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Super Edge-Magic Labeling of 5-Uniform and 6-Uniform Hypergraphs Generated by Arbitrary Simple Graphs

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Abstract : The super edge-magic (SEM) labeling on hypergraphs is the extension of the SEM labeling on graphs. For a hypergraph H with vertex set V_H and hyperedge set E_H, we call a bijective mapping $f : V_H \cup E_H \to \{1, 2, 3, \dots, |V_H| + |E_H|\}$ as an SEM labeling of H if and only if (i) there is an integer Λ such that for every $e \in E_H$, $f(e) + \sum_{v\in e} f(v) = \Lambda$ and (ii) $f(V_H) = \{1, 2, 3, \dots, |V_H|\}$. In this article, we define 5-uniform $H^{(5)}(G)$ and 6-uniform $H^{(6)}(G)$ hypergraphs from an arbitrary simple graph G and show that $H^{(5)}(G)$ is always an SEM hypergraph. However, if G has odd number of edges, then $H^{(6)}(G)$ is an SEM hypergraph. Unfortunately, if G has even number of edges, the SEM labeling for $H^{(6)}(G)$ depends on the structure of the hypergraph. Thus, an example of SEM labeling of $H^{(6)}(nC_4)$, which has even number of edges, is given. Finally, if H is a k-uniform SEM hypergraph, then we can show that H', obtained from H by adding more vertices, is $(k + 2m)$-uniform SEM hypergraph.

Keywords : Super edge-magic labeling; Hypergraph labeling.
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1 Introduction

In this paper, we consider simple graphs G having no isolated vertices. A *hypergraph* H is the pair (V_H, E_H) where V_H is a finite set and E_H is a subset of the power set of V_H. The sets V_H and E_H are called *vertex set* and *hyperedge set* of H, respectively, see [1]. Moreover, if every element of E_H has the same cardinality k, then H is said to be *k-uniform* and denoted by $H^{(k)}$. In this paper, we construct 5-uniform hypergraphs and 6-uniform hypergraphs from simple graphs as follow.

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Definition 1.1. *Let G be a simple graph with the vertex set $\{v_1, v_2, v_3, \ldots, v_p\}$ and the edge set $\{e_1, e_2, e_3, \ldots, e_q\}$. Construct an additional vertex set $\{v'_1, v'_2, v'_3, \ldots, v'_p\}$. For each $e_k = v_iv_j$ of G, define $E_k = \{e_k, v_i, v'_i, v_j, v'_j\}$. Then, a hypergraph whose vertex set and hyperedge set are $\bigcup_{k=1}^{q} E_k$ and $\bigcup_{k=1}^{q}\{E_k\}$, respectively, is called the 5-uniform hypergraph generated by G and denoted by $H^{(5)}(G)$.*

Remark 1.2. *From Definition 1.1, $H^{(5)}(G)$ has $2p+q$ vertices and q hyperedges. Moreover, $V_{H^{(5)}(G)} = \{v_1, v_2, v_3, \ldots, v_p\} \cup \{v'_1, v'_2, v'_3, \ldots, v'_p\} \cup \{e_1, e_2, e_3, \ldots, e_q\}$ and $E_{H^{(5)}(G)} = \{E_1, E_2, E_3, \ldots, E_q\}$.*

Example 1.3. *Let $K_{1,4}$ be the complete bipartite graph (as known as the star graph S_4). Then, $H^{(5)}(K_{1,4})$ can be represented as in Figure 1.*

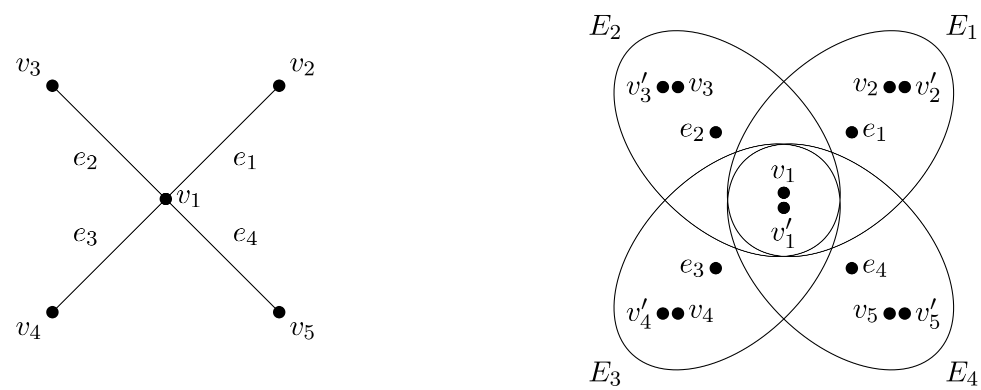

Figure 1: $K_{1,4}$ and $H^{(5)}(K_{1,4})$

Definition 1.4. *Let G be a simple graph with the vertex set $\{v_1, v_2, v_3, \ldots, v_p\}$ and the edge set $\{e_1, e_2, e_3, \ldots, e_q\}$. Construct additional vertex sets $\{v'_1, v'_2, v'_3, \ldots, v'_p\}$ and $\{e'_1, e'_2, e'_3, \ldots, e'_q\}$. For each edge $e_k = v_iv_j$ of G, define $E_k = \{e_k, e'_k, v_i, v'_i, v_j, v'_j\}$. Then, a hypergraph whose vertex set and hyperedge set are $\bigcup_{k=1}^{q} E_k$ and $\bigcup_{k=1}^{q}\{E_k\}$, respectively, is called the 6-uniform hypergraph generated by G and denoted by $H^{(6)}(G)$.*

Remark 1.5. *From Definition 1.4, $H^{(5)}(G)$ has $2p+q$ vertices and q hyperedges.*

Moreover, $V_{H^{(6)}(G)} = \{v_1, v_2, v_3, \ldots, v_p\} \cup \{v'_1, v'_2, v'_3, \ldots, v'_p\} \cup \{e_1, e_2, e_3, \ldots, e_q\} \cup \{e'_1, e'_2, e'_3, \ldots, e'_q\}$ and $E_{H^{(6)}(G)} = \{E_1, E_2, E_3, \ldots, E_q\}$.

Example 1.6. *Let P_3 be the path graph of size 3. Then, $H^{(6)}(P_3)$ can be represented as in Figure 2.*

Thus, every simple graph G has the 5-uniform and 6-uniform hypergraphs generated by it.

The concept of super edge-magic (SEM) labelings in a graph was first introduced in 1998 by Enomoto et al. [2]. Later, Boonklurb et al. [3] generalized this concept to SEM labeling in hypergraph as stated in Definition 1.7.

Definition 1.7. [3] *For a hypergraph H, the SEM labeling of H is a bijection $f : V_H \cup E_H \to \{1, 2, 3, \ldots, |V_H| + |E_H|\}$ satisfying*

1. *there exists a constant Λ such that for all $e \in E_H, f(e) + \sum_{v\in e} f(v) = \Lambda$ and*
2. *$f(V_H) = \{1, 2, 3, \ldots, |V_H|\}$.*

A hypergraph admitting the SEM labeling is called SEM hypergraph. Note that this notation agrees in the case that H is a simple graph.

Example 1.8. *Consider $H = H^{(5)}(C_5 \cup C_6)$ whose $|V_H| = 33$ and $|E_H| = 11$ as shown in the Figure 3. In each hyperedge, the sum of 5 vertex-labels and their incident hyperedge-labels is equal to 113. Since all vertex-labels are $1, 2, 3, \ldots, 33$, we have that $H = H^{(5)}(C_5 \cup C_6)$ is SEM.*

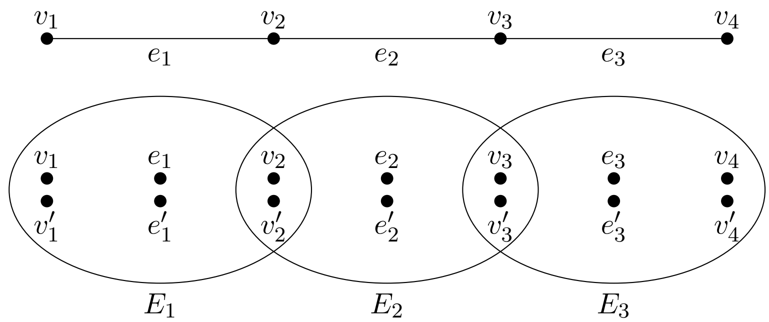

Figure 2: P_3 and $H^{(6)}(P_3)$

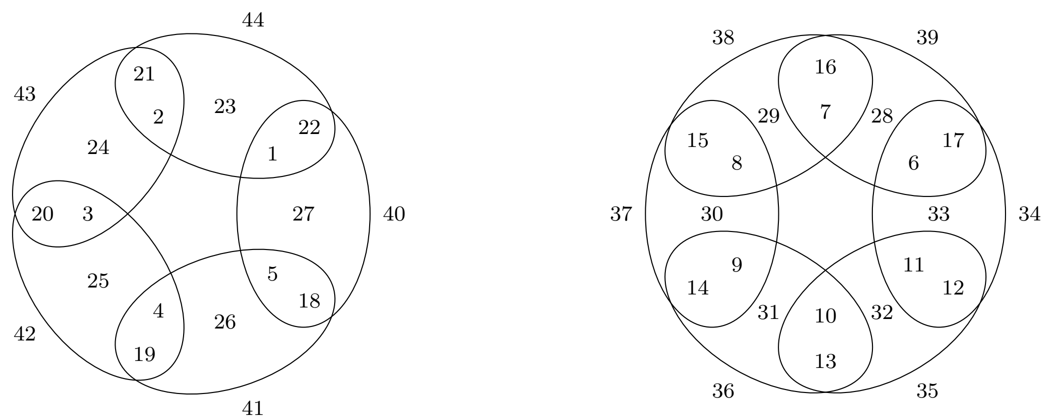

Figure 3: An SEM labeling of $H^{(5)}(C_5 \cup C_6)$ with $\Lambda = 113$

2 The SEM labeling of $H^{(5)}(G)$

Assume that a simple graph G has p vertices and q edges. By Definition 1.1, the hypergraph $H^{(5)}(G)$ has $2p+q$ vertices and q hyperedges. Since each hyperedge of $H^{(5)}(G)$ is of form $E_k = \{e_k, v_i, v_j, v_i', v_j'\}$ where $e_k = v_iv_j$, we can give an SEM labeling for $H^{(5)}(G)$ as shown in the following theorem.

Theorem 2.1. *Let G be a simple graph with the vertex set $\{v_1, v_2, v_3, \ldots, v_p\}$ and the edge set $\{e_1, e_2, e_3, \ldots, e_q\}$. There exists an SEM labeling for $H^{(5)}(G)$.*

Proof. We define a function $f : V_{H^{(5)}(G)} \cup E_{H^{(5)}(G)} \to \{1, 2, 3, \ldots, 2p+q, 2p+q+1, \ldots, 2p+2q\}$ by

$$\begin{aligned} f(v_i) &= i && \text{for } i \in \{1,2,3,\ldots,p\}, \\ f(v_i') &= 2p+1-i && \text{for } i \in \{1,2,3,\ldots,p\}, \\ f(e_i) &= 2p+i && \text{for } i \in \{1,2,3,\ldots,q\}, \\ f(E_i) &= 2p+2q+1-i && \text{for } i \in \{1,2,3,\ldots,q\}. \end{aligned}$$

It is straight forward to prove that f is a bijection. Consider for each $k \in \{1, 2, 3, \ldots\}$, we have that

$$\begin{aligned} f(E_k) + \sum_{v \in E_k} f(v) &= f(E_k) + f(e_k) + f(v_i) + f(v_i') + f(v_j) + f(v_j') \\ &= (2p + 2q + 1 - k) + (2p + k) + i + (2p + 1 - i) \\ &\quad + j + (2p + 1 - j) \\ &= 8p + 2q + 3, \end{aligned}$$

where $e_k = v_i v_j$. Since $f(V_{H^{(5)}(G)}) = \{f(v_i) | i \in \{1, 2, 3, \ldots, p\}\} \cup \{f(v_i') | i \in \{1, 2, 3, \ldots, p\}\} \cup \{f(e_i) | i \in \{1, 2, 3, \ldots, q\}\} = \{1, 2, 3, \ldots, p\} \cup \{p + 1, p + 2, p + 3, \ldots, 2p\} \cup \{2p + 1, 2p + 2, 2p + 3, \ldots, 2p + q\} = \{1, 2, 3, \ldots, 2p + q\}$, f is an SEM labeling. Thus, $H^{(5)}(G)$ is SEM. □

Example 2.2. *Let C_5 be a cycle with $p = 5$ and $q = 5$. We represent $H^{(5)}(C_5)$ in the middle and use Theorem 2.1 to give an SEM labeling for it as shown in Figure 4. Furthermore, $\Lambda = 8p + 2q + 3 = 53$.*

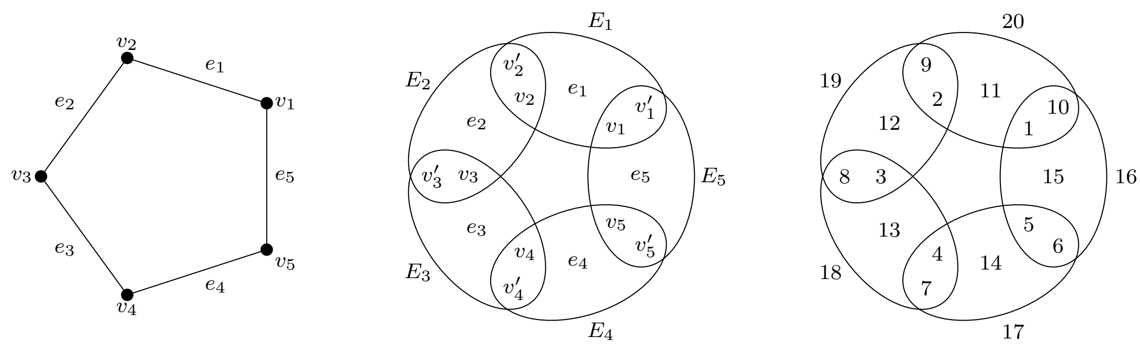

Figure 4: $C_5, H^{(5)}(C_5)$ and an SEM labeling for $H^{(5)}(C_5)$ with $\Lambda = 53$

Especially, if a graph G admits an SEM labeling, namely f_G, then we can extend f_G to an SEM labeling f for $H^{(5)}(G)$ as shown in Theorem 2.3.

Theorem 2.3. *Let G be a simple graph with the vertex set $V_G = \{v_1, v_2, v_3, \ldots, v_p\}$ and the edge set $E_G = \{e_1, e_2, e_3, \ldots, e_q\}$. If a graph G admits an SEM labeling f_G, then the SEM labeling f for $H^{(5)}(G)$ exists. Moreover, $f|_{V_G \cup E_G} = f_G$.*

Proof. Assume that G is an SEM graph with the SEM labeling f_G. Note that $f_G(V_G) = \{1, 2, 3, \ldots, p\}$, $f_G((E_G) = \{p + 1, p + 2, p + 3, \ldots, p + q\}$ and there is constant λ such that for every $e_k = v_i v_j \in E_G$, $f_G(v_i) + f_G(v_j) + f_G(e_k) = \lambda$. To construct SEM labeling for $H^{(5)}(G)$ which has the vertex set $V_{H^{(5)}(G)}$ and the hyperedge set $E_{H^{(5)}(G)}$, we define a function $f : V_{H^{(5)}(G)} \cup E_{H^{(5)}(G)} \to \{1, 2, 3, \ldots, 2p + 2q\}$ by

$$\begin{aligned} f(v_i) &= f_G(v_i) && \text{for } i \in \{1, 2, 3, \ldots, p\}, \\ f(v_i') &= 2p + q + 1 - f_G(v_i) && \text{for } i \in \{1, 2, 3, \ldots, p\}, \\ f(e_i) &= f_G(e_i) && \text{for } i \in \{1, 2, 3, \ldots, q\}, \\ f(E_i) &= 3p + 2q + 1 - f_G(e_i) && \text{for } i \in \{1, 2, 3, \ldots, q\}. \end{aligned}$$

It is easy to see that $f|_{V_G \cup E_G} = f_G$. Consider for each $k \in \{1, 2, 3, \ldots, q\}$, we have

$$\begin{aligned} f(E_k) + \sum_{v \in E_k} f(v) &= f(E_k) + f(e_k) + f(v_i) + f(v_i') + f(v_j) + f(v_j') \\ &= (3p + 2q + 1 - f_G(e_k)) + f_G(e_k) \\ &\quad + f_G(v_i) + (2p + q + 1 - f_G(v_i)) \\ &\quad + f_G(v_j) + (2p + q + 1 - f_G(v_j)) \\ &= 7p + 4q + 3 \end{aligned}$$

is constant, where $e_k = v_i v_j$. Since $f(V_{H^{(5)}(G)}) = \{f(v_i) | i \in \{1,2,3,\ldots,p\}\} \cup \{f(v_i') | i \in \{1,2,3,\ldots,p\}\} \cup \{f(e_i) | i \in \{1,2,3,\ldots,q\}\} = \{1,2,3,\ldots,p\} \cup \{2p+1, 2p+2, 2p+3, \ldots, 2p+q\} \cup \{p+1, p+2, p+3, \ldots, 2p\} = \{1,2,3,\ldots,2p+q\}$, f is an SEM labeling. Thus, $H^{(5)}(G)$ is SEM. □

Example 2.4. *In Figure 5, C_5 is SEM by f_G. Since C_5 has $p = 5$ vertices and $q = 5$ edges, by Theorem 2.3, we have $H^{(5)}(C_5)$ admitting an SEM labeling with $\Lambda = 7p + 4q + 3 = 58$.*

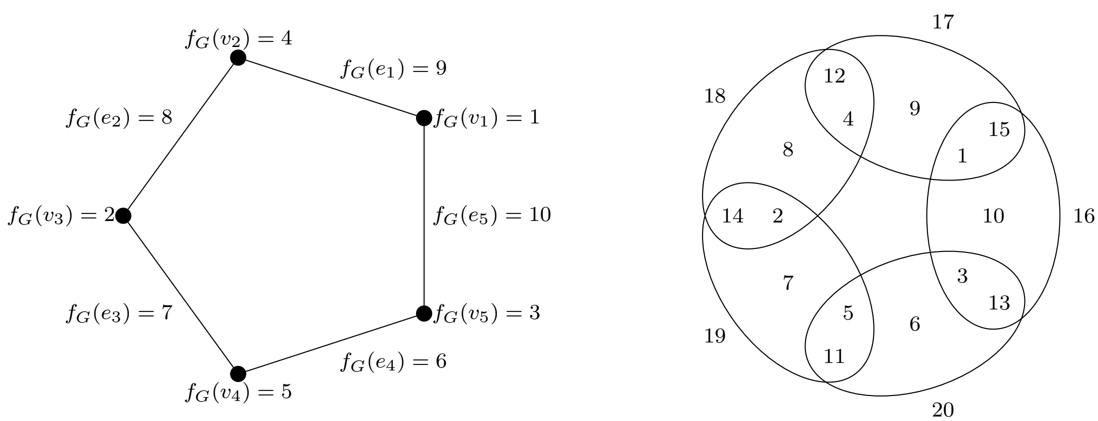

Figure 5: An SEM labeling f_G of C_5 and an SEM labeling of $H^{(5)}(C_5)$ given by Theorem 2.3

3 The SEM labeling of $H^{(6)}(G)$

In section 2, we use the technique that the even consecutive integers can be paired in such a way that their sum in each pair is the same constant. In this section, we deal with a hypergraph $H^{(6)}(G)$ having $2p+2q$ vertices and q hyperedge. Since each hyperedge of $H^{(6)}(G)$ is of the form $E_k = \{e_k, e_k', v_i, v_i', v_j, v_j'\}$, we first think about how to label e_k, e_k' and E_k so that $f(e_k) + f(e_k') + f(E_k)$ is the same constant for all $i \in \{1,2,3,\ldots,q\}$. Fortunately, this task can be done in general if q is odd.

Theorem 3.1. *Let G be a simple graph with the vertex set $\{v_1, v_2, v_3, \ldots, v_p\}$ and the edge set $\{e_1, e_2, e_3, \ldots, e_q\}$. If q is odd, then there exists an SEM labeling for $H^{(6)}(G)$.*

Proof. Let q be an odd positive integer. Define a function $f : V_{H^{(6)}(G)} \cup E_{H^{(6)}(G)} \to \{1,2,3,\ldots,2p+2q, 2p+2q+1, \ldots, 2p+3q\}$ by

$$
\begin{aligned}
f(v_i) &= i && \text{for } i \in \{1,2,3,\ldots,p\},\\
f(v_i') &= 2p+1-i && \text{for } i \in \{1,2,3,\ldots,p\},\\
f(e_i) &= 2p+i && \text{for } i \in \{1,2,3,\ldots,q\},\\
f(e_i') &= 2p+\tfrac{3q-1}{2}+i && \text{for } i \in \{1,2,3,\ldots,\tfrac{q+1}{2}\},\\
f(e_i') &= 2p+\tfrac{q-1}{2}+i && \text{for } i \in \{\tfrac{q+3}{2},\tfrac{q+5}{2},\tfrac{q+7}{2},\ldots,q\},\\
f(E_i) &= 2p+3q-2(i-1) && \text{for } i \in \{1,2,3,\ldots,\tfrac{q+1}{2}\},\\
f(E_i) &= 2p+3q-1-2(i-\tfrac{q+3}{2}) && \text{for } i \in \{\tfrac{q+3}{2},\tfrac{q+5}{2},\tfrac{q+7}{2},\ldots,q\}.
\end{aligned}
$$

We can easily check that f is a bijection and $f(V_{H^{(6)}(G)}) = \{1,2,3,\ldots,2p+2q\}$. Consider each E_k,

- if $k \in \{1, 2, 3, \dots, \frac{q+1}{2}\}$, then

$$\begin{aligned} f(E_k) + \textstyle\sum_{v \in E_k} f(v) &= f(E_k) + f(e_k) + f(e'_k) + f(v_i) + f(v'_i) \\ &\quad + f(v_j) + f(v'_j) \\ &= (2p + 3q - 2(k-1)) + (2p + k) + (2p + \tfrac{3q-1}{2} + k) \\ &\quad + i + (2p + 1 - i) + j + (2p + 1 - j) \\ &= 10p + \tfrac{9q-1}{2} + 4; \end{aligned}$$

- if $k \in \{\frac{q+3}{2}, \frac{q+5}{2}, \frac{q+7}{2}, \dots, q\}$, then

$$\begin{aligned} f(E_k) + \textstyle\sum_{v \in E_k} f(v) &= f(E_k) + f(e_k) + f(e'_k) + f(v_i) + f(v'_i) \\ &\quad + f(v_j) + f(v'_j) \\ &= (2p + 3q - 1 - 2(k - \tfrac{q+3}{2})) + (2p + k) \\ &\quad + (2p + \tfrac{q-1}{2} + k) + i + (2p + 1 - i) + j + (2p + 1 - j) \\ &= 10p + \tfrac{9q-1}{2} + 4; \end{aligned}$$

where $e_k = v_i v_j$. Since $f(E_k) + \sum_{v \in E_k} f(v)$ is the same constant, $H^{(6)}(G)$ is SEM. □

Example 3.2. *Let C_5 be a cycle with $p = 5$ and $q = 5$. We represent $H^{(6)}(C_5)$ and use Theorem 3.1 to label it as shown in Figure 6. Furthermore, $\Lambda = 10p + \frac{9q-1}{2} + 4 = 76$.*

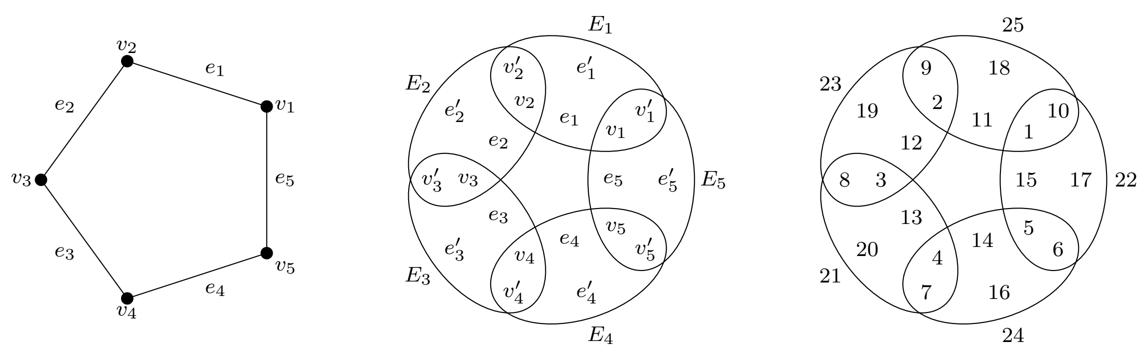

Figure 6: $C_5, H^{(6)}(C_5)$ and an SEM labeling for $H^{(6)}(C_5)$ with $\Lambda = 76$

In the case when q is even, it is impossible to distribute $3q$ consecutive integers into n 3-subsets so that the sum of elements of each 3-subsets is the same constant. Thus, constructing a labeling for $H^{(6)}(G)$, where G is a simple graph with even size, depends on a structure of its hypergraph. However, some hypergraphs are justified to be SEM. For example, in [3], they gave the SEM labelings of $H^{(6)}(C_n)$ and $H^{(6)}(P_n)$ (note that in [3], they defined $H^{(6)}(C_n)$ and $H^{(6)}(P_n)$ in terms of ${}^2C_n^{(6)}$ and ${}^2P_n^{(6)}$, respectively). In the next section, we show that $H^{(6)}(nC_4)$ is SEM.

4 The SEM labeling of $H^{(6)}(nC_4)$

Firstly, we represent nC_4 as shown in Figure 7. Note that v_{ij} denote the jth vertex of ith cycle. Furthermore, $e_{k1} = v_{k1}v_{k2}, e_{k2} = v_{k2}v_{k3}, e_{k3} = v_{k3}v_{k4}$ and $e_{k4} = v_{k4}v_{k1}$ are 4 edges of the kth cycle where $k \in \{1, 2, 3, \dots, n\}$.

Thus, by the Definition 1.4, $H^{(6)}(nC_4)$ can be illustrated by Figure 8. Note that $H^{(6)}(nC_4)$ has $16n$ vertices and $4n$ hyperedges. Before constructing the labeling, we prove the following lemma.

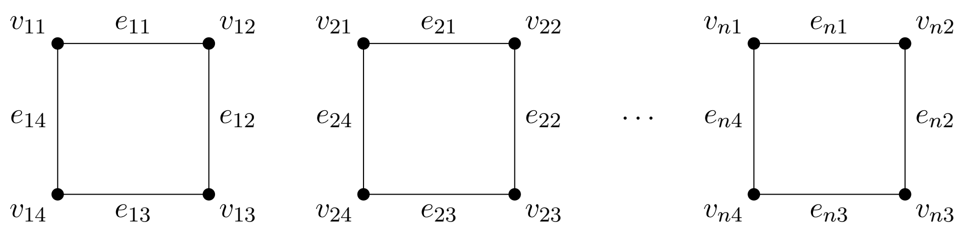

Figure 7: nC_4

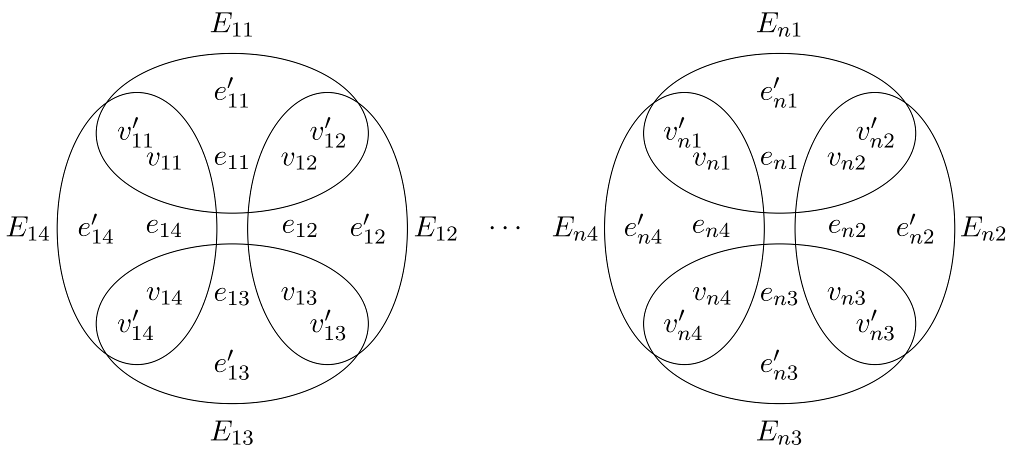

Figure 8: $H^{(6)}(nC_4)$

Lemma 4.1. *Let n be a positive integer. The list of consecutive integers $1, 2, 3, \ldots, 16n$ can be paired into $8n$ doubleton in such a way that the sums in each doubletons are $6n+2, 6n+3, 6n+4, \ldots, 10n+1, 22n+1, 22n+2, 22n+3, \ldots, 26n$.*

Proof. We define doubletons as the following

1. $\{1, 10n\}$,
2. $\{2, 6n\}, \{3, 6n+1\}, \{4, 6n+2\}, \ldots, \{2n+1, 8n-1\}$,
3. $\{2n+2, 4n+1\}, \{2n+3, 4n+2\}, \{2n+4, 4n+3\}, \ldots, \{4n, 6n-1\}$,
4. $\{8n, 14n+1\}, \{8n+1, 14n+2\}, \{8n+2, 14n+3\}, \ldots, \{10n-1, 16n\}$,
5. $\{10n+1, 12n+1\}, \{10n+2, 12n+2\}, \{10n+3, 12n+3\}, \ldots, \{12n, 14n\}$.

Then, the result follows immedietly. □

Now, we are ready to show that $H^{(6)}(nC_4)$ is SEM.

Theorem 4.2. *$H^{(6)}(nC_4)$ is SEM.*

Proof. Let S_k be the doubleton whose sum of both elements is k. Then, by Lemma 4.1, we have $S_{6n+2}, S_{6n+3}, S_{6n+4}, \ldots, S_{10n+1}, S_{22n+1}, S_{22n+2}, S_{22n+3}, \ldots, S_{26n}$. To construct an SEM labeling f :

$V_{H^{(6)}(nC_4)} \cup E_{H^{(6)}(nC_4)} \to \{1,2,3,\ldots,20n\}$, we define the bijective mapping in such a way that

$$\begin{array}{rcll}
f(\{v_{i1}, v'_{i1}\}) & = & S_{6n+1+i} & \text{for } i \in \{1,2,3,\ldots,n\},\\
f(\{v_{i2}, v'_{i2}\}) & = & S_{8n+1+i} & \text{for } i \in \{1,2,3,\ldots,n\},\\
f(\{v_{i3}, v'_{i3}\}) & = & S_{8n+2-i} & \text{for } i \in \{1,2,3,\ldots,n\},\\
f(\{v_{i4}, v'_{i4}\}) & = & S_{10n+2-i} & \text{for } i \in \{1,2,3,\ldots,n\},\\
f(\{e_{i1}, e'_{i1}\}) & = & S_{26n+1-i} & \text{for } i \in \{1,2,3,\ldots,n\},\\
f(\{e_{i2}, e'_{i2}\}) & = & S_{22n+i} & \text{for } i \in \{1,2,3,\ldots,n\},\\
f(\{e_{i3}, e'_{i3}\}) & = & S_{24n+i} & \text{for } i \in \{1,2,3,\ldots,n\},\\
f(\{e_{i4}, e'_{i4}\}) & = & S_{23n+i} & \text{for } i \in \{1,2,3,\ldots,n\},\\
f(E_{i1}) & = & 18n+1-i & \text{for } i \in \{1,2,3,\ldots,n\},\\
f(E_{i2}) & = & 20n+1-i & \text{for } i \in \{1,2,3,\ldots,n\},\\
f(E_{i3}) & = & 16n+i & \text{for } i \in \{1,2,3,\ldots,n\},\\
f(E_{i4}) & = & 19n+1-i & \text{for } i \in \{1,2,3,\ldots,n\}.
\end{array}$$

It is clear that $f(V_{H^{(6)}(nC_4)}) = \{1,2,3,\ldots,16n\}$ by the prove of Lemma 4.1. Also, we have

$$\begin{array}{rcl}
f(v_{i1}) + f(v'_{i1}) & = & 6n+1+i,\\
f(v_{i2}) + f(v'_{i2}) & = & 8n+1+i,\\
f(v_{i3}) + f(v'_{i3}) & = & 8n+2-i,\\
f(v_{i4}) + f(v'_{i4}) & = & 10n+2-i,\\
f(e_{i1}) + f(e'_{i1}) & = & 26n+1-i,\\
f(e_{i2}) + f(e'_{i2}) & = & 22n+i,\\
f(e_{i3}) + f(e'_{i3}) & = & 24n+i,\\
f(e_{i4}) + f(e'_{i4}) & = & 23n+i.
\end{array}$$

To verify that f is an SEM labeling, we consider E_{ij} for all $i \in \{1,2,3,\ldots,n\}$,

- if $j = 1$, then

$$\begin{array}{rcl}
f(E_{i1}) + \sum_{v\in E_{i1}} f(v) & = & f(E_{i1}) + (f(e_{i1}) + f(e'_{i1})) + (f(v_{i1}) + f(v'_{i1}))\\
& & +(f(v_{i2}) + f(v'_{i2}))\\
& = & (18n+1-i) + (26n+1-i) + (6n+1+i)\\
& & +(8n+1+i)\\
& = & 58n+4;
\end{array}$$

- if $j = 2$, then

$$\begin{array}{rcl}
f(E_{i2}) + \sum_{v\in E_{i2}} f(v) & = & f(E_{i2}) + (f(e_{i2}) + f(e'_{i2})) + (f(v_{i2}) + f(v'_{i2}))\\
& & +(f(v_{i3}) + f(v'_{i3}))\\
& = & (20n+1-i) + (22n+i) + (8n+1+i)\\
& & +(8n+2-i)\\
& = & 58n+4;
\end{array}$$

- if $j = 3$, then

$$\begin{array}{rcl}
f(E_{i3}) + \sum_{v\in E_{i3}} f(v) & = & f(E_{i3}) + (f(e_{i3}) + f(e'_{i3})) + (f(v_{i3}) + f(v'_{i3}))\\
& & +(f(v_{i4}) + f(v'_{i4}))\\
& = & (16n+i) + (24n+i) + (8n+2-i)\\
& & +(10n+2-i)\\
& = & 58n+4;
\end{array}$$

- if $j = 4$, then

$$\begin{aligned} f(E_{i4}) + \sum_{v \in E_{i4}} f(v) &= f(E_{i4}) + (f(e_{i4}) + f(e'_{i4})) + (f(v_{i4}) + f(v'_{i4})) \\ &\quad +(f(v_{i1}) + f(v'_{i1})) \\ &= (19n + 1 - i) + (23n + i) + (10n + 2 - i) \\ &\quad +(6n + 1 + i) \\ &= 58n + 4. \end{aligned}$$

□

Thus, the sum $f(E_{ij}) + \sum_{v \in E_{ij}} f(v)$ is the same constant for every hyperedge E_{ij} of $H^{(6)}(nC_4)$. Therefore, $H^{(6)}(nC_4)$ is SEM.

Example 4.3. *To construct an SEM labeling for* $H^{(6)}(2C_4)$*. Notice that* $n = 2$ *and* $|V_{H^{(6)}(2C_4)}| = 16n = 32$*. By using Lemma 4.1, we have doubletons,*

- $\{2, 12\}, \{6, 9\}, \{3, 13\}, \{7, 10\}, \{4, 14\}, \{8, 11\}, \{5, 15\}\{1, 20\}$,
- $\{16, 29\}, \{21, 25\}, \{17, 30\}, \{22, 26\}, \{18, 31\}, \{23, 27\}, \{19, 32\}\{24, 28\}$,

whose sums in each doubleton are $14, 15, 16, \ldots, 21$ *and* $45, 46, 47, \ldots, 52$*, orderly. By Theorem 4.2, we give labeling as follow,*

$v_{11} \to 2$,	$e_{12} \to 16$,	$v_{23} \to 3$,	$e_{24} \to 22$,
$v'_{11} \to 12$,	$e'_{12} \to 29$,	$v'_{23} \to 13$,	$e'_{24} \to 26$,
$v_{12} \to 4$,	$e_{13} \to 18$,	$v_{24} \to 5$,	$E_{11} \to 36$,
$v'_{12} \to 14$,	$e'_{13} \to 31$,	$v'_{24} \to 15$,	$E_{12} \to 40$,
$v_{13} \to 7$,	$e_{14} \to 17$,	$e_{21} \to 19$,	$E_{13} \to 33$,
$v'_{13} \to 10$,	$e'_{14} \to 30$,	$e'_{21} \to 32$,	$E_{14} \to 38$,
$v_{14} \to 1$,	$v_{21} \to 6$,	$e_{22} \to 21$,	$E_{21} \to 35$,
$v'_{14} \to 20$,	$v'_{21} \to 9$,	$e'_{22} \to 25$,	$E_{22} \to 39$,
$e_{11} \to 24$,	$v_{22} \to 8$,	$e_{23} \to 23$,	$E_{23} \to 34$,
$e'_{11} \to 28$,	$v'_{22} \to 11$,	$e'_{23} \to 27$,	$E_{24} \to 37$.

We illustrate the labeling as shown in Figure 9. Moreover, $\Lambda = 58n + 4 = 120$.

5 Conclusion an Discussion

In this article, we construct hypergraphs $H^{(5)}(G)$ and $H^{(6)}(G)$ from an arbitrary simple graph G. Even if G is or is not SEM graph, we still can prove that $H^{(5)}(G)$ is always an SEM 5-uniform hypergraph and $H^{(6)}(G)$ is an SEM 6-uniform hypergraph if G has even number of edges. For $H^{(6)}(G)$ with even number of edges, we give only example of an SEM labeling for $H^{(6)}(nC_4)$. There is a way to add more vertices to an SEM hypergraph H and preserve its SEM property. This method is similar to the one in [3] and we give a short proof here.

Theorem 5.1. *Let H be a hypergraph with p vertices and q hyperedges. If H is SEM, then there exists an SEM hypergraph with $p + 2q$ vertices and q hyperedges.*

Proof. Assume that H admits the SEM labeling f and has hyperedge set $E_H = \{E_1, E_2, E_3, \ldots, E_q\}$. Thus, $f(E_H) = \{p+1, p+2, p+3, \ldots, p+q\}$. Let $V' = \{v_1, v_2, v_3, \ldots, v_{2q}\}$ be the set of new vertices. Define

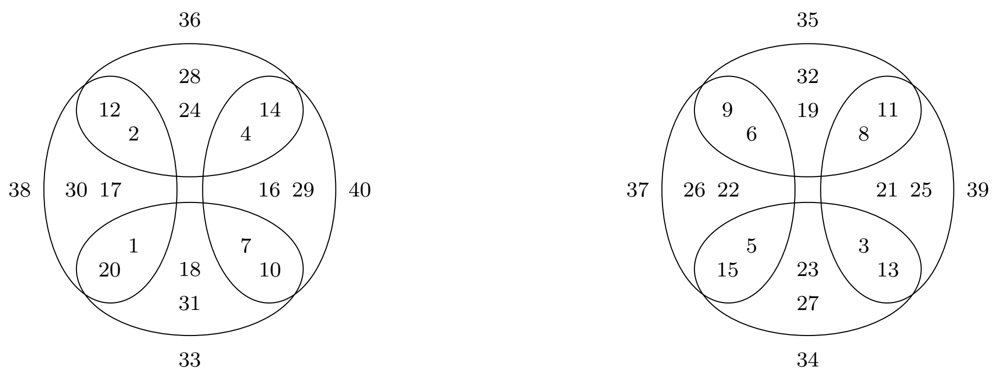

Figure 9: an SEM labeling for $H^{(6)}(2C_4)$ with $\Lambda = 120$

$E_i' = E_i \cup \{v_i, v_{2q-i}\}$ for $i \in \{1, 2, 3, \ldots, q\}$. To show that hypergraph H' with vertex set $V_{H'} = V_H \cup V'$ and hyperedge set $E_{H'} = \cup_{i=1}^{q}\{E_i'\}$ is SEM, we give a mapping f' by

$$\begin{array}{rcll} f'(v) & = & f(v) & \text{for all } v \in V_H, \\ f'(v_i) & = & p+i & \text{for all } i \in \{1,2,3,\ldots,2q\}, \\ f'(E_i') & = & f(E_i)+2q & \text{for all } i \in \{1,2,3,\ldots,q\}. \end{array}$$

It is straight forward to check that $f'(V_H \cup V') = \{1, 2, 3, \ldots, p + 2q\} = |V_{H'}|$ and $f(e) + \sum_{v \in e} f(v)$ is the same constant for each $e \in E_{H'}$. Thus, H' is SEM. □

By Theorem 5.1, if H is a k-uniform hypergraph then H' is a $(k + 2)$-uniform hypergraph. Hence, we can obatain a $(k + 2m)$-uniform SEM hypergraph by iterating the process in Theorem 5.1 m times.

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