf{u}$, where $\mathbf{U}^{(m)} = [U^{(m)}(\bar{x}_1), U^{(m)}(\bar{x}_2), U^{(m)}(\bar{x}_3), ..., U^{(m)}(\bar{x}_N)]^T$.

2.2 The Proposed Numerical Algorithm

Now, we apply the modified FIM using Chebyshev polynomial to construct the numerical algorithm for solving the one-dimensional nonlinear Burgers' equation with shock wave to achieve high-performance computations. Consider the following one-dimensional nonlinear Burgers' equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2}, \quad x \in (a, b), \quad t \in (t_0, T], \tag{2.4}$$

subject to the initial condition:

$$u(x, t_0) = \phi(x), \quad x \in [a, b] \tag{2.5}$$

and the boundary conditions:

$$u(a, t) = \psi_1(t),\ u(b, t) = \psi_2(t), \quad t \in (t_0, T], \tag{2.6}$$

where t and x represent time and space variables, respectively, $\nu > 0$ is the coefficient of kinematic viscosity defined by $\nu = \frac{1}{Re}$, Re is the Reynolds number and $\phi(x)$ is a given sufficiently smooth function.

First, we transform the space variable $x \in [a, b]$ into $\bar{x} \in [-1, 1]$ by using the transformation $\bar{x} = \frac{2x-a-b}{b-a}$ and let $h = \frac{2}{b-a}$. Then, (2.4) becomes

$$\frac{\partial u(\bar{x}, t)}{\partial t} + hu(\bar{x}, t)\frac{\partial u(\bar{x}, t)}{\partial \bar{x}} = h^2\nu\frac{\partial^2 u(\bar{x}, t)}{\partial \bar{x}^2}. \tag{2.7}$$

Next, we linearize (2.7) by determining the iterations and using the first order forward difference quotient for the time derivative. Thus, for $m \geq 1$, we have

$$\frac{u^m(\bar{x}) - u^{m-1}(\bar{x})}{\Delta t} + hu^{m-1}(\bar{x})\frac{\partial u^m(\bar{x})}{\partial \bar{x}} = h^2\nu\frac{\partial^2 u^m(\bar{x})}{\partial \bar{x}^2}, \tag{2.8}$$

where Δt is a time step, u^{m-1} and u^m are numerical values in the $(m-1)^{th}$ and m^{th} iterations, respectively. Applying the modified FIM to eliminate derivative out of (2.8) by taking double layer integration. Then, we obtain the following equation at point $\bar{x}_k$ defined in (2.2),

$$\int_{-1}^{\bar{x}_k}\int_{-1}^{\eta}\left(\frac{u^m - u^{m-1}}{\Delta t}\right)d\xi d\eta + h\int_{-1}^{\bar{x}_k}\int_{-1}^{\eta}\left(u^{m-1}\frac{\partial u^m}{\partial \xi}\right)d\xi d\eta = h^2\nu u^m(\bar{x}_k) + c_1\bar{x}_k + c_2, \tag{2.9}$$

where c_1 and c_2 are the arbitrary constants of integration.

Let $\int_{-1}^{\bar{x}_k}\int_{-1}^{\eta}\left(u^{m-1}\frac{\partial u^m}{\partial \xi}\right)d\xi d\eta := q(\bar{x}_k)$. Using the technique of integration by parts to obtain

$$\begin{aligned}
q(\bar{x}_k) &= \int_{-1}^{\bar{x}_k}\int_{-1}^{\eta}\left(\sum_{n=0}^{N-1} c_n^{m-1}T_n(\xi)\right)\frac{\partial u^m}{\partial \xi}\,d\xi d\eta \\
&= \int_{-1}^{\bar{x}_k}\sum_{n=0}^{N-1} c_n^{m-1}\left(T_n(\eta)u^m(\eta) - \int_{-1}^{\eta} T_n'(\xi)u^m(\xi)\,d\xi\right)d\eta \\
&= \int_{-1}^{\bar{x}_k}\sum_{n=0}^{N-1} c_n^{m-1}T_n(\eta)u^m(\eta)\,d\eta - \int_{-1}^{\bar{x}_k}\int_{-1}^{\eta}\sum_{n=0}^{N-1} c_n^{m-1}T_n'(\xi)u^m(\xi)\,d\xi d\eta \\
&= \int_{-1}^{\bar{x}_k} u^{m-1}(\eta)u^m(\eta)\,d\eta - \int_{-1}^{\bar{x}_k}\int_{-1}^{\eta}\mathbf{T}'(\xi)\mathbf{c}^{m-1}u^m(\xi)\,d\xi d\eta \\
&= \int_{-1}^{\bar{x}_k} u^{m-1}(\eta)u^m(\eta)\,d\eta - \int_{-1}^{\bar{x}_k}\int_{-1}^{\eta}\mathbf{T}'(\xi)\mathbf{T}^{-1}\mathbf{u}^{m-1}u^m(\xi)\,d\xi d\eta, \qquad (2.10)
\end{aligned}$$

where $\mathbf{T}'(\xi) = [T_0'(\xi), T_1'(\xi), T_2'(\xi), ..., T_{N-1}'(\xi)]$ and $\mathbf{c}^{m-1} = \mathbf{T}^{-1}\mathbf{u}^{m-1}$ as defined in Section 2.1. Thus, for $k \in \{1, 2, 3, ..., N\}$, (2.10) can be expressed in matrix form as

$$\mathbf{q} = \mathbf{A}\text{diag}\left(\mathbf{u}^{m-1}\right)\mathbf{u}^m - \mathbf{A}^2\text{diag}\left(\mathbf{T}'\mathbf{T}^{-1}\mathbf{u}^{m-1}\right)\mathbf{u}^m, \qquad (2.11)$$

where $\mathbf{q} = [q(\bar{x}_1), q(\bar{x}_2), q(\bar{x}_3), ..., q(\bar{x}_N)]^T$. Consequently, by consuming (2.11) and the idea of Boonklurb et al. [9], we can convert (2.9) into the matrix form as follow.

$$\left(\frac{1}{\Delta t}\mathbf{A}^2 + h\mathbf{A}\text{diag}\left(\mathbf{u}^{m-1}\right) - h\mathbf{A}^2\text{diag}\left(\mathbf{T}'\mathbf{T}^{-1}\mathbf{u}^{m-1}\right) - h^2\nu\mathbf{I}\right)\mathbf{u}^m - c_1\mathbf{x} - c_2\mathbf{i} = \frac{1}{\Delta t}\mathbf{A}^2\mathbf{u}^{m-1}, \qquad (2.12)$$

where $\mathbf{I}$ is the identity matrix with size $N \times N$, $\mathbf{A} = \bar{\mathbf{T}}\mathbf{T}^{-1}$, $\mathbf{x} = [\bar{x}_1, \bar{x}_2, \bar{x}_3, ..., \bar{x}_N]^T$, $\mathbf{i} = [1, 1, 1, ..., 1]^T$, $\mathbf{u}^m = [u^m(\bar{x}_1), u^m(\bar{x}_2), u(\bar{x}_3), ..., u^m(\bar{x}_N)]^T$, $\mathbf{u}^{m-1} = [u^{m-1}(\bar{x}_1), u^{m-1}(\bar{x}_2), u^{m-1}(\bar{x}_3), ..., u^{m-1}(\bar{x}_N)]^T$ and

$$\mathbf{T}' = \begin{bmatrix}\mathbf{T}'(\bar{x}_1)\\ \mathbf{T}'(\bar{x}_2)\\ \vdots \\ \mathbf{T}'(\bar{x}_N)\end{bmatrix} = \begin{bmatrix} T_0'(\bar{x}_1) & T_1'(\bar{x}_1) & \cdots & T_{N-1}'(\bar{x}_1)\\ T_0'(\bar{x}_2) & T_1'(\bar{x}_2) & \cdots & T_{N-1}'(\bar{x}_2)\\ \vdots & \vdots & \ddots & \vdots \\ T_0'(\bar{x}_N) & T_1'(\bar{x}_N) & \cdots & T_{N-1}'(\bar{x}_N)\end{bmatrix}.$$

From the given boundary conditions (2.6), we can change them into the vector forms by using linear combination of Chebyshev polynomial at m^{th} iterations as following:

$$u^m(-1) = \sum_{n=0}^{N-1} c_n^m T_n(-1) = \sum_{n=0}^{N-1} c_n^m(-1)^n := \mathbf{t}_l\mathbf{c}^m = \mathbf{t}_l\mathbf{T}^{-1}\mathbf{u}^m = \psi_1(t_m), \qquad (2.13)$$

$$u^m(1) = \sum_{n=0}^{N-1} c_n^m T_n(1) = \sum_{n=0}^{N-1} c_n^m(1)^n := \mathbf{t}_r\mathbf{c}^m = \mathbf{t}_r\mathbf{T}^{-1}\mathbf{u}^m = \psi_2(t_m), \qquad (2.14)$$

where $t_m = t_0 + m\Delta t$ for $m \in \mathbb{N}$, $\mathbf{t}_l = [1, -1, 1, ..., (-1)^{N-1}]$ and $\mathbf{t}_r = [1, 1, 1, ..., 1]$.

Finally, from (2.12), (2.13) and (2.14), we can construct the following system of iterative linear equations for a total of $N+2$ unknowns containing $\mathbf{u}^m$, c_1 and c_2

$$\left[\begin{array}{c|cc}\mathbf{K} & -\mathbf{x} & -\mathbf{i}\\ \hline \mathbf{t}_l\mathbf{T}^{-1} & 0 & 0\\ \mathbf{t}_r\mathbf{T}^{-1} & 0 & 0\end{array}\right]\left[\begin{array}{c}\mathbf{u}^m\\ \hline c_1\\ c_2\end{array}\right] = \left[\begin{array}{c}\frac{1}{\Delta t}\mathbf{A}^2\mathbf{u}^{m-1}\\ \hline \psi_1(t_m)\\ \psi_2(t_m)\end{array}\right], \qquad (2.15)$$

where $\mathbf{K} = \frac{1}{\Delta t}\mathbf{A}^2 + h\mathbf{A}\text{diag}\left(\mathbf{u}^{m-1}\right) - h\mathbf{A}^2\text{diag}\left(\mathbf{T}'\mathbf{T}^{-1}\mathbf{u}^{m-1}\right) - h^2\nu\mathbf{I}$. Then, the solution $\mathbf{u}^m$ can be approximated by solving the system (2.15) with strating from the given initial condition (2.5) that is $\mathbf{u}^0 = [\phi(x_1), \phi(x_2), \phi(x_3), ..., \phi(x_N)]^T$, where $x_k = \frac{1}{2}[(b-a)\bar{x}_k + a + b]$ for $k \in \{1, 2, 3, ..., N\}$.

Note that when we want to calculate a numerical solution u at arbitrary $x \in [a, b]$ for the terminal time T, we can find it from (2.3) with corresponding to $\bar{x} = \frac{2x-a-b}{b-a}$ as follow

$$u(x, T) = \sum_{n=0}^{N-1} c_n T_n(\bar{x}) = \mathbf{T}(\bar{x})\mathbf{c} = \mathbf{T}(\bar{x})\mathbf{T}^{-1}\mathbf{u}^m, \tag{2.16}$$

where $\mathbf{T}(\bar{x}) = [T_0(\bar{x}), T_1(\bar{x}), T_2(\bar{x}), ..., T_{N-1}(\bar{x})]$ and $\mathbf{u}^m$ is the final m^{th} iterative solution of (2.15).

Algorithm 2.1 Algorithm to find a numerical solution of the Burgers' equation by modified FIM using Chebyshev polynomial

Input: $a, b, x, \nu, t_0, T, N, \Delta t, \phi(x), \psi_1(t), \psi_2(t)$.
Output: An approximate solution $u(x, T)$.

```
Set x̄_k = −cos((2k−1)/(2N) π) for k ∈ {1, 2, 3, ..., N}.
Set x_k = ½[(b − a)x̄_k + a + b] for k ∈ {1, 2, 3, ..., N}.
Compute h, x, i, t_l, t_r, I, T′, T̄, T⁻¹, A.
Construct u⁰ = [φ(x_1), φ(x_2), φ(x_3), ..., φ(x_N)]ᵀ.
Set m = 0.
while t_m ≤ T do
    Set m = m + 1.
    Set t_m = t_0 + mΔt.
    Find uᵐ by solving the linear system (2.15).
end while
Find u(x, T) = T(x̄)T⁻¹uᵐ, where x̄ = (2x−a−b)/(b−a).
```

3 Numerical Examples

In this section, we have applied the proposed Algorithm 2.1 based on the modified FIM using Chebyshev polynomial for finding the approximate solutions of one-dimensional nonlinear Burgers' equations with shock wave in order to illustrate the efficiency and accuracy. In the following Examples 3.1 and 3.2, the analytical solutions were obtained by using the Hopf-Cole transformation that Benton and Platzman [11] have surveyed. Their analytic solutions involve an infinite series, which may converge very slowly for the small viscosity ν. Then, Miller [12] has shown that these problems produce oscillations and instabilities for $\nu < 0.01$. We see from our result that our proposed Algorithm 2.1 can reduce the effect of these problems, especially for small ν. The presented method also can be performed on Examples 3.3 and 3.4 that contain shock waves in their exact solutions. The accuracy of the numerical results is measured in term of error norms L_∞, L_1, L_2 and absolute error E_a.

Example 3.1. Consider the Burgers' equation (2.4) with initial and boundary conditions

$$\begin{aligned} u(x, 0) &= \sin(\pi x), \quad x \in [0, 1], \\ u(0, t) &= u(1, t) = 0, \quad t > 0. \end{aligned} \tag{3.1}$$

The analytical solution given by Cole [13] of this equation is

$$u(x, t) = \frac{2\pi\nu \sum_{n=1}^{\infty} a_n \exp(-n^2\pi^2\nu t) n \sin(n\pi x)}{a_0 + \sum_{n=1}^{\infty} a_n \exp(-n^2\pi^2\nu t)\cos(n\pi x)}, \tag{3.2}$$

where the Fourier coefficients a_0 and a_n are

$$a_0 = \int_0^1 \exp\{-(2\pi\nu)^{-1}[1-\cos(\pi x)]\}dx,$$
$$a_n = 2\int_0^1 \exp\{-(2\pi\nu)^{-1}[1-\cos(\pi x)]\}\cos(n\pi x)dx, \quad n \in \{1,2,3,...\}.$$

The numerical solutions $u(x,T)$ of Example 3.1 achieved by the proposed Algorithm 2.1 for choosing $\Delta t = 0.0001$ and $N = 80$ with the viscosity $\nu = 0.01$ are shown in Table 3.1. They have been compared with the numerical results obtained by the FEM [14], FDM [13], traditional FIM [15] and their analytical solutions measured by the absolute error E_a.

x	T	Exact	FEM [14]		FDM [13]		Traditional FIM [15]		Modified FIM CBS	
			$u(x,T)$	E_a	$u(x,T)$	E_a	$u(x,T)$	E_a	$u(x,T)$	E_a
0.25	0.4	0.34191	0.34819	6.28×10^{-3}	0.34244	5.30×10^{-4}	0.34183	8.00×10^{-5}	0.34191	1.1647×10^{-6}
	0.6	0.26896	0.27536	6.40×10^{-3}	0.26905	9.00×10^{-5}	0.26891	5.00×10^{-5}	0.26896	5.2590×10^{-7}
	0.8	0.22148	0.22752	6.04×10^{-3}	0.22145	3.00×10^{-5}	0.22145	3.00×10^{-5}	0.22148	2.8243×10^{-7}
	1.0	0.18819	0.19375	5.56×10^{-3}	0.18813	6.00×10^{-5}	0.18817	2.00×10^{-5}	0.18819	1.7484×10^{-7}
	3.0	0.07511	0.07754	2.43×10^{-3}	0.07509	2.00×10^{-5}	0.07510	1.00×10^{-5}	0.07511	2.5503×10^{-8}
0.50	0.4	0.66071	0.66543	4.72×10^{-3}	0.67152	1.08×10^{-2}	0.66054	1.70×10^{-4}	0.66070	1.4588×10^{-5}
	0.6	0.52942	0.53525	5.83×10^{-3}	0.53406	4.64×10^{-3}	0.52931	1.10×10^{-4}	0.52941	6.4394×10^{-6}
	0.8	0.43914	0.44526	6.12×10^{-3}	0.44143	2.29×10^{-3}	0.43906	8.00×10^{-5}	0.43914	3.2064×10^{-6}
	1.0	0.37442	0.38047	6.05×10^{-3}	0.37568	1.26×10^{-3}	0.37437	5.00×10^{-5}	0.37442	1.7819×10^{-6}
	3.0	0.15018	0.15362	3.44×10^{-3}	0.15020	2.00×10^{-3}	0.15017	1.00×10^{-5}	0.15018	9.2911×10^{-8}
0.75	0.4	0.91026	0.91201	1.75×10^{-3}	0.94675	3.65×10^{-2}	0.90998	2.80×10^{-4}	0.91019	7.2687×10^{-5}
	0.6	0.76724	0.77132	4.08×10^{-3}	0.78474	1.75×10^{-2}	0.76705	1.90×10^{-4}	0.76721	3.0208×10^{-5}
	0.8	0.64740	0.65254	5.14×10^{-3}	0.65659	9.19×10^{-3}	0.64727	1.30×10^{-4}	0.64738	1.3926×10^{-5}
	1.0	0.55605	0.56157	5.52×10^{-3}	0.56135	5.30×10^{-3}	0.55596	9.00×10^{-5}	0.55604	7.2688×10^{-6}
	3.0	0.22481	0.22874	3.92×10^{-3}	0.22502	2.10×10^{-3}	0.22483	2.00×10^{-5}	0.22481	1.8491×10^{-7}

Table 3.1: Comparison results of Example 3.1 for $\Delta t = 0.0001$, $\nu = 0.01$ and $N = 80$ at different time T

Example 3.2. Consider the Burgers' equation (2.4) with boundary condition (3.1) and initial condition

$$u(x,0) = 4x(1-x), \quad x \in [0,1].$$

The analytical solution of this equation is given by (3.2) with the Fourier coefficients

$$a_0 = \int_0^1 \exp\{-x^2(3\nu)^{-1}(3-2x)\}dx,$$
$$a_n = 2\int_0^1 \exp\{-x^2(3\nu)^{-1}(3-2x)\}\cos(n\pi x)dx, \quad n \in \{1,2,3,...\}.$$

In order to compare the approximate solutions $u(x,T)$ attained by Algorithm 2.1 of Example 3.2 at the different times T with the numerical solutions from Asaithambi [16], Kutluay [14], Xu [17] and Ganaie [18] and their analytical solutions by using the parameters $\Delta t = 0.0001$ and $N = 80$ with the viscosity $\nu = 0.01$ are shown in Table 3.2 which measured by the absolute error E_a.

Example 3.3. Consider (2.4) for $t > 0$ with boundary condition (3.1) and initial condition

$$u(x,0) = \frac{2\pi\nu\sin(\pi x)}{\sigma + \cos(\pi x)}, \quad x \in [0,1].$$

x	T	Exact	Asai [16]	Kutluay [14]	Xu [17]	Ganaie [18]	Modified FIM CBS	
							$u(x,T)$	E_a
0.25	0.4	0.36226	0.36232	0.36911	0.3622	0.36225	0.36226	2.9384×10^{-6}
	0.6	0.28204	0.28209	0.28903	0.2820	0.28204	0.28204	1.3140×10^{-6}
	0.8	0.23045	0.23049	0.23703	0.2304	0.23047	0.23045	6.8861×10^{-7}
	1.0	0.19469	0.19472	0.20069	0.1947	0.19472	0.19469	4.1704×10^{-7}
	3.0	0.07613	0.07614	0.07865	0.0761	0.07613	0.07613	5.5231×10^{-8}
0.50	0.4	0.68368	0.68380	0.68818	0.6836	0.68368	0.68366	2.0152×10^{-5}
	0.6	0.54832	0.54840	0.55425	0.5483	0.54836	0.54831	1.0251×10^{-5}
	0.8	0.45371	0.45377	0.46011	0.4537	0.45373	0.45371	5.5454×10^{-6}
	1.0	0.38568	0.38572	0.39206	0.3856	0.38568	0.38567	3.2320×10^{-6}
	3.0	0.15218	0.15219	0.15576	0.1522	0.15218	0.15218	1.8408×10^{-7}
0.75	0.4	0.92050	0.92101	0.92194	0.9205	0.92050	0.92043	7.0462×10^{-5}
	0.6	0.78299	0.78324	0.78676	0.7830	0.78300	0.78296	3.4963×10^{-5}
	0.8	0.66272	0.66285	0.66777	0.6627	0.66269	0.66270	1.8200×10^{-5}
	1.0	0.56932	0.56940	0.57491	0.5693	0.56932	0.56931	1.0371×10^{-5}
	3.0	0.22774	0.22786	0.23183	0.2277	0.22774	0.22774	3.9847×10^{-7}

Table 3.2: Comparison results of Example 3.2 for $\Delta t = 0.0001$, $\nu = 0.01$ and $N = 80$ at different time T

The analytical solution given by Wood [19] for an arbitrary constant σ of this equation is

$$u(x,t) = \frac{2\pi\nu \exp(-\pi^2 \nu t)\sin(\pi x)}{\sigma + \exp(-\pi^2 \nu t)\cos(\pi x)}.$$

The approximate results of Example 3.3 obtained by the presented Algorithm 2.1 for $\sigma = 2$, $N = 40$, $T = 0.001$ and $\Delta t = 0.0001$ with the different viscosity values $\nu = 0.5, 0.2, 0.1$ are compared with those achieved by Mittal [20] and Ganaie [18]. We can see in Table 3.3 that the modified FIM has lower both of L_∞ and L_2 error norms than the other two methods. In Table 3.4, we compare our solutions versus the solutions reported in Mittal [20] and Rahman [21] for $\sigma = 100$, $T = 1$ and $\Delta t = 0.01$ with the viscosity $\nu = 0.005$ at the different nodal numbers $N = 10, 20, 40, 80$ which observe that the L_∞ and L_2 errors of our proposed FIM still provides much less than the errors of both Mittal and Rahman.

ν	Mittal [20]		Ganaie [18]		Modified FIM CBS		
	L_∞	L_2	L_∞	L_2	L_∞	L_2	L_1
0.5	7.44×10^{-5}	2.79×10^{-5}	2.00×10^{-5}	3.54×10^{-6}	1.2721×10^{-5}	2.1025×10^{-6}	2.0417×10^{-6}
0.2	1.22×10^{-5}	4.57×10^{-6}	3.00×10^{-6}	5.24×10^{-7}	8.2543×10^{-7}	3.9663×10^{-7}	2.6183×10^{-7}
0.1	3.08×10^{-6}	1.15×10^{-6}	2.00×10^{-6}	3.54×10^{-7}	1.0395×10^{-7}	4.9837×10^{-8}	3.2863×10^{-8}

Table 3.3: Comparison results of Example 3.3 for $\sigma = 2$, $N = 40$, $T = 0.001$, $\Delta t = 0.0001$ at various viscosity values $\nu = 0.5, 0.2, 0.1$

N	Mittal [20]		Rahman [21]		Modified FIM CBS		
	L_∞	L_2	L_∞	L_2	L_∞	L_2	L_1
10	1.215×10^{-7}	8.631×10^{-8}	1.2458×10^{-7}	8.8189×10^{-8}	3.6359×10^{-9}	2.5761×10^{-9}	2.0901×10^{-9}
20	3.062×10^{-8}	2.153×10^{-8}	3.3944×10^{-8}	2.4029×10^{-8}	3.6387×10^{-9}	2.5760×10^{-9}	2.2032×10^{-9}
40	7.644×10^{-9}	5.378×10^{-9}	1.1249×10^{-8}	7.9424×10^{-9}	3.6485×10^{-9}	2.5760×10^{-9}	2.2604×10^{-9}
80	1.917×10^{-9}	1.345×10^{-9}	5.5490×10^{-9}	3.9178×10^{-9}	3.6485×10^{-9}	2.5760×10^{-9}	2.2892×10^{-9}

Table 3.4: Comparison results of Example 3.3 for $\sigma = 100$, $\nu = 0.005$, $T = 1$, $\Delta t = 0.01$ at different number of nodes $N = 10, 20, 40, 80$

Example 3.4. Consider the Burgers' equation (2.4) for $t \geq 1$ with initial condition

$$u(x,1) = \frac{x}{1+\exp\left(\frac{4x^2-1}{16\nu}\right)}, \quad x \in [0,1].$$

The analytical solution given by Harris [22] with $t_0 = \exp\left(\frac{1}{8\nu}\right)$ of this equation is

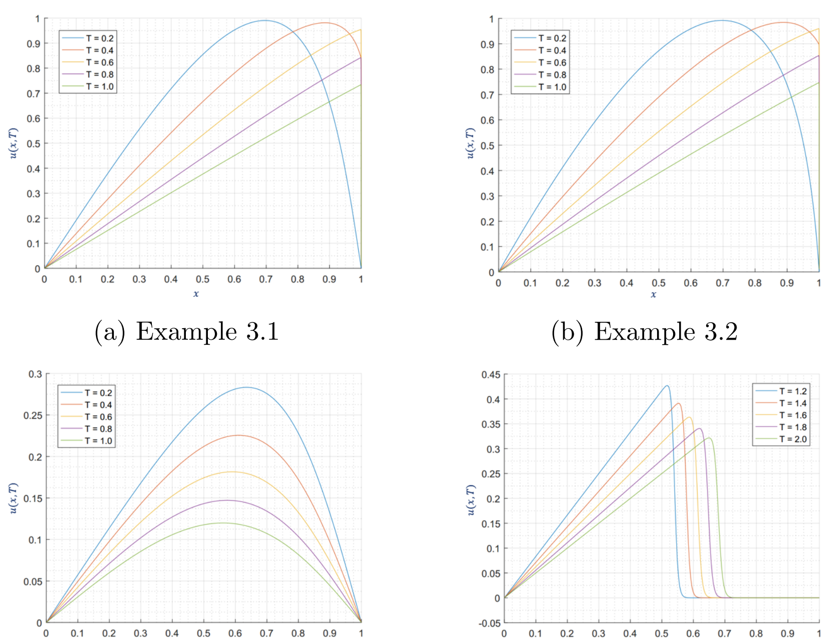
$$u(x,t) = \frac{\frac{x}{t}}{1+\sqrt{\frac{t}{t_0}}\exp\left(\frac{x^2}{4\nu t}\right)}.$$

In our computation of Example 3.4 for the various times $T = 1.7, 2.4, 3.1$ by using Algorithm 2.1, we choose $N = 100$, $\Delta t = 0.001$ with the viscosity $\nu = 0.005$. The L_∞ and L_2 errors are compared with the numerical solutions obtained by procedures of Ashpazzadeh [23] and Dogan [24] as shown in Table 3.5. From this table, it is clearly seen that our method produces much better solutions than [23] and [24].

T	Ashpazzadeh [23]		Dogan [24]		Modified FIM CBS		
	L_∞	L_2	L_∞	L_2	L_∞	L_2	L_1
1.7	2.943×10^{-3}	1.117×10^{-3}	8.099×10^{-3}	2.107×10^{-3}	1.9019×10^{-3}	4.8257×10^{-4}	1.7867×10^{-4}
2.4	2.081×10^{-3}	9.830×10^{-4}	1.165×10^{-2}	3.345×10^{-3}	1.1086×10^{-3}	5.9616×10^{-4}	2.4819×10^{-4}
3.1	4.790×10^{-3}	2.191×10^{-3}	1.587×10^{-2}	4.820×10^{-3}	2.0850×10^{-3}	6.3400×10^{-4}	2.8144×10^{-4}

Table 3.5: Comparison results of Example 3.4 for $N = 100, \Delta t = 0.001, \nu = 0.005$ at $T = 1.7, 2.4, 3.1$

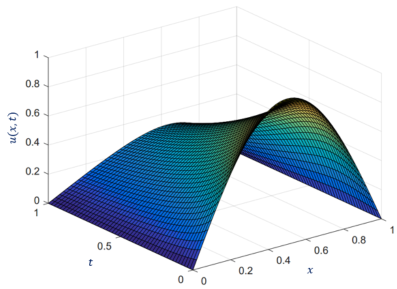
In Figure 1, each subfigure displays the numerical solutions of all Examples 3.1-3.4 at different times T, respectively. The graphical representations in Figure 2 show the correctly physical behavior of numerical solutions through these problems obtained by our modified FIM for small kinematic viscosity values $\nu = 0.0001$ when $\Delta t = 0.01$ and $N = 400$.



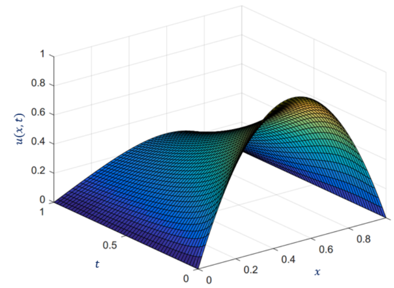
(a) Example 3.1 (b) Example 3.2

(c) Example 3.3 (d) Example 3.4

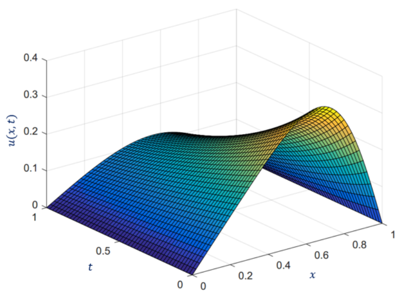
Figure 1: Our numerical solutions at different times of Examples 3.1-3.4 for $\nu = 0.0001$



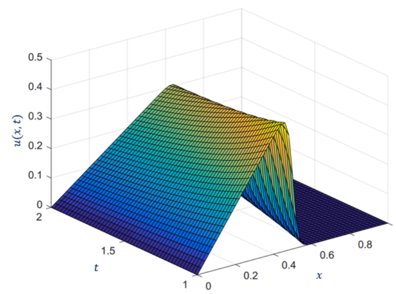
(a) Example 3.1



(b) Example 3.2



(c) Example 3.3



(d) Example 3.4

Figure 2: Physical behavior of our numerical solutions of Examples 3.1-3.4 for $\nu = 0.0001$

4 Conclusion and Discussion

In this paper, we proposed the numerical algorithm based on the modified FIM by using Chebyshev polynomial expansion for solving one-dimensional nonlinear Burgers' equation with shock wave to demonstrate the efficiency and accuracy of the procedure without the adaptively discretizing node nearby the peak. The present Algorithm 2.1 can reduce the oscillations and instabilities for small viscosity ν that can observe from the graphical behavior of several numerical examples in Section 3. Our algorithm significantly improves those traditional FIM in terms of accuracy under the same parameters and conditions that gives higher accuracy than other methods. The current implementation deal with a one-dimensional Burgers' equation, we believe that the proposed Algorithm 2.1 is directly extendable to higher dimensional nonlinear PDEs with the time derivative that is confidently taken in our future works.

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