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Vertex-Magic Labelings for Complete 3-Uniform Hypergraphs with $4n$ Vertices where n is Odd

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Abstract : Let H be a hypergraph with the vertex set V_H and the hyperedge set E_H. For $v \in V_H$, denote $\text{nbhd}(v) = \{e \in E_H \mid v \in e\}$. We generalize the definition of vertex-magic labeling in graph into the definition of vertex-magic labeling in hypergraph as follow. A vertex-magic labeling of H is a bijective mapping $f : V_H \cup E_H \to \{1, 2, 3, \ldots, |V_H| + |E_H|\}$ with a vertex-magic constant Λ such that for every $v \in V_H, f(v) + \sum_{e\in \text{nbhd}(v)} f(e) = \Lambda$. This paper constructs some magic rectangle sets and applies them to determine a vertex-magic labeling for a complete 3-uniform hypergraph with $4n$ vertices where $n \in \{1, 3, 5, \ldots\}$.

Keywords : Hypergraphs; Complete hypergraphs; Vertex-magic labeling; Magic rectangle sets.
2010 Mathematics Subject Classification : 05C78.

1 Introduction and Preliminary Results

First of all, let us introduce the vertex-magic labeling of a graph.

Definition 1.1. [1] *Let G be a simple graph with the vertex set V_G and the edge set E_G. For $v \in V_G$, denote $N(v) = \{u \in V_G \mid uv \in E_G\}$. A vertex-magic labeling of G is a bijective mapping $f : V_G \cup E_G \to \{1, 2, 3, \ldots, |V_G| + |E_G|\}$ with a constant λ such that for every vertex $v \in V_G$, $f(v) + \sum_{u\in N(v)} f(uv) = \lambda$. A graph which admits this labeling is said to be vertex-magic.*

This labeling was first defined in [1] by MacDougall et al. Plenty of graphs were studied whether they are vertex-magic or not. For instance, the cycle C_n where $n > 3$ and the path P_n where $n > 2$ are vertex-magic, see [1].

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A *hypergraph* is the generalization of graphs with the property that each edge (or hyperedge) may consist of any number of vertices. If every hyperedge has the same number of vertices k, then it is called *k-uniform*. The hypergraph with n vertices and has a property that every m vertices lie in exactly one hyperedge, is called a *complete k-uniform hypergraph*, denoted by $K_n^{(m)}$.

In Figure 1, we represent $nK_4^{(3)}$, which consists of n copies of $K_4^{(3)}$, by n top-viewed tetrahedrons. The vertices are in the same hyperedge if they appear on the same face of tetrahedron.

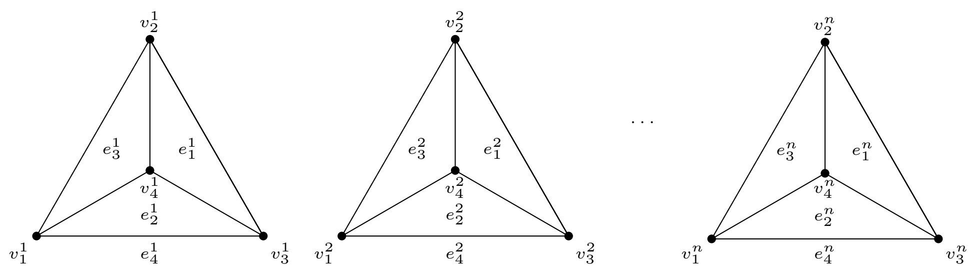

Figure 1: $nK_4^{(3)}$

Note that v_j^i denotes the jth vertex of ith tetrahedron. Furthermore, $v_1^i \in e_2^i \cap e_3^i \cap e_4^i, v_2^i \in e_1^i \cap e_3^i \cap e_4^i, v_3^i \in e_1^i \cap e_2^i \cap e_4^i$ and $v_4^i \in e_1^i \cap e_2^i \cap e_3^i$, for all $i \in \{1, 2, 3, \dots, n\}$.

To generalize the concept of the vertex-magic labelings in graphs, we define the vertex-magic labeling for hypergraphs in the same sense. By maintaining the sums of vertex-label and its incident hyperedge-labels, we have a new version of vertex-magic labeling as defined in Definition 1.2.

Definition 1.2. *Let H be a hypergraph with the vertex set V_H and the hyperedge set E_H. For $v \in V_H$, denote* $\mathrm{nbhd}(v) = \{e \in E_H \mid v \in e\}$. *A vertex-magic labeling of H is a bijective mapping $f : V_H \cup E_H \to \{1, 2, 3, \dots, |V_H|+|E_H|\}$ with a constant Λ such that for every vertex $v \in V_G$, $f(v)+\sum_{e\in \mathrm{nbhd}(v)} f(e) = \Lambda$. A hypergraph which admits this labeling is said to be vertex-magic.*

The purpose of this article is to give a vertex-magic labeling for $K_{4n}^{(3)}$. However, it is worth to show that $nK_4^{(3)}$ is vertex-magic.

Theorem 1.3. *For all $n \in \mathbb{N}$, $nK_4^{(3)}$ is vertex-magic.*

Proof. Let $V_{nK_4^{(3)}} = \{v_j^i | i \in \{1, 2, 3, \dots, n\}$ and $j \in \{1, 2, 3, 4\}\}$ and $E_{nK_4^{(3)}} = \{e_j^i | i \in \{1, 2, 3, \dots, n\}$ and $j \in \{1, 2, 3, 4\}\}$ be the vertex set and the hyperedge set of $nK_4^{(3)}$, respectively. Let $v_1^i \in e_2^i \cap e_3^i \cap e_4^i, v_2^i \in e_1^i \cap e_3^i \cap e_4^i, v_3^i \in e_1^i \cap e_2^i \cap e_4^i$ and $v_4^i \in e_1^i \cap e_2^i \cap e_3^i$, for all $i \in \{1, 2, 3, \dots, n\}$.

Notice that $nK_4^{(3)}$ has $4n$ vertices and $4n$ hyperedges. Define $f : V_{nK_4^{(3)}} \cup E_{nK_4^{(3)}} \to \{1, 2, 3, \dots, 8n\}$ by

$$
\begin{array}{rcll}
e_1^i & = & 2i-1 & \text{for } i \in \{1,2,3,\dots,n\}, \\
e_2^i & = & 2i & \text{for } i \in \{1,2,3,\dots,n\}, \\
e_3^i & = & 4n+1-2i & \text{for } i \in \{1,2,3,\dots,n\}, \\
e_4^i & = & 4n+2-2i & \text{for } i \in \{1,2,3,\dots,n\}, \\
v_1^i & = & 4n-1+2i & \text{for } i \in \{1,2,3,\dots,n\}, \\
v_2^i & = & 4n+2i & \text{for } i \in \{1,2,3,\dots,n\}, \\
v_3^i & = & 8n+1-2i & \text{for } i \in \{1,2,3,\dots,n\}, \\
v_4^i & = & 8n+2-2i & \text{for } i \in \{1,2,3,\dots,n\}.
\end{array}
$$

It is easy to see that f is bijective. To check vertex-magic property of f, let us consider for all $i \in \{1, 2, 3, \dots, n\}$,

- at v_1^i;

$$\begin{aligned} f(v_1^i) + \textstyle\sum_{e \in \mathrm{nbhd}(v_1^i)} f(e) &= f(v_1^i) + f(e_2^i) + f(e_3^i) + f(e_4^i) \\ &= (4n-1+2i) + 2i + (4n+1-2i) + (4n+2-2i) \\ &= 12n+2, \end{aligned}$$

- at v_2^i;

$$\begin{aligned} f(v_2^i) + \textstyle\sum_{e \in \mathrm{nbhd}(v_2^i)} f(e) &= f(v_2^i) + f(e_1^i) + f(e_3^i) + f(e_4^i) \\ &= (4n+2i) + (2i-1) + (4n+1-2i) + (4n+2-2i) \\ &= 12n+2, \end{aligned}$$

- at v_3^i;

$$\begin{aligned} f(v_3^i) + \textstyle\sum_{e \in \mathrm{nbhd}(v_3^i)} f(e) &= f(v_3^i) + f(e_1^i) + f(e_2^i) + f(e_4^i) \\ &= (8n+1-2i) + (2i-1) + 2i + (4n+2-2i) \\ &= 12n+2, \qquad \text{and} \end{aligned}$$

- at v_3^i;

$$\begin{aligned} f(v_4^i) + \textstyle\sum_{e \in \mathrm{nbhd}(v_4^i)} f(e) &= f(v_4^i) + f(e_1^i) + f(e_2^i) + f(e_3^i) \\ &= (8n+2-2i) + (2i-1) + 2i + (4n+1-2i) \\ &= 12n+2. \end{aligned}$$

These conclude that f is a vertex-magic labeling for $nK_4^{(3)}$ with $\Lambda = 12n+2$. □

In 2009, Krishnappa et al. [2] used the existence of magic squares to conclude that a complete graph K_n is vertex-magic, except K_2. We assure the readers a notion of magic-square, the $n \times n$ array whose elements are $1, 2, 3, \ldots, n^2$ such that all column-sums, row-sums and both diagonal-sums are the same integer, says *magic constant*. In fact, the magic property of both diagonal-sums of a magic square is inessential for constructing a vertex-magic labeling for K_n. Thus, Froncek [3] introduced a new collection of arrays called a *magic rectangle set*.

Definition 1.4. [3] *A magic rectangle set $\mathcal{M} = \mathrm{MRS}(a, b; c)$ is a collection of c arrays $(a \times b)$ whose entries are elements of $\{1, 2, 3, \ldots, abc\}$, each appearing once, with all row-sums in every rectangle equals to the same constant ρ and all column-sums in every rectangle equals to the same constant σ.*

Observe that, adding the number k to every entry of $\mathcal{M}$ gives the new arrays which maintain the row-sums and column-sums properties. We summarize this observation in Lemma 1.5.

Lemma 1.5. *Let $\mathcal{M} = \mathrm{MRS}(a, b; c)$ be a magic rectangle set with ρ and σ. For the number k, denote $k + \mathcal{M}$ be the collection of c arrays whose constructed by adding k to each entry of $\mathcal{M}$. Then, $k + \mathcal{M}$ has these properties;*

1. *entries of $k + \mathcal{M}$ are elements of $\{k+1, k+2, k+3, \ldots, k+abc\}$, and*
2. *row-sums and column-sums become $\rho + kb$ and $\sigma + ka$, respectively.*

In [3], Froncek gave an algorithm for constructing $\mathrm{MRS}(a, b; c)$ where $a \equiv b \equiv 0 \pmod 2$ and $b \geq 4$. Although his algorithm is not suitable for our purpose, we prove our own results on the existence of $\mathrm{MRS}(6n-6, 4; n)$ for $n \in \{3, 5, 7, \ldots\}$ and $\mathrm{MRS}(4, 4; \binom{n}{3})$ for $n \in \{3, 4, 5, \ldots\}$ in Lemmas 1.6 and 1.7, respectively.

Lemma 1.6. *If $n \in \{3, 5, 7, \ldots\}$, then $\mathrm{MRS}(6n-6, 4; n)$ exists.*

Proof. For convenience, let $\alpha = 24n^2 - 24n$ and x^s_{ij} be the entry in ith row and jth column of sth array. Define x^1_{ij} by

$$
\begin{array}{lcll}
x^1_{i1} & = & i & \text{for } i \in \{1, 3, 5, \ldots, 3n-4\}, \\
x^1_{i1} & = & \alpha + 2 - i & \text{for } i \in \{2, 4, 6, \ldots, 3n-3\}, \\
x^1_{i1} & = & \alpha - i & \text{for } i \in \{3n-2, 3n, 3n+2, \ldots, 6n-5\}, \\
x^1_{i1} & = & i & \text{for } i \in \{3n-1, 3n+1, 3n+3, \ldots, 6n-6\}, \\
x^1_{i2} & = & \alpha - i & \text{for } i \in \{1, 3, 5, \ldots, 3n-4\}, \\
x^1_{i2} & = & i & \text{for } i \in \{2, 4, 6, \ldots, 3n-3\}, \\
x^1_{i2} & = & i & \text{for } i \in \{3n-2, 3n, 3n+2, \ldots, 6n-5\}, \\
x^1_{i2} & = & \alpha + 2 - i & \text{for } i \in \{3n-1, 3n+1, 3n+3, \ldots, 6n-6\}, \\
x^1_{i3} & = & 6n - 5 + i & \text{for } i \in \{1, 3, 5, \ldots, 3n-4\}, \\
x^1_{i3} & = & \alpha - 6n + 7 - i & \text{for } i \in \{2, 4, 6, \ldots, 3n-3\}, \\
x^1_{i3} & = & \alpha - 6n + 7 - i & \text{for } i \in \{3n-2, 3n, 3n+2, \ldots, 6n-5\}, \\
x^1_{i3} & = & 6n - 7 + i & \text{for } i \in \{3n-1, 3n+1, 3n+3, \ldots, 6n-6\}, \\
x^1_{i4} & = & \alpha - 6n + 7 - i & \text{for } i \in \{1, 3, 5, \ldots, 3n-4\}, \\
x^1_{i4} & = & 6n - 7 + i & \text{for } i \in \{2, 4, 6, \ldots, 3n-3\}, \\
x^1_{i4} & = & 6n - 5 + i & \text{for } i \in \{3n-2, 3n, 3n+2, \ldots, 6n-5\}, \\
x^1_{i4} & = & \alpha - 6n + 7 - i & \text{for } i \in \{3n-1, 3n+1, 3n+3, \ldots, 6n-6\}.
\end{array}
$$

For $s > 1$, the remaining entry x^s_{ij} are defined recursively by

$$
\begin{array}{lcll}
x^s_{ij} & = & x^{s-1}_{ij} + 12n - 12 & \text{if } x^{s-1}_{ij} < 12n^2 - 12n \text{ for } s \in \{2, 3, 4, \ldots, n\}, \\
x^s_{ij} & = & x^{s-1}_{ij} - 12n + 12 & \text{if } x^{s-1}_{ij} > 12n^2 - 12n \text{ for } s \in \{2, 3, 4, \ldots, n\}.
\end{array}
$$

Notice that each of these numbers $1, 2, 3, \ldots, 24n^2 - 24n$ appears in a unique array once. It follows by direct calculation that $\rho = 2\alpha + 2$ and $\sigma = (3n-3)(\alpha + 1)$. Thus, $\mathrm{MRS}(6n-6, 4; n)$ exists. □

Lemma 1.7. *If $n \in \{3, 4, 5, \ldots\}$, then $\mathrm{MRS}(4, 4; 4\binom{n}{3})$ exists.*

Proof. For convenience, let $\alpha = 64\binom{n}{3}$ and x^s_{ij} be the entry in ith row and jth column of sth array. Define $[x^1_{ij}]$ by

$$
\begin{bmatrix}
1 & \alpha - 1 & 8 & \alpha - 6 \\
\alpha & 2 & \alpha - 7 & 7 \\
3 & \alpha - 3 & 6 & \alpha - 4 \\
\alpha - 2 & 4 & \alpha - 5 & 5
\end{bmatrix}.
$$

For $s > 1$, the remaining entry x^s_{ij} are defined recursively by

$$
\begin{array}{lcll}
x^s_{ij} & = & x^{s-1}_{ij} + 8 & \text{if } x^{s-1}_{ij} < 32\binom{n}{3} \text{ for } s \in \{2, 3, 4, \ldots, \binom{n}{3}\}, \\
x^s_{ij} & = & x^{s-1}_{ij} - 8 & \text{if } x^{s-1}_{ij} > 32\binom{n}{3} \text{ for } s \in \{2, 3, 4, \ldots, \binom{n}{3}\}.
\end{array}
$$

Notice that each of these numbers $1, 2, 3, \ldots, 16\binom{n}{3}$ appears in a unique array once. It follows by direct calculation that $\rho = \sigma = 2\alpha + 2$. Thus, $\mathrm{MRS}(4, 4; 4\binom{n}{3})$ exists. □

2 Vertex-Magic Labeling for $K^{(3)}_{4n}$

By the definition of $K^{(3)}_{4n}$, we can construct $K^{(3)}_{4n}$ from $nK^{(3)}_4$ with some additional hyperedges. Thus, according to Section 1, we have shown that $nK^{(3)}_4$ is vertex-magic. To construct a vertex-magic labeling for $K^{(3)}_{4n}$, we use a vertex-magic labeling of $nK^{(3)}_4$ and then, by the aids of $\mathrm{MRS}(6n-6, 4; n)$ and $\mathrm{MRS}(4, 4; 4\binom{n}{3})$, we can give labels to the additional hyperedges in such the way that those labels preserve the vertex-magic property. Since $nK_4^{(3)}$ has n components from n copies of $K_4^{(3)}$ and each of them has 4 vertices, the additional hyperedges of $K_{4n}^{(3)}$ are of the followings 2 cases;

1. hyperedges of type-1 are of form $\{u, v, w\}$ where u and v come from the same component of $nK_4^{(3)}$ while w comes from the others components,
2. hyperedges of type-2 are of form $\{u, v, w\}$ where u, v and w come from different components of $nK_4^{(3)}$.

Fortunately, the number of hyperedges of type-1 and type-2 are $24n^2 - 24n$ and $64\binom{n}{3}$ which are equal to the number of entries in MRS$(6n-6, 4; n)$ and MRS$(4, 4; 4\binom{n}{3})$, respectively. In Theorem 2.1, we prove that if n is odd, then $K_{4n}^{(3)}$ admitting a vertex-magic labeling by applying MRS$(6n-6, 4; n)$ and MRS$(4, 4; 4\binom{n}{3})$ to those additional hyperedge-labels.

For simplification purpose, let us define these notations.

1. Let γ_1 be a function such that $\gamma_1(1) = 1, \gamma_1(2) = 1, \gamma_1(3) = 1, \gamma_1(4) = 2, \gamma_1(5) = 2, \gamma_1(6) = 3$.
2. Let γ_2 be a function such that $\gamma_2(1) = 2, \gamma_2(2) = 3, \gamma_2(3) = 4, \gamma_2(4) = 3, \gamma_2(5) = 4, \gamma_2(6) = 4$.
3. Let δ be the dictionary order of $X = \{(x, y, z) \mid x, y, z \in \{1, 2, 3, \ldots, n\}$ and $x < y < z\}$, i.e., $\delta(1, 2, 3) = 1, \delta(1, 2, 4) = 2, \ldots, \delta(1, 2, n) = n-2, \delta(1, 3, 4) = n-1, \ldots, \delta(n-2, n-1, n) = \binom{n}{3}$. Note that $\delta : X \to \{1, 2, 3, \ldots, \binom{n}{3}\}$ is bijective and δ^{-1} exists.
4. Let $\pi_1, \pi_2, \pi_3 : \mathbb{R}^3 \to \mathbb{R}$ be the function defined by $\pi_1(x, y, z) = x, \pi_2(x, y, z) = y$ and $\pi_3(x, y, z) = z$.

Theorem 2.1. *If $n \in \{1, 3, 5, \ldots\}$, then $K_{4n}^{(3)}$ is vertex-magic.*

Proof. It is clear by Theorem 1.3 that $K_4^{(3)}$ is vertex-magic. Suppose that $n \in \{3, 5, 7, \ldots\}$. Then, $\mathcal{M}_1 = $ MRS$(6n-6, 4; n)$ and $\mathcal{M}_2 = $ MRS$(4, 4; 4\binom{n}{3})$ exist by Lemmas 1.6 and 1.7. Since $K_{4n}^{(3)}$ is a combination of $nK_4^{(3)}$ and hyperedges of type-1 and type-2 , we separate the task into 3 steps.

1. Label $1, 2, 3, \ldots, 8n$ to the vertices and hyperedges of $nK_4^{(3)}$ by using Theorem 1.3. Note that each vertex-label and its incident hyperedge-labels add up to $12n + 2$.

2. Consider the type-1 hyperedges. Let y_{ij}^s be the entry in ith row, jth column and sth array of $8n + \mathcal{M}_1$. For all $i \in \{1, 2, 3, \ldots, 6n-6\}$, write $i = 6p + q$ where $q \in \{1, 2, 3, 4, 5, 6\}$.

(a) If $p < s - 1$, then let y_{ij}^s be the label of $\{v_j^s, v_{\gamma_1(q)}^{p+1}, v_{\gamma_2(q)}^{p+1}\}$.

(b) If $p \geq s - 1$, then let y_{ij}^s be the label of $\{v_j^s, v_{\gamma_1(q)}^{p+2}, v_{\gamma_2(q)}^{p+2}\}$.

We can represent the sth arrays of $8n + \mathcal{M}_1$ as shown in Figure 2.

Consequently, the hyperedges which are incident to vertex v_j^i receives the labels from exactly 1 column (from the ith array) and $3n-3$ rows (3 rows from the other $n-1$ arrays) of $8n+\mathcal{M}_1$. By the property of $8n+\mathcal{M}_1$, hyperedge-labels incident to each vertex add up to the same constant $(\sigma+ka)+(3n-3)(\rho+kb) = [(3n-3)(24n^2-24n+1)+8n(6n-6)]+(3n-3)[(48n^2-48n+2)+(8n)(4)] = 216n^3-288n^2+81n-9$. Note that these labels used are $8n+1, 8n+2, 8n+3, \ldots, 24n^2-16n$.

3. Consider the type-2 hyperedges. Let z_{ij}^s be the entry in ith row, jth column and sth array of $(24n^2-16n)+\mathcal{M}_2$. For all $s \in \{1, 2, 3, \ldots, \binom{n}{3}\}$, write $s = 4p+q$ where $q \in \{1, 2, 3, 4\}$. Then, let z_{ij}^s be the label of $\{v_q^{\pi_1(\gamma^{-1}(p+1))}, v_i^{\pi_2(\gamma^{-1}(p+1))}, v_j^{\pi_3(\gamma^{-1}(p+1))}\}$. We can represent the sth arrays of $(24n^2-16n)+\mathcal{M}_2$ as shown in Figure 3.

Consequently, the hyperedges which are incident to vertex v_j^i receive labels from

(a) all entries of $\binom{n-i}{2}$ arrays,

(b) all entries in $4(i-1)(n-i)$ columns, and

(c) all entries in $4\binom{i-1}{2}$ rows.

	v_1^s	v_2^s	v_3^s	v_4^s
$v_1^1v_2^1$				
$v_1^1v_3^1$				
$v_1^1v_4^1$				
$v_2^1v_3^1$				
$v_2^1v_4^1$				
$v_3^1v_4^1$				
$v_1^2v_2^2$				
$v_1^2v_3^2$				
$v_1^2v_4^2$				
$v_2^2v_3^2$				
$v_2^2v_4^2$				
$v_3^2v_4^2$				
$\vdots$				
$v_1^{s-1}v_2^{s-1}$				
$v_1^{s-1}v_3^{s-1}$				
$v_1^{s-1}v_4^{s-1}$				
$v_2^{s-1}v_3^{s-1}$	sth array of $8n + \mathcal{M}_1$			
$v_2^{s-1}v_4^{s-1}$				
$v_3^{s-1}v_4^{s-1}$				
$v_1^{s+1}v_2^{s+1}$				
$v_1^{s+1}v_3^{s+1}$				
$v_1^{s+1}v_4^{s+1}$				
$v_2^{s+1}v_3^{s+1}$				
$v_2^{s+1}v_4^{s+1}$				
$v_3^{s+1}v_4^{s+1}$				
$\vdots$				
$v_1^nv_2^n$				
$v_1^nv_3^n$				
$v_1^nv_4^n$				
$v_2^nv_3^n$				
$v_2^nv_4^n$				
$v_3^nv_4^n$				

Figure 2: the sth array of $8n + \mathcal{M}_1$

$v_q^{\pi_1(\gamma^{-1}(p+1))}$	$v_1^{\pi_2(\gamma^{-1}(p+1))}$	$v_2^{\pi_2(\gamma^{-1}(p+1))}$	$v_3^{\pi_2(\gamma^{-1}(p+1))}$	$v_4^{\pi_2(\gamma^{-1}(p+1))}$
$v_1^{\pi_3(\gamma^{-1}(p+1))}$				
$v_2^{\pi_3(\gamma^{-1}(p+1))}$	sth array of $(24n^2 - 16n) + \mathcal{M}_2$			
$v_3^{\pi_3(\gamma^{-1}(p+1))}$				
$v_4^{\pi_3(\gamma^{-1}(p+1))}$				

Figure 3: the sth array of $(24n^2 - 16n) + \mathcal{M}_2$

Since the row-sums and column-sums of $(24n^2 - 16n) + \mathcal{M}_2$ are equal, hyperedge-labels at each vertex add up to the same constant $(\sigma + ka)(4\binom{n-i}{2} + 4(i-1)(n-i) + 4\binom{i-1}{2}) = [(128\binom{n}{3} + 2) + (24n^2 - 16n)(4)](2n^2 - 6n + 4) = \frac{128}{3}n^5 - 64n^4 - \frac{448}{3}n^3 + 260n^2 - \frac{292}{3}n + 8$. Note that these labels used are $24n^2 - 16n + 1, 24n^2 - 16n + 2, 24n^2 - 16n + 3, \ldots, 24n^2 - 16n + 64\binom{n}{3} = 4n + \binom{4n}{3}$.

These concludes that $K_{4n}^{(3)}$ is vertex-magic when n is odd. Moreover, in case of $n \in \{3, 5, 7, \ldots\}$, the vertex-magic constant $\Lambda = (12n+2)+(216n^3-288n^2+81n-9)+(\frac{128}{3}n^5-64n^4-\frac{448}{3}n^3+260n^2-\frac{292}{3}n+8) = \frac{128}{3}n^5 - 64n^4 + \frac{200}{3}n^3 - 28n^2 - \frac{13}{3}n + 1$.

$\square$

Example 2.1. *To show that $K_{12}^{(3)}$ whose having 12 vertices and $\binom{12}{3} = 220$ hyperedges is vertex-magic, we first give a vertex-magic labeling for $3K_4^{(3)}$ as shown in Figure 4.*

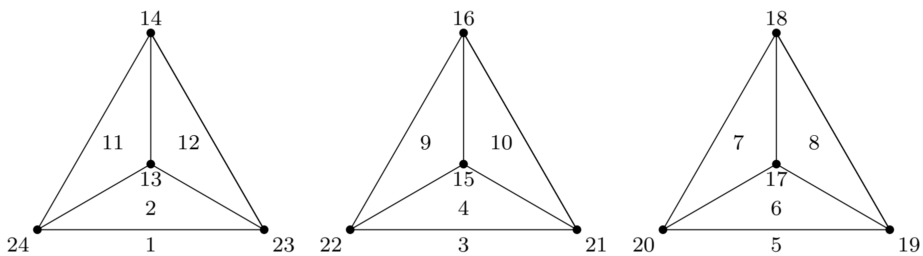

Figure 4: a vertex-magic labeling of $3K_4^{(3)}$

Then, we construct $24 + \mathrm{MRS}(12, 4; 3)$ and $168 + \mathrm{MRS}(4, 4; 4)$ as follow.

	v_1^1	v_2^1	v_3^1	v_4^1
$v_1^2v_2^2$	25	167	38	156
$v_1^2v_3^2$	168	26	155	37
$v_1^2v_4^2$	27	165	40	154
$v_2^2v_3^2$	166	28	153	39
$v_2^2v_4^2$	29	163	42	152
$v_3^2v_4^2$	164	30	151	41
$v_1^3v_2^3$	161	31	150	44
$v_1^3v_3^3$	32	162	43	149
$v_1^3v_4^3$	159	33	148	46
$v_2^3v_3^3$	34	160	45	147
$v_2^3v_4^3$	157	35	146	48
$v_3^3v_4^3$	36	158	47	145

	v_1^2	v_2^2	v_3^2	v_4^2
$v_1^1v_2^1$	49	143	62	132
$v_1^1v_3^1$	144	50	131	61
$v_1^1v_4^1$	51	141	64	130
$v_2^1v_3^1$	142	52	129	63
$v_2^1v_4^1$	53	139	66	128
$v_3^1v_4^1$	140	54	127	65
$v_1^3v_2^3$	137	55	126	68
$v_1^3v_3^3$	56	138	67	125
$v_1^3v_4^3$	135	57	124	70
$v_2^3v_3^3$	58	136	69	123
$v_2^3v_4^3$	133	59	122	72
$v_3^3v_4^3$	60	134	71	121

	v_1^3	v_2^3	v_3^3	v_4^3
$v_1^1v_2^1$	73	119	86	108
$v_1^1v_3^1$	120	74	107	85
$v_1^1v_4^1$	75	117	88	106
$v_2^1v_3^1$	118	76	105	87
$v_2^1v_4^1$	77	115	90	104
$v_3^1v_4^1$	116	78	103	89
$v_1^2v_2^2$	113	79	102	92
$v_1^2v_3^2$	80	114	91	101
$v_1^2v_4^2$	111	81	100	94
$v_2^2v_3^2$	82	112	93	99
$v_2^2v_4^2$	109	83	98	96
$v_3^2v_4^2$	84	110	95	97

v_1^1	v_1^2	v_2^2	v_3^2	v_4^2
v_1^3	169	231	176	226
v_2^3	232	170	225	175
v_3^3	171	229	174	228
v_4^3	230	172	227	173

v_2^1	v_1^2	v_2^2	v_3^2	v_4^2
v_1^3	177	223	184	218
v_2^3	224	178	217	183
v_3^3	179	221	182	220
v_4^3	222	180	219	181

v_3^1	v_1^2	v_2^2	v_3^2	v_4^2
v_1^3	185	215	194	210
v_2^3	216	186	209	191
v_3^3	187	213	190	212
v_4^3	214	188	221	189

v_4^1	v_1^2	v_2^2	v_3^2	v_4^2
v_1^3	193	207	200	202
v_2^3	208	194	201	199
v_3^3	195	205	198	204
v_4^3	206	196	203	197

These arrays inform the labels of type-1 and type-2 hyperedges, for example $\{v_1^1, v_2^1, v_3^2\}$ and $\{v_3^1, v_2^2, v_4^3\}$ receive labels 62 and 188, respectively. Moreover, $\Lambda = 6720$.

3 Conclusion and Discussion

The existence of $\mathrm{MRS}(6n-6, 4; n)$ and $\mathrm{MRS}(4, 4; 4\binom{n}{3})$ implies the existence of a vertex-magic labeling of $K_{4n}^{(3)}$. However, we still cannot construct $\mathrm{MRS}(6n-6, 4; n)$ for n is even. Thus, our future work is to complete the vertex-magic labeling for $K_{4n}^{(3)}$ and possibly other $K_m^{(3)}$ for $4 \nmid m$.

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